A Closed-Form Sensitivity Analysis of Transmission Lines

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Abstract – A new approach for frequency-domain sensitivity analysis of transmission lines is presented. The propagation problem for the voltage sensitivity is considered and solved in terms of the closed-form Green's function of the 1-D wave propagation problem. This leads to a closed-form solution for the voltage sensitivity. The accuracy of the proposed method is verified by comparison with the perturbation approach.

Keywords: Transmission lines, interconnects, sensitivity analysis, Green's function.

I. INTRODUCTION

With the rapid increase in operating speeds, density, and complexity of modern electronics, the effects of interconnects such as delay, ringing and distortion have become a dominant factor. As the rise-times in nowadays interconnects may be few tens of picoseconds, distributed lossy transmission line models must be used assuming the quasi-TEM mode as the dominant one [1, 2].

At the design stage, it may be useful to compute not only the response of the line to a given excitation, but also its sensitivity with respect to a physical or geometrical parameter. From this perspective, it is frequently required that designers make the proper trade-off, often between conflicting design requirements using optimization techniques, to obtain the best possible performance [3].

Sensitivity analysis has been widely used in control and circuit theory [4, 5]. An extensive research work has been done over the recent years in the implementation of sensitivity analysis techniques with full-wave electromagnetic solvers for high frequency problems [6–14].

The knowledge of response derivatives is also crucial for macro-modeling purposes [15]. In fact, recent advancements in macro-modeling techniques have demonstrated that the use of response derivatives is effective to speed-up the generation of macromodels of linear systems while preserving the accuracy [16–18].

Although the application of full-wave techniques to compute transmission lines sensitivity is surely feasible, efficient use of computing resources is always to be preferred and, when possible, analytical solutions are to be considered. In this paper we present a new approach to frequency-domain sensitivity of transmission lines which is based on the use of the Green's function of the 1-D wave propagation problem. Telegrapher's equations are modified inorder to incorporate port currents as external sources to the system. In [19] it has been shown that such a technique allows to treat the Telegrapher's equation as a Sturm-Liouville problem for the voltage which can be directly written in terms of the Green's function of the 1-D wave propagation [20]. Voltage and current sensitivities satisfy the same Sturm-Liouville problem as voltages and currents but with a different forcing term. Hence, the same Green's function can be adopted. The knowledge of the closed-form Green's function for the transmission line problem permits to compute the voltage sensitivity analytically, thus avoiding any numerical processing and pawns the way to an accurate and efficient sensitivity analysis. The proposed methodology is well suited to be extended to the computation of higher-order sensitivities.

The paper is organized as follows. In Section II the formulation is presented leading to the computation of the voltage sensitivity in terms of the closed-form Green's function. Section III presents the computation of derivatives and voltage sensitivities in a closed-form. Two numerical examples are described in Section IV confirming the capability of the proposed approach to provide a fast and reliable method to sensitivity of transmission lines. The conclusions are drawn in Section V.

II. SENSITIVITY FORMULATION

Let us consider the transmission line illustrated in Fig. 1.

The physics of transmission lines under the quasi-TEM hypothesis is captured by the Telegrapher's equations [1],

$$\frac{\partial}{\partial z}v(z,t) = -R\ i(z,t) - L\ \frac{\partial}{\partial t}i(z,t) \tag{1a}$$

$$\frac{\partial}{\partial z}i(z,t) = -G\ v(z,t) - C\ \frac{\partial}{\partial t}v(z,t) + i_S(z,t), \tag{1b}$$

where $R \in \Re$, $L \in \Re$, $C \in \Re$ and $G \in \Re$ are the per-unit-length (p.u.l.) parameters of the transmission line, $v(z,t) \in \Re$ and $i(z,t) \in \Re$ represent the voltage and current as a function of position z and time t, and $i_S(z,t)$ describe a distributed current source along the line. Differentiating (1a) and (1b) with respect to a

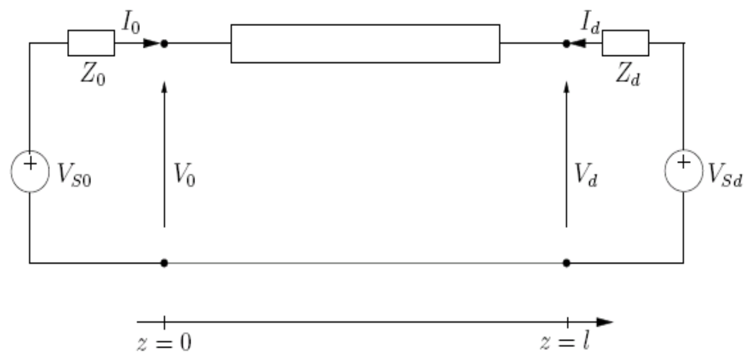

Fig. 1. Transmission line with linear terminations.

parameter λ (where λ represents any electrical or physical parameter of interest of the transmission line) yields the following equations,

$$\frac{\partial}{\partial z}\hat{v}(z,t) = -R\,\hat{i}(z,t) - L\,\frac{\partial}{\partial t}\hat{i}(z,t) - \left(\frac{\partial R}{\partial \lambda}\,i(z,t) + \frac{\partial L}{\partial \lambda}\frac{\partial}{\partial t}i(z,t)\right) \tag{2a}$$

$$\frac{\partial}{\partial z}\hat{i}(z,t) = -G\,\hat{v}(z,t) - C\,\frac{\partial}{\partial t}\hat{v}(z,t) - \left(\frac{\partial G}{\partial \lambda}\,v(z,t) + \frac{\partial C}{\partial \lambda}\frac{\partial}{\partial t}v(z,t)\right) + \frac{\partial}{\partial \lambda}i_S(z,t), \tag{2b}$$

where the sensitivity variables in equations (2a) and (2b) are defined as,

$$\hat{v}(z,t) = \frac{\partial}{\partial \lambda}v(z,t) \qquad \hat{i}(z,t) = \frac{\partial}{\partial \lambda}i(z,t). \tag{3}$$

Transforming equations (1a), (1b), (2a) and (2b) in the Laplace domain, we obtain,

$$\frac{\partial}{\partial z}V(z,s) = -Z_s(s)I(z,s) \tag{4a}$$

$$\frac{\partial}{\partial z}I(z,s) = -Y_p(s)V(z,s) + I_S(z,s), \tag{4b}$$

$$\frac{\partial}{\partial z}\widehat{V}(z,s) = -Z_s(s)\widehat{I}(z,s) - \frac{\partial Z_s(s)}{\partial \lambda}I(z,s), \tag{4c}$$

$$\frac{\partial}{\partial z}\widehat{I}(z,s) = -Y_p(s)\widehat{V}(z,s) - \frac{\partial Y_p(s)}{\partial \lambda}V(z,s) + \frac{\partial}{\partial \lambda}I_S(z,s), \tag{4d}$$

where the series impedance $Z_s(s)$ and the parallel admittance $Y_p(s)$ of the line are defined as,

$$Z_s(s) = R + sL \tag{5a}$$

$$Y_p(s) = G + sC. \tag{5b}$$

Differentiating equation (4c) with respect to z we obtain,

$$\frac{\partial^2}{\partial z^2}\widehat{V}(z,s) = -Z_s(s)\frac{\partial}{\partial z}\widehat{I}(z,s) - \frac{\partial Z_s(s)}{\partial \lambda}\frac{\partial}{\partial z}I(z,s). \tag{6}$$

If we substitute equations (4b) and (4d) in equation (6), we can write,

$$\frac{\partial^2}{\partial z^2}\widehat{V}(z,s) - \gamma^2(s)\,\widehat{V}(z,s) = \frac{\partial \gamma^2}{\partial \lambda}V(z,s) - Z_s(s)\frac{\partial I_S(z,s)}{\partial \lambda} - \frac{\partial Z_s(s)}{\partial \lambda}I_S(z,s) \tag{7}$$

where

$$\gamma^2 = Z_s(s)Y_p(s)$$
$$\frac{\partial(\gamma^2)}{\partial \lambda} = \left(Z_s(s)\frac{\partial Y_p(s)}{\partial \lambda} + Y_p(s)\frac{\partial Z_s(s)}{\partial \lambda}\right). \tag{8}$$

In the following, the current sources $I_S(z,s)$ are assumed to be located only in correspondence of the terminations, yielding,

$$I_S(z,s) = I_0(z,s)\delta(z) + I_l(z,s)\delta(z-l) \tag{9}$$

where $\delta(z)$ represents the Dirac delta function.

Equation (7) represents a Helmholtz equation whose formal solution can be obtained by using the Green's function approach. The computation of the forcing term in equation (7) requires the evaluation of the derivative of the p.u.l. parameters with respect to λ, the port currents I_S and the voltage distribution $V(z,s)$. The latter expression can be obtained through the standard transmission line technique while the expression of the derivative of I_S is to be computed.

The transmission line can be represented as a multiport system with port voltages and currents at $z=0$ and $z=l$ related by,

$$\begin{bmatrix} V_0 \\ V_l \end{bmatrix} = \begin{bmatrix} Z_{11} & Z_{12} \\ Z_{21} & Z_{22} \end{bmatrix} \begin{bmatrix} I_0 \\ I_l \end{bmatrix} \tag{10}$$

where V_0 and V_l are the voltage at the port $z=0$ and $z=l$. Furthermore, the termination conditions at the port $z=0$ and $z=d$, as shown in Fig. 1, read,

$$\begin{bmatrix} V_0 \\ V_l \end{bmatrix} = \begin{bmatrix} V_{S0} \\ V_{Sd} \end{bmatrix} - \begin{bmatrix} Z_0 & 0 \\ 0 & Z_l \end{bmatrix} \begin{bmatrix} I_0 \\ I_l \end{bmatrix} \tag{11}$$

Combining equations (10) and (11) we obtain,

$$\left[\begin{array}{c} I_0 \\ I_l \end{array}\right] = \left(\left[\begin{array}{cc} Z_0 & 0 \\ 0 & Z_l \end{array}\right] + \left[\begin{array}{cc} Z_{11} & Z_{12} \\ Z_{21} & Z_{22} \end{array}\right]\right)^{-1} \left[\begin{array}{c} V_{S0} \\ V_{Sd} \end{array}\right] \tag{12}$$

The evaluation of the forcing term equation (7) requires computing the port current sensitivities,

$$\frac{\partial}{\partial\lambda}\left[\begin{array}{c} I_0 \\ I_l \end{array}\right] = \frac{\partial}{\partial\lambda}\left(\left[\begin{array}{cc} Z_0 & 0 \\ 0 & Z_l \end{array}\right] + \left[\begin{array}{cc} Z_{11} & Z_{12} \\ Z_{21} & Z_{22} \end{array}\right]\right)^{-1} \left[\begin{array}{c} V_{S0} \\ V_{Sd} \end{array}\right] \tag{13}$$

where voltage sources are assumed not depending on the parameter λ. Although it can be obtained by first computing the inverse of the global impedance matrix and then evaluating the derivative, an elegant way to do that is to separate the derivative from the matrix inverse. It can be done observing that,

$$\boldsymbol{A}(\lambda)\boldsymbol{A}^{-1}(\lambda) = \boldsymbol{I}.$$

Hence, by the chain rule,

$$\frac{d\boldsymbol{A}(\lambda)}{d\lambda}\boldsymbol{A}^{-1}(\lambda) + \boldsymbol{A}(\lambda)\frac{d\boldsymbol{A}^{-1}(\lambda)}{d\lambda} = \boldsymbol{0}$$

we obtain

$$\frac{d\boldsymbol{A}^{-1}(\lambda)}{d\lambda} = -\boldsymbol{A}^{-1}(\lambda)\frac{d\boldsymbol{A}(\lambda)}{d\lambda}\boldsymbol{A}^{-1}(\lambda).$$

This identity allows to calculate the current sensitivity equation (12) as follows,

$$\frac{\partial}{\partial\lambda}\left[\begin{array}{c} I_0 \\ I_l \end{array}\right] = -\left(\left[\begin{array}{cc} Z_0 & 0 \\ 0 & Z_l \end{array}\right] + \left[\begin{array}{cc} Z_{11} & Z_{12} \\ Z_{21} & Z_{22} \end{array}\right]\right)^{-1} \frac{\partial}{\partial\lambda}\left(\left[\begin{array}{cc} Z_0 & 0 \\ 0 & Z_l \end{array}\right] + \left[\begin{array}{cc} Z_{11} & Z_{12} \\ Z_{21} & Z_{22} \end{array}\right]\right) \left(\left[\begin{array}{cc} Z_0 & 0 \\ 0 & Z_l \end{array}\right] + \left[\begin{array}{cc} Z_{11} & Z_{12} \\ Z_{21} & Z_{22} \end{array}\right]\right)^{-1} \left[\begin{array}{c} V_{S0} \\ V_{Sd} \end{array}\right]. \tag{14}$$

Equation (14) can be simplified since Z_0 and Z_l are not depending from λ,

$$\frac{\partial}{\partial\lambda}\left[\begin{array}{c} I_0 \\ I_l \end{array}\right] = -\left(\left[\begin{array}{cc} Z_0 & 0 \\ 0 & Z_l \end{array}\right] + \left[\begin{array}{cc} Z_{11} & Z_{12} \\ Z_{21} & Z_{22} \end{array}\right]\right)^{-1} \frac{\partial}{\partial\lambda}\left[\begin{array}{cc} Z_{11} & Z_{12} \\ Z_{21} & Z_{22} \end{array}\right] \left(\left[\begin{array}{cc} Z_0 & 0 \\ 0 & Z_l \end{array}\right] + \left[\begin{array}{cc} Z_{11} & Z_{12} \\ Z_{21} & Z_{22} \end{array}\right]\right)^{-1} \left[\begin{array}{c} V_{S0} \\ V_{Sd} \end{array}\right]. \tag{15}$$

Again, although the derivative of the impedance matrix $[Z]$ can be computed relying on the standard transmission line theory [1], a different approach can be adopted which is based on the Green's function method [20] and pawns the way to the extension of the proposed method to time-domain.

Assuming boundary condition of the Neumann type, having incorporated port currents into sources $I_S(z,s)$, a two-conductor transmission line is characterized by the following closed-form Green's function [20],

$$G(z,z',s) = -\frac{\cosh[\gamma(s)(l-z_>)]\cosh[\gamma(s)z_<]}{\gamma(s)\sinh[\gamma(s)l]} \tag{16}$$

where $z_>$ and $z_<$ indicate the greater and lesser of the pair (z,z'), respectively, and γ is defined as,

$$\gamma = \sqrt{Z_s Y_p}. \tag{17}$$

It can be proved that the impedance matrix $[Z]$ can be expressed in terms of the Green's function as,

$$[Z] = \left[\begin{array}{cc} G(0,0,s)(-Z_s(s)) & G(0,l,s)(-Z_s(s)) \\ G(l,0,s)(-Z_s(s)) & G(l,l,s)(-Z_s(s)) \end{array}\right]. \tag{18}$$

Next, differentiating equation (18) with respect to λ, we obtain the derivative of the Z matrix. The result is shown at the top of the next page in equation (19).

The voltage distribution $V(z,s)$ in equation (7) can also be computed through the use of the Green's function as a function of the port currents,

$$V(z,s) = \left[G(z,0,s)(-Z_s(s)) \; G(z,l,s)(-Z_s(s))\right]\left[\begin{array}{c} I_0 \\ I_l \end{array}\right] \tag{20}$$

The differential problem equation (7) can be regarded as a Sturm-Liouville problem for the voltage sensitivity $\widehat{V}(z,s)$ satisfying the same boundary conditions of Neumann type as the voltage $V(z,s)$. Indeed, the following identities hold,

$$\left.\frac{\partial}{\partial z}\widehat{V}(z,s)\right|_{z=0} = \left.\frac{\partial}{\partial z}\frac{\partial}{\partial\lambda}V(z,s)\right|_{z=0} = \left.\frac{\partial}{\partial\lambda}\frac{\partial}{\partial z}V(z,s)\right|_{z=0} = 0 \tag{21a}$$

$$\left.\frac{\partial}{\partial z}\widehat{V}(z,s)\right|_{z=d} = \left.\frac{\partial}{\partial z}\frac{\partial}{\partial\lambda}V(z,s)\right|_{z=d} = \left.\frac{\partial}{\partial\lambda}\frac{\partial}{\partial z}V(z,s)\right|_{z=d} = 0, \tag{21b}$$

where the order of derivatives has been exchanged according to Schwarz's theorem being the partial derivatives continuous. Hence, it can be claimed that the Green's function for the voltage sensitivity is the same as for the voltage along the line. Hence, the voltage sensitivity can be computed as convolution between the Green's function and the forcing term $f(z')$,

$$\widehat{V}(z,s) = \int_0^l G(z,z')f(z')\mathrm{d}z' \tag{22}$$

where

$$f(z') = \frac{\partial(\gamma^2)}{\partial\lambda}V(z',s) - Z_s(s)\frac{\partial I_S(z',s)}{\partial\lambda} - \frac{\partial Z_s(s)}{\partial\lambda}I_S(z',s). \tag{23}$$

$$\frac{\mathrm{d}}{\mathrm{d}\lambda}[Z] = \begin{bmatrix} \frac{\mathrm{d}}{\mathrm{d}\lambda}[G(0,0,s)](-Z_s(s)) + G(0,0,s)\frac{\mathrm{d}}{\mathrm{d}\lambda}[(-Z_s(s))] & \frac{\mathrm{d}}{\mathrm{d}\lambda}[G(0,l,s)](-Z_s(s)) + G(0,l,s)\frac{\mathrm{d}}{\mathrm{d}\lambda}[(-Z_s(s))] \\ \frac{\mathrm{d}}{\mathrm{d}\lambda}[G(l,0,s)](-Z_s(s)) + G(l,0,s)\frac{\mathrm{d}}{\mathrm{d}\lambda}[(-Z_s(s))] & \frac{\mathrm{d}}{\mathrm{d}\lambda}[G(l,l,s)](-Z_s(s)) + G(l,l,s)\frac{\mathrm{d}}{\mathrm{d}\lambda}[(-Z_s(s))] \end{bmatrix} \quad (19)$$

III. COMPUTATION OF DERIVATIVES

Using equations (18) and (19), we are able to compute the $[Z]$ matrix and its derivative $\frac{\mathrm{d}}{\mathrm{d}\lambda}[Z]$ required for the calculation of equation (15). For this purpose, we can compute the derivative of Green's function analytically,

$$\frac{\mathrm{d}}{\mathrm{d}\lambda}[G(0,0,s)] = \frac{\mathrm{d}}{\mathrm{d}\lambda}[G(l,l,s)] = \frac{\mathrm{d}\gamma/\mathrm{d}\lambda}{\gamma^2\sinh^2(\gamma l)}[l\gamma + \sinh(\gamma l)\cosh(\gamma l)] \quad (24a)$$

$$\frac{\mathrm{d}}{\mathrm{d}\lambda}[G(0,l,s)] = \frac{\mathrm{d}}{\mathrm{d}\lambda}[G(l,0,s)] = \frac{\mathrm{d}\gamma/\mathrm{d}\lambda}{\gamma^2\sinh^2(\gamma l)}[\sinh(\gamma l) + l\gamma\cosh(\gamma l)]. \quad (24b)$$

The function to be integrated in (22) reads,

$$F(z,z',s) = G(z,z',s)\frac{\partial(\gamma^2)}{\partial\lambda}V(z',s) - G(z,z',s)\left[-Z_s(s)\frac{\partial I_S(z',s)}{\partial\lambda} - \frac{\partial Z_s(s)}{\partial\lambda}I_S(z',s)\right] \quad (25)$$

The computation of equation (25) can be split in two terms $F(z,z',s) = F_1(z,z',s) + F_2(z,z',s)$, where $V(z',s)$ is given by equation (20),

$$F_1(z,z',s) = \frac{\partial(\gamma^2)}{\partial\lambda}I_0(-Z_s(s))\cdot G(z,z',s)\cdot G(z',0,s) + \frac{\partial(\gamma^2)}{\partial\lambda}I_l(-Z_s(s))\cdot G(z,z',s)\cdot G(z',l,s) \quad (26a)$$

$$F_2(z,z',s) = \left[-Z_s(s)\frac{\partial I_0}{\partial\lambda}\delta(z') - Z_s(s)\frac{\partial I_l}{\partial\lambda}\delta(z'-l) - \frac{\partial Z_s(s)}{\partial\lambda}I_0\delta(z') - \frac{\partial Z_s(s)}{\partial\lambda}I_l\delta(z'-l)\right]\cdot G(z,z',s), \quad (26b)$$

where z is the abscissa wherein we compute the sensitivity and z' is the integration variable. Hence, the calculation of the integral of $F_1(z,z',s)$ depends on the products,

$$G(z,z') \cdot G(z',0) \quad (27a)$$

$$G(z,z') \cdot G(z',l) \quad (27b)$$

Being interested to the voltage sensitivity at abscissa $z = 0$ and $z = l$, it is useful exploiting the dependence of equation (27) on the z' coordinate, yielding,

$$G(0,z')\cdot G(z',0) = K\cdot\left[\cosh(2\gamma l)\cosh(2\gamma z') - \sinh(2\gamma l)\sinh(2\gamma z') + 1\right] \quad (28a)$$

$$G(0,z')\cdot G(z',l) = K\cdot\left[\cosh(\gamma l) + \cosh(\gamma l)\cosh(2\gamma z') - \sinh(2\gamma z')\sinh(\gamma l)\right], \quad (28b)$$

$$G(l,z')\cdot G(z',0) = K\cdot\left[\cosh(\gamma l) + \cosh(\gamma l)\cosh(2\gamma z') - \sinh(2\gamma z')\sinh(\gamma l)\right], \quad (28c)$$

$$G(l,z')\cdot G(z',l) = K\cdot\left[1 + \cosh(2\gamma z')\right], \quad (28d)$$

where

$$K = \frac{1}{2\gamma^2\sinh^2(\gamma l)}. \quad (29)$$

The integration of equations (28a) to (28d) is straightforward. Equations (30a) to (30d) show the definite integrals,

$$\int_0^l G(0,z')\cdot G(z',0)\mathrm{d}z' = K\cdot\left[\frac{1}{2\gamma}\sinh(2\gamma z')\cosh(2\gamma l) - \frac{1}{2\gamma}\cosh(2\gamma z')\sinh(2\gamma l) + z'\right]\Bigg|_0^l \quad (30a)$$

$$\int G(0,z')\cdot G(z',l)\mathrm{d}z' = K\cdot\left[\cosh(\gamma l)\cdot z' + \frac{1}{2\gamma}\sinh(2\gamma z')\cosh(\gamma l) - \frac{1}{2\gamma}\cosh(2\gamma z')\sinh(\gamma l)\right]\Bigg|_0^l, \quad (30b)$$

$$\int G(l,z')\cdot G(z',0)\mathrm{d}z' = K\cdot\left[\cosh(\gamma l)\cdot z' + \frac{1}{2\gamma}\sinh(2\gamma z')\cosh(\gamma l) - \frac{1}{2\gamma}\cosh(2\gamma z')\sinh(\gamma l)\right]\Bigg|_0^l, \quad (30c)$$

$$\int G(l,z')\cdot G(z',l)\mathrm{d}z' = K\cdot\left[z' + \frac{1}{2\gamma}\sinh(2\gamma z')\right]\Bigg|_0^l. \quad (30d)$$

The second term $F_2(z,z',s)$ in equation (26b) is finally considered.

Its contribution to the overall voltage sensitivity, taking the delta Dirac function sampling property into account [21], is given by,

$$\int_0^l F_2(z,z',s)dx' = -Z_s(s)\frac{\partial I_0}{\partial\lambda}G(z,0,s) - Z_s(s)\frac{\partial I_l}{\partial\lambda}G(z,l,s) - \frac{\partial Z_s(s)}{\partial\lambda}I_0G(z,0,s) - \frac{\partial Z_s(s)}{\partial\lambda}I_lG(z,l,s). \quad (31)$$

Finally, the voltage sensitivity with respect to parameter λ at abscissa $z = 0$ and $z = l$ can be computed as,

$$\widehat{V}(0,s) = \int_0^l \left(F_1(0,z',s) + F_2(0,z',s)\right)\mathrm{d}z' \quad (32a)$$

$$\widehat{V}(l,s) = \int_0^l \left(F_1(l,z',s) + F_2(l,z',s)\right)\mathrm{d}z'. \quad (32b)$$

Space limitations do not permit the description of the time-domain analysis here but it can be easily obtained working with linear loads from the inverse fast Fourier transform (IFFT).

A. Higher-order sensitivities

The voltage sensitivities of equation (32) correspond to first-order sensitivities of port voltages with respect to parameter λ. The evaluation of higher-order sensitivities can be performed using the same approach outlined in Section II. In fact, the governing equation for the k-order voltage sensitivity is,

$$\frac{\partial^2}{\partial z^2}\widehat{V}_k(z,s) - \gamma^2(s)\ \widehat{V}_k(z,s) = k\frac{\partial \gamma^2}{\partial \lambda}\widehat{V}_{k-1}(z,s) + \frac{\partial^k \gamma^2}{\partial \lambda^k}V(z,s) - \frac{\partial^k}{\partial \lambda^k}\left(Z_s(s)I_S(z,s)\right). \quad (33)$$

The Sturm-Liouville problem for the k-order voltage sensitivity $\widehat{V}_k(z,s)$ admits the same Green's function as the voltage distribution $V(z,s)$ and, as a consequence, it can be computed as,

$$\widehat{V}_k(z,s) = \int_0^l G(z,z')f_k(z')\mathrm{d}z' \quad (34)$$

where

$$f_k(z') = k\frac{\partial \gamma^2}{\partial \lambda}\widehat{V}_{k-1}(z',s) + \frac{\partial^k \gamma^2}{\partial \lambda^k}V(z',s) - \frac{\partial^k}{\partial \lambda^k}\left(Z_s(s)I_S(z',s)\right). \quad (35)$$

Each term of the forcing term (35) can be analytically computed as well as the integrand function in equation (34) and its integral, leading to a closed-form k-order voltage sensitivity.

IV. NUMERICAL RESULTS

In this section we present two examples of transmission lines whose voltage sensitivity with respect to geometrical parameters are computed by using the proposed methodology. For the sake of comparison, the voltage sensitivity is also computed by the perturbative approach by giving a small perturbation $\Delta\lambda$ to the parameter λ and computing the sensitivity as,

$$\widehat{V}^p(z,s) = \frac{V(z,s,\lambda+\Delta\lambda) - V(z,s,\lambda)}{\Delta\lambda}. \quad (36)$$

A. Two-conductor transmission line

Let us consider a couple of conductors of radius $r_0 = 2$ mm and length $l = 0.1$ m, at a distance $d = 1$ cm, in the free space (permittivity $\varepsilon_0 = 8.854$ pF/m and permeability $\mu_0 = 0.4\pi$ mH/m). The p.u.l. parameters of the line are [1],

$$R = \frac{\rho}{\pi r_0^2} \qquad G = 0$$
$$L = \frac{\mu_0}{2\pi}\ln\frac{2d}{r_0} \qquad C = \frac{2\pi\varepsilon_0}{\ln\frac{2d}{r_0}}. \quad (37)$$

The voltage sensitivity is computed with respect to the distance between the conductors $\lambda = d$. First we can compute the series impedance Z_s and the parallel admittance Y_p and their derivatives with respect to sensitivity parameter d,

$$Z_s(d,s) = R + sL \quad (38a)$$
$$Y_p(d,s) = G + sC, \quad (38b)$$
$$\frac{\partial}{\partial d}Z_s(d,s) = s\cdot\frac{\mu_0}{2\pi}\frac{1}{d}, \quad (38c)$$
$$\frac{\partial}{\partial d}Y_p(d,s) = s\cdot\left(-\frac{2\pi\varepsilon_0}{d\ \ln^2\left[\frac{2d}{r_0}\right]}\right). \quad (38d)$$

The circuit is excited by a voltage pulse V_s with 800 ps width and 500-ps rise and fall times, whose magnitude spectrum is shown in Fig. 2. The frequency range of analysis is $0-5$ GHz.

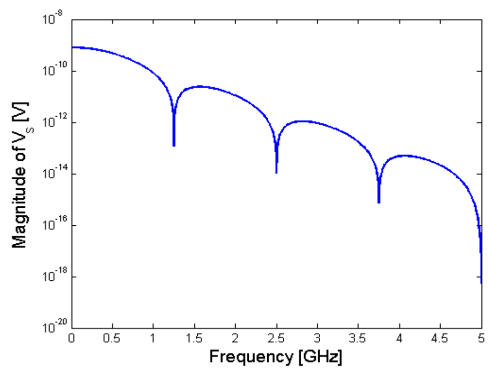

Fig. 2. Magnitude spectrum of the voltage source (example IV-A).

In Figs. 3 and 4 it is shown the magnitude and phase spectra of the voltage sensitivity at $z = 0$ and $z = l$ as computed by using equations (32a) and (32b) and compared with the perturbative approach.

No noticeable difference can be observed between the proposed and perturbative approach.

B. Microstrip

As a second example we consider a microstrip transmission line. It can be characterized by geometrical and physical parameters such as width of the strip W, height of the dielectric substrate H, strip thickness T, permeability and dielectric constants. The p.u.l. capacitance in free space is given by [22],

$$C_a = \begin{cases} \varepsilon_0\left[\frac{W_e}{H} + 1.393 + 0.667\cdot \ln\left(\frac{W_e}{H} + 1.444\right)\right] & W/H > 1 \\ \frac{2\pi\varepsilon_0}{\ln\left(\frac{8H}{W_e} + \frac{W_e}{4H}\right)} & W/H \leq 1 \end{cases} \quad (39)$$

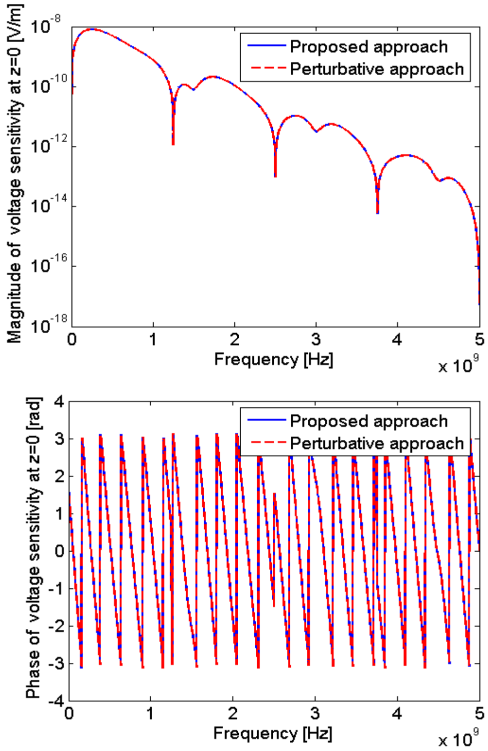

Fig. 3. Magnitude and phase spectra of voltage sensitivity at $z = 0$ (example IV-A).

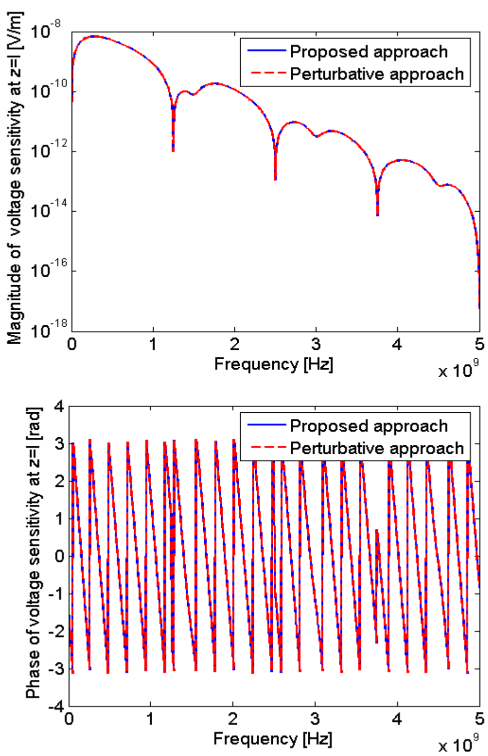

Fig. 4. Magnitude and phase spectra of voltage sensitivity at $z = l$ (example IV-A).

where the effect of the finite thickness can be included using effective width W_e instead of W [22],

$$W_e = \begin{cases} W + 0.398T\left(1 + \ln\dfrac{2\pi}{T}\right) & W/H > \dfrac{1}{2\pi} \\ W + 0.398T\left(1 + \ln\dfrac{4\pi W}{T}\right) & W/H \leq \dfrac{1}{2\pi} \end{cases} \tag{40}$$

The effective dielectric constant, ε_{eff}, for a microstrip line with an isotropic substrate is given by [22],

$$\varepsilon_{eff} = \frac{\varepsilon_r + 1}{2} + \frac{\varepsilon_r - 1}{2}\left(1 + \frac{12H}{W_e}\right)^{-\frac{1}{2}} + F(\varepsilon_r, H) - 0.217(\varepsilon_r - 1)\frac{T}{\sqrt{W_e H}} \tag{41}$$

where

$$F(\varepsilon_r, H) = \begin{cases} 0 & W/H > 1 \\ 0.02(\varepsilon_r - 1)\left(1 - \dfrac{W_e}{H}\right)^2 & W/H \leq 1 \end{cases} \tag{42}$$

The p.u.l capacitance and inductance, assuming an homogeneous medium, is given by [22],

$$C = C_a \varepsilon_{eff} \tag{43a}$$

$$L = \frac{\mu_0 \varepsilon_0 \varepsilon_{eff}}{C} = \frac{\mu_0 \varepsilon_0}{C_a}. \tag{43b}$$

In order to obtain the voltage sensitivity, the derivatives of equations (43a) and (43b) are needed. In the following both the voltage sensitivity with respect the width of the strip W as well as the height of the dielectric substrate H are computed. The derivatives read,

$$\frac{\mathrm{d}C}{\mathrm{d}W} = \frac{\mathrm{d}C_a}{\mathrm{d}W}\varepsilon_{eff} + C_a\frac{\mathrm{d}\varepsilon_{eff}}{\mathrm{d}W} \tag{44a}$$

$$\frac{\mathrm{d}L}{\mathrm{d}W} = -\frac{\varepsilon_0\mu_0}{C_a^2}\frac{\mathrm{d}C_a}{\mathrm{d}W}, \tag{44b}$$

$$\frac{\mathrm{d}C}{\mathrm{d}H} = \frac{\mathrm{d}C_a}{\mathrm{d}H}\varepsilon_{eff} + C_a\frac{\mathrm{d}\varepsilon_{eff}}{\mathrm{d}H}, \tag{44c}$$

$$\frac{\mathrm{d}L}{\mathrm{d}H} = -\frac{\varepsilon_0\mu_0}{C_a^2}\frac{\mathrm{d}C_a}{\mathrm{d}H}, \tag{44d}$$

where

$$\frac{dC_a}{dW} = \begin{cases} \dfrac{\varepsilon_0 W_e'}{H}\left[1 + 0.667\dfrac{1}{W_e/H + 1.444}\right] & W/H > 1 \\ -\dfrac{2\pi\varepsilon_0\left(-\dfrac{8W_e'H}{W_e^2} + \dfrac{W_e'}{4H}\right)}{\ln^2\left(\dfrac{8H}{W_e} + \dfrac{W_e}{4H}\right)\cdot\left(\dfrac{8H}{W_e} + \dfrac{W_e}{4H}\right)} & W/H \leq 1 \end{cases} \tag{45}$$

$$\frac{dC_a}{dH} = \begin{cases} -\varepsilon_0\dfrac{W_e}{H^2}\left[1 + 0.667\dfrac{1}{W_e/H + 1.444}\right] & W/H > 1 \\ -\dfrac{2\pi\varepsilon_0\left(\dfrac{8}{W_e} - \dfrac{W_e}{4H^2}\right)}{\ln^2\left(\dfrac{8H}{W_e} + \dfrac{W_e}{4H}\right)\cdot\left(\dfrac{8H}{W_e} + \dfrac{W_e}{4H}\right)} & W/H \leq 1 \end{cases} \tag{46}$$

$$\frac{\mathrm{d}\varepsilon_{eff}}{\mathrm{d}W} = \frac{\varepsilon_r - 1}{2}\,\frac{6HW_e^{'}}{W_e^2}\left(1 + \frac{12H}{W_e}\right)^{-\frac{3}{2}} + \frac{\mathrm{d}F(\varepsilon_r, H)}{\mathrm{d}W} + \\ +0.217(\varepsilon_r - 1)\frac{HTW_e^{'}}{2}(W_e H)^{-\frac{3}{2}}, \quad (47)$$

$$\frac{\mathrm{d}\varepsilon_{eff}}{\mathrm{d}H} = -\frac{\varepsilon_r - 1}{2}\,\frac{6}{W_e}\left(1 + \frac{12H}{W_e}\right)^{-\frac{3}{2}} + \frac{\mathrm{d}F(\varepsilon_r, H)}{\mathrm{d}H} + \\ +0.217(\varepsilon_r - 1)\frac{W_e T}{2}(W_e H)^{-\frac{3}{2}}, \quad (48)$$

$$\frac{\mathrm{d}W_e}{\mathrm{d}W} = W_e^{'} = \begin{cases} 1 & W/H > \frac{1}{2\pi} \\ 1 + 0.398\frac{T}{W} & W/H \leq \frac{1}{2\pi} \end{cases} \quad (49)$$

$$\frac{\mathrm{d}F(\varepsilon_r, H)}{\mathrm{d}W} = \begin{cases} 0 & W/H > 1 \\ 0.02(\varepsilon_r - 1)\frac{2W_e^{'}}{H}\left(\frac{W_e}{H} - 1\right) & W/H \leq 1 \end{cases} \quad (50)$$

$$\frac{\mathrm{d}F(\varepsilon_r, H)}{\mathrm{d}H} = \begin{cases} 0 & W/H > 1 \\ 0.02(\varepsilon_r - 1)\frac{2W_e}{H^2}\left(1 - \frac{W_e}{H}\right) & W/H \leq 1 \end{cases} \quad (51)$$

where $W_e^{'}$ stands for the derivative of W_e with respect to W, since it does not depend on H.

As a numerical test, a microstrip line has been considered with length $l = 5$ cm, width of the strip $W = 2$ mm, height of the dielectric substrate $H = 1$ mm and strip thickness $T = 0.5\mathrm{mm}$. The relative permittivity of the substrate is $\varepsilon_r = 3$.

The voltage sensitivity has been computed using equations (32a) and (32b), considering as parameter λ the width of the strip W and the height of the dielectric substrate H. The circuit input is the same of the previous example (Fig. 2) and the frequency range of analysis is $0 - 5$ GHz. The magnitude and phase spectra of the voltage sensitivity with respect to W are shown in Figs. 5 and 6, while those of the voltage sensitivity with respect to H are shown in Figs. 7 and 8.

As before, a very good agreement is achieved between the proposed and the perturbative approach. For the sake of comparison the sensitivities have also been computed numerically. The computation has been performed on a machine equipped with AMD Athlon 64 processor. It took about 10 s to be completed by using the proposed technique and 73 s computing the sensitivities numerically, for 3751 frequency samples, leading to a speed-up of 7.3.

V. CONCLUSIONS

In this paper we have proposed a new approach to analyze frequency-domain sensitivity of transmission lines. It is based on the closed-form Green's function of the 1-D wave propagation problem. Relying on the knowledge of the Green's function for the transmission line problem, a closed-form solution for the voltage sensitivity with respect to either physical or geometrical parameters is readily computed. The proposed technique is well suited to be extended to the computation of higher-order

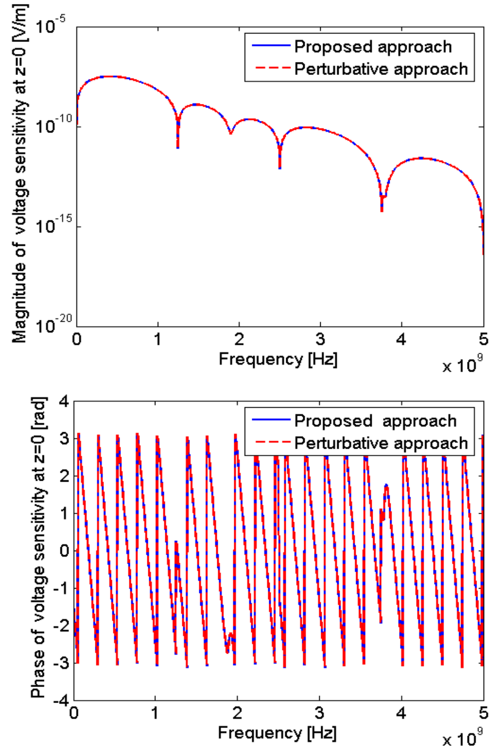

Fig. 5. Magnitude and phase of voltage sensitivity with respect to W at $z = 0$ (example IV-B).

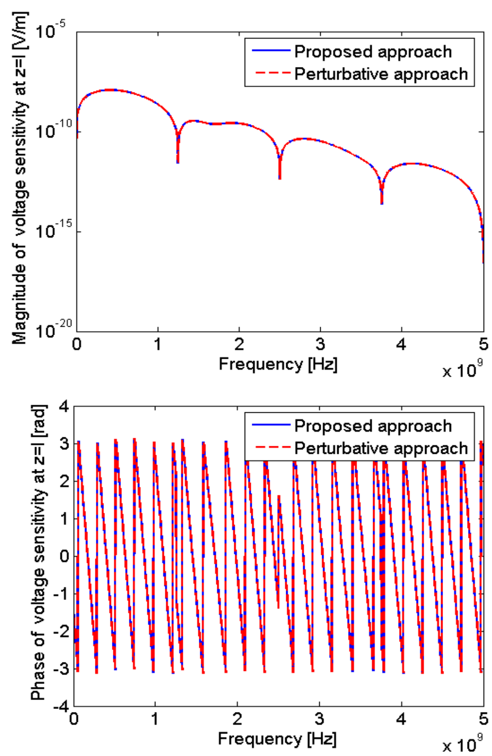

Fig. 6. Magnitude and phase of the voltage sensitivity with respect to W at $z = l$ (example IV-B).

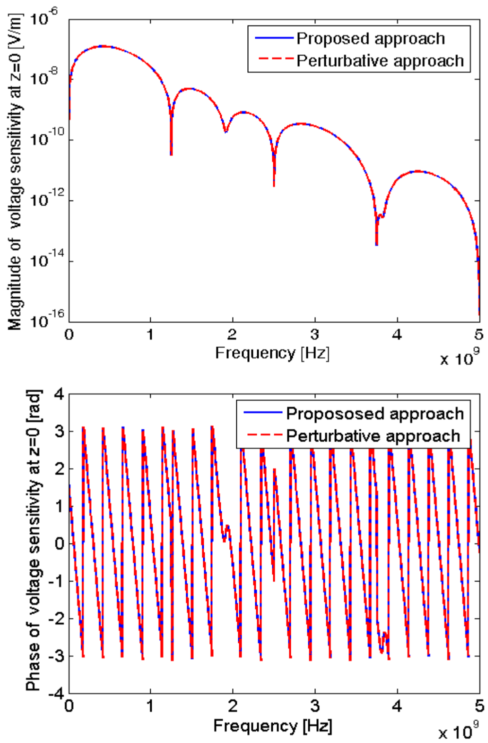

Fig. 7. Magnitude and phase spectra of voltage sensitivity with respect to H at $z = 0$ (example IV-B).

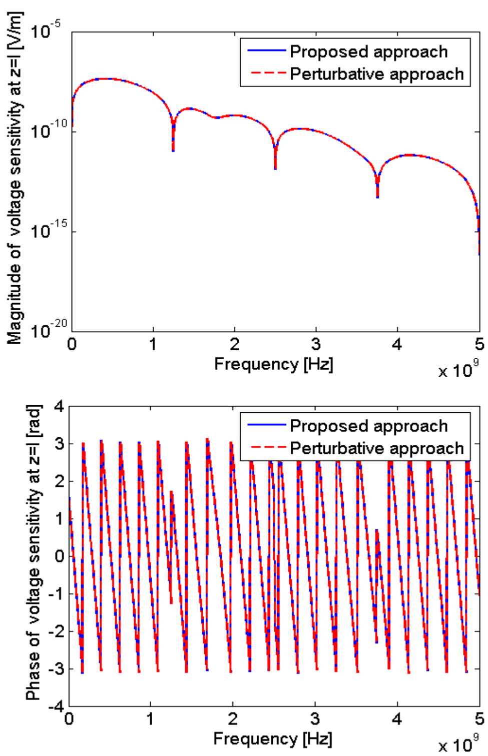

Fig. 8. Magnitude and phase spectra of voltage sensitivity with respect to H at $z = l$ (example IV-B).

sensitivity. Its implementation is straightforward and does not require any numerical processing. Two examples were presented showing the accuracy of the method compared to the perturbation technique.

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