Three dimensional magnetostatics using the magnetic vector potential with nodal and edge finite elements

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Abstract -- **In this paper, the three dimensional vector potential magnetostatic problem is solved using nodal and edge finite elements. The influence of the gauge condition A.w=0 in the characteristics of the edge-element generated matrix is analyzed. Three gauge conditions are studied: no gauge, the complete A.w=0 gauge and the incomplete A.w=0 gauge condition.**

I. Introduction

The 3D nodal finite-element magnetostatic formulations are usually based on scalar potentials. However, these formulations present some problems such as the cancellation errors in highly permeable regions and difficulties to treat multiply connected regions [4]. These problems do not appear when the magnetic vector potential is used. However, there are also some computational drawbacks in this case, due to the use of three unknowns per node and due to the necessity of imposing a gauge condition.

A new kind of finite-element that is being nowadays used is the edge element [1]. This element is very interesting from the computational point of view. Its degrees of freedom are line integrals of the vector potential along the edges. These elements automatically impose the tangential continuity of the interpolated variable between elements and let the normal component free. These elements are also adequate to impose the gauge **A.w=0**, where **w** is an arbitrary vector field that does not possess closed lines.

The main objective of this paper is to compare the use of the edge and nodal finite elements in the solution of 3D magnetic vector potential static problems. The number of unknowns, the number of non zero elements in the matrix and the number of ICCG iterations are analyzed. The influence of the gauge **A.w=0**, for edge elements is also analyzed.

II. Mathematical Formulation

The magnetic vector potential **A**, defined by

$$\vec{B} = \vec{\nabla} \times \vec{A}, \qquad (1)$$

satisfies the following differential equation:

$$\vec{\nabla} \times (\nu \vec{\nabla} \times \vec{A}) = \vec{J} \qquad (2)$$

where **B** is the flux density, ν is the magnetic reluctivity and **J** is the current density vector.

This problem is not completely defined yet. The interface conditions between regions of different material characteristics, the boundary and gauge conditions must be specified. The interface conditions are based on the tangential continuity of the magnetic field **H** and on the continuity of the normal component of the magnetic induction **B** on the interface between regions of different characteristics, that is:

$$\hat{n} \times (\nu(\vec{\nabla} \times \vec{A})_1) = \hat{n} \times (\nu(\vec{\nabla} \times \vec{A})_2)$$
$$(\vec{\nabla} \times \vec{A})_1 \cdot \hat{n} = (\vec{\nabla} \times \vec{A})_2 \cdot \hat{n} \qquad (3)$$

Here **n** is a unit vector normal to the interface.

The boundary conditions can be specified as:

$$\nu \vec{\nabla} \times \vec{A} \times \hat{n} = 0 \quad \textit{where} \quad \vec{H} \times \hat{n} = 0$$
$$\vec{A} \times \hat{n} = 0 \quad \textit{where} \quad \vec{B} \cdot \hat{n} = 0 \qquad (4)$$

where **n** is a unit vector normal to the boundary.

Using the Galerkin method and considering the interface and boundary conditions, the following weak form can be obtained [4]:

$$\int_{\Omega} \nu \vec{\nabla} \times \vec{A} \cdot \vec{\nabla} \times \vec{W} \, d\Omega = \int_{\Omega} \vec{W} \cdot \vec{J} \, d\Omega \qquad \forall \vec{W} \qquad (5)$$

where **W** is a vector weighting function.

Equation (2) associated with conditions (3) and (4) does not assure the uniqueness of the solution **A**. If $\mathbf{A}_1$ is one solution, other solutions can be generated adding an arbitrary gradient function, that is:

$$\vec{B} = \vec{\nabla} \times \vec{A}_1 = \vec{\nabla} \times (\vec{A}_2 + \vec{\nabla}\phi) = \vec{\nabla} \times \vec{A}_2 \qquad (6)$$

A gauge condition must be imposed so that the magnetic vector potential is uniquely determined. The strategy to apply this gauge is different if we consider nodal or edge elements. This is discussed in the following section:

A. *Nodal finite-elements*

The Coulomb gauge is imposed to guarantee the solution uniqueness, that is:

$$\vec{\nabla}\cdot\vec{A}=0 \qquad (7)$$

Now, the weak form becomes [4]:

$$\int_{\Omega}\nu(\vec{\nabla}\times\vec{A}\cdot\vec{\nabla}\times\vec{W}+\vec{\nabla}\cdot\vec{A}\,\vec{\nabla}\cdot\vec{W})\,d\Omega=\int_{\Omega}\vec{W}\cdot\vec{J}\,d\Omega \qquad (8)$$

The magnetic vector potential is approximated by nodal finite elements, that is:

$$\vec{A}^h=\sum_{i=1}^{NNOS}(A_{xi}\hat{i}+A_{yi}\hat{j}+A_{zi}\hat{k})N_i \qquad (9)$$

where NNOS is the number of nodes of the element, (A_{xi}, A_{yi}, A_{zi}) are the components of **A** at the node i, and N_i is the nodal shape function associated with the node i.

B. *Edge finite elements*

The magnetic vector potential is now approximated by vector shape functions, defined over the element edges, that is:

$$\vec{A}^h=\sum_{i=1}^{NAR}A_i\vec{W}_i \qquad (10)$$

where NAR is the number of edges of the element. The A_i's are the line integrals of **A** along the edges and the $\mathbf{W}_i$'s are the vector shape functions. The line integral of $\mathbf{W}_i$ along the edge where it is defined is equal to one and along the other edges it is equal to zero[7].

As already mentioned, a gauge condition must be imposed to guarantee the solution uniqueness. The adopted gauge for edge elements is $\mathbf{A}\cdot\mathbf{w}=\mathbf{0}$ where **w** is an arbitrary vector field without closed lines. In [2] it is proved that this gauge guarantees the solution uniqueness.

If the finite element mesh is seen as a connected graph, there exists a very interesting way to apply this gauge. The discretized version of $\mathbf{A}\cdot\mathbf{w}=\mathbf{0}$ is obtained choosing the direction **w** as an arbitrary tree of the mesh graph [2]. Then, the degrees of freedom associated with the tree are zeroed and only the edges corresponding to the co-tree must be evaluated.

As the tree is arbitrary, there are some cases where it can generate an ill conditioned matrix and this can result in a big number of ICCG iterations [5]. To avoid this problem, an incomplete gauge condition has been investigated. In this case, the edges that are zeroed form an incomplete tree, that is, a path connecting two arbitrary nodes cannot exist. In this work, the construction of this incomplete tree is based on the idea that the edges where **A** is significant should not be zeroed [3].

III. Numerical Results

The problem shown in Fig. 1 has been solved to compare the two finite element types. The problem consists of a cube with relative permeability equal to 1000. A magnetic induction of 1 T is applied in the z direction. Hexahedral elements are used to discretize the geometry. The solution of the generated matrix system is obtained by the ICCG method. The convergence criterion for the ICCG is reached when the Euclidean norm of the residual is less than 1E-7. The problem does not present an analytic solution. So, **B** is evaluated at the point x=y=z=10mm so that the calculated values can be compared with the values presented in [5].

Different meshes are used. The discretization characteristics are presented in Table I. Table II shows the results obtained using nodal elements. Tables III, IV and V show the results obtained with edge elements without using any gauge, using the **A.w=0** gauge and the incomplete **A.w=0** gauge, respectively. In this case, the incomplete gauge was applied zeroing all the edge unknowns in the z direction.

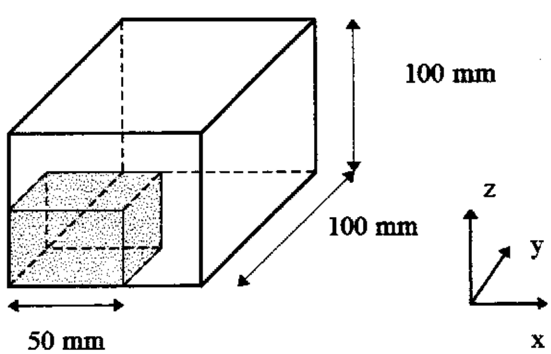

Fig. 1:A magnetic cube in an uniform field

Table I.
Discretization Characteristics

Divisions	Number of Elements	Number of Nodes	Number of Edges
4x4x4	64	125	300
6x6x6	216	343	882
8x8x8	512	729	1944

Table II.
Nodal Elements

Divisions	Matrix coeffic.	Equations	ICCG iterations	**B** |T|
4x4x4	3630	177	2	2.5860
6x6x6	16224	615	3	2.6114
8x8x8	43826	1477	3	2.6288

Table III.
Edge Elements without gauge

Divisions	Matrix coeffic.	Equations	ICCG iterations	**B** |T|
4x4x4	1680	156	8	2.6007
6x6x6	7294	570	10	2.5912
8x8x8	19356	1400	13	2.5963

Table IV.
Edge Elements with the **A**·**w** =0 gauge.

Divisions	Matrix coeffic.	Equations	ICCG iterations	**B** |T|
4x4x4	851	111	31	2.6009
6x6x6	3475	395	80	2.6040
8x8x8	8971	959	131	2.6116

Table V.
Edge elements with the incomplete **A**·**w** =0 gauge

Divisions	Matrix coeffic.	Equations	ICCG iterations	**B** |T|
4x4x4	892	120	10	2.6007
6x6x6	3592	420	16	2.5914
8x8x8	9204	1008	21	2.5964

The results were obtained through an Object Oriented Program written in the C++ language [6] using a 486, 50 MHz, PC.

When no gauge is applied, edge elements generate a matrix system with dimension approximately equal to the dimension of the nodal elements generated matrix. However, the first matrix is much more sparse.

The nodal elements' system has a very fast ICCG convergence, as compared to the edge elements. It can be seen in Table III that the formulation without gauge presents the lowest number of ICCG iterations for the edge element formulations. However, the number of equations and non zero elements is much bigger than the ones presented in Tables IV and V. Comparing the data in Tables IV and V, it can be seen that the incomplete gauge formulation reduces significantly the number of ICCG iterations, as compared to the complete gauge formulation. It can also be seen that the number of unknowns and of non zero coefficients is only a little greater than in the complete gauge formulation.

The number of ICCG iterations can be reduced if the "Shifted Incomplete Cholesky Factorization" [8] is used. In this method the standard Incomplete Cholesky Factorization is modified including a shift factor γ to scale the diagonal elements. The case of $\gamma=1$ corresponds to the Standard Incomplete Cholesky Factorization. The effectiveness of the preconditioning method changes with γ. Fig. 2 shows the influence of this factor in the number of ICCG iterations for all the edge element formulations.

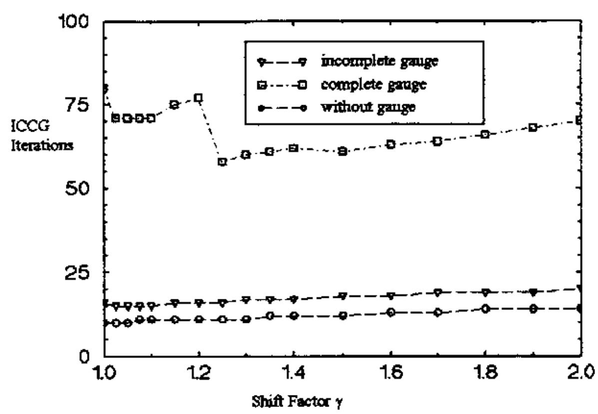

Fig. 2: Number of ICCG iterations as influenced by the Incomplete Cholesky Factorization Shift Factor

It can be seen in Fig. 2 that for the no gauge and the Incomplete Gauge formulations the shift factor does not have a big influence on the ICCG convergence and the standard ICCG can be used. However, for the complete gauge formulation, we must use a shift factor greater than one.

The flux density **B** is almost the same for the four different formulations and converges to the value presented in [5]. Fig. 3 confirms this, showing the values of **B** along the z direction evaluated with nodal and edge elements with the complete gauge.

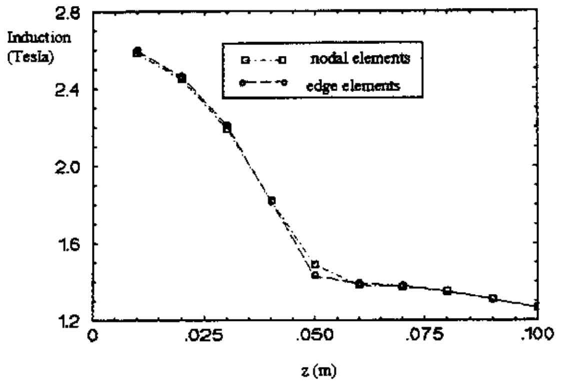

Fig. 3: |B| in the z direction (x=y=0.01)

IV. Conclusions

In this paper we presented a comparison between nodal and edge finite elements for 3D vector potential magnetostatic formulations. From the results presented in the previous sections, the following conclusions can be obtained:

- Nodal elements present better convergence for the ICCG method;
- The edge elements matrix is more sparse than the matrix generated by the nodal elements matrix;
- The application of the complete gauge condition reduces considerably the number of equations and the number of non-zero elements in the system matrix. However, it increases a lot the number of ICCG iterations;
- The complete gauge formulation is very sensitive to the value of the shift factor in the Incomplete Cholesky Factorization. The standard factorization ($\gamma = 1$) must be avoided in this case, because the number of ICCG iterations is very high;
- The incomplete gauge seems to be the best of the edge element formulations if we consider the analyzed aspects of memory requirements and the number of ICCG iterations. However, it must be emphasized that this conclusion is limited to the simple structure treated in this paper. For more complicated structures additional work must be done to guarantee that this is still valid.

V. References

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