Efficient FDTD Implementation of the ADE-Based CN-PML for the Two-Dimensional TMz Waves

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Abstract **─** An efficient, unsplit-field and unconditional stable implementation of the stretched coordinate perfectly matched layer (SC-PML) is proposed for terminating the finite-difference time-domain (FDTD) method. Via incorporating the Crank-Nicolson Douglas-Gunn (CNDG) and the auxiliary differential equation (ADE) methods, respectively, the proposed PML formulations can take advantage of the unconditional stability of the CNDG method which has smaller numerical anisotropy than the existing alternately direction implicit (ADI) method. A numerical test carried out in a 2D free space FDTD domain is provided to validate the proposed CNDG-based PML. It has been shown that the proposed PML can not only overcome the Courant-Friedrich-Levy (CFL) stability constraint, but attenuate the propagating waves efficiently.

Index Terms **─** Auxiliary differential equation (ADE), Crank-Nicolson Douglas-Gunn (CNDG), finite-difference time-domain (FDTD), perfectly matched layer (PML).

I. INTRODUCTION

The finite-difference time-domain (FDTD) method plays an important role in the design and simulation of electromagnetic behaviors [1]. As an explicit numerical method, the Yee's FDTD is conditionally stable, which means that the FDTD time-step is constrained by the Courant-Friedrich-Levy (CFL) limit to maintain stability and makes the FDTD method not very efficient in analyzing electrically small structures [1]. In order to remove the CFL stability constraint on time step and improve computational efficiency, unconditionally stable methods such as the alternating-direction implicit FDTD (ADI-FDTD) scheme and the Crank-Nicolson FDTD (CN-FDTD) scheme have been introduced in [2-6]. As pointed in [5], the ADI's accuracy is inferior to that of CN scheme. The CN-FDTD with Douglas-Gunn (DG) algorithm (denoted as CNDG FDTD method) is developed in [6] to overcome the drawbacks that the CN-FDTD with a huge irreducible matrix is hardly to be solved without approximate algorithms.

In addition, one of the greatest challenges of applying the FDTD method is the development of absorbing boundary conditions (ABCs) which truncate open region problems to simulate the extension of the computational domain to infinity [1]. It has been shown that the perfectly matched layer (PML), introduced by Berenger, is one of the most effective ABCs [7]. The stretched coordinate PML (SC-PML) has the advantage of simple implementation in the corners and edges of the PML regions [8].

To our knowledge, there is only one literature about the formulation of the 2D unconditionally stable PML based on an approximate CN scheme [9]. The method in [9] is a split-field PML for 2D TEz waves.

In this paper, an alternative efficient, unconditionally stable and unsplit-field PML, denoted as ADE CNDG-PML, is constructed for 2D TMz waves. The formulation is based upon incorporating the CNDG algorithm and auxiliary differential equation (ADE) method into the PML implementation.

II. FORMULATION

For simplicity, the PML is constructed for 2D TMz waves only for truncating the free space. The frequency-domain modified Maxwell's equations in the SC-PML can be written as:

$$-j\omega H_x = c \cdot S_y^{-1} \frac{\partial E_z}{\partial y}, \quad (1)$$

$$j\omega H_y = c \cdot S_x^{-1} \frac{\partial E_z}{\partial x}, \quad (2)$$

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$$j\omega E_z = c\cdot S_x^{-1}\frac{\partial H_y}{\partial x} - c\cdot S_y^{-1}\frac{\partial H_x}{\partial y}, \tag{3}$$

where c is the free-space wave-propagation velocity, S_η ($\eta=x,y$) is the stretched coordinate variables chosen within the PML region as:

$$S_\eta^{-1} = \frac{1}{1+\sigma_\eta/(j\omega\varepsilon_0)} = 1 - \frac{\sigma_\eta/\varepsilon_0}{j\omega+\sigma_\eta/\varepsilon_0}, \tag{4}$$

where σ_η is the conductivity profile along the η direction in the PML region [8], ε_0 is the free-space permittivity.

Using (4) and the inverse Fourier translation, (1)-(3) can be written in the time domain as:

$$-\frac{\partial}{\partial t}H_x = c\cdot\frac{\partial E_z}{\partial y} - g_{xy}, \tag{5}$$

$$\frac{\partial}{\partial t}H_y = c\cdot\frac{\partial E_z}{\partial x} - g_{yx}, \tag{6}$$

$$\frac{\partial}{\partial t}E_z = c\cdot\frac{\partial H_y}{\partial x} - c\cdot\frac{\partial H_x}{\partial y} - f_{zx} + f_{zy}, \tag{7}$$

where f_{zx}, f_{zy}, g_{xy} and g_{yx} are given by:

$$\frac{\partial}{\partial t}f_{zx} + \frac{\sigma_x}{\varepsilon_0}f_{zx} = c\cdot\frac{\sigma_x}{\varepsilon_0}\frac{\partial H_y}{\partial x}, \tag{8}$$

$$\frac{\partial}{\partial t}f_{zy} + \frac{\sigma_y}{\varepsilon_0}f_{zy} = c\cdot\frac{\sigma_y}{\varepsilon_0}\frac{\partial H_x}{\partial y}, \tag{9}$$

$$\frac{\partial}{\partial t}g_{xy} + \frac{\sigma_y}{\varepsilon_0}g_{xy} = c\cdot\frac{\sigma_y}{\varepsilon_0}\frac{\partial E_z}{\partial y}, \tag{10}$$

$$\frac{\partial}{\partial t}g_{yx} + \frac{\sigma_x}{\varepsilon_0}g_{yx} = c\cdot\frac{\sigma_x}{\varepsilon_0}\frac{\partial E_z}{\partial x}. \tag{11}$$

Applying the CN scheme to discretize (5)-(11), we have the following discrete equations as:

$$H_{x_{i,j+1/2}}^{n+1} = H_{x_{i,j+1/2}}^{n} - \chi_y\cdot\Gamma_y(E_{z_{i,j}}^{n}) + \Delta t_h\cdot(g_{xy_{i,j+1/2}}^{n+1} + g_{xy_{i,j+1/2}}^{n}), \tag{12}$$

$$H_{y_{i+1/2,j}}^{n+1} = H_{y_{i+1/2,j}}^{n} + \chi_x\cdot\Gamma_x(E_{z_{i,j}}^{n}) - \Delta t_h\cdot(g_{yx_{i+1/2,j}}^{n+1} + g_{yx_{i+1/2,j}}^{n}), \tag{13}$$

$$E_{z_{i,j}}^{n+1} = E_{z_{i,j}}^{n} + \chi_x\cdot\Gamma_x(H_{y_{i+1/2,j}}^{n}) - \chi_y\cdot\Gamma_y(H_{x_{i,j+1/2}}^{n}) + \Delta t_h\cdot(f_{zy_{i,j}}^{n+1} + f_{zy_{i,j}}^{n}) - \Delta t_h\cdot(f_{zx_{i,j}}^{n+1} + f_{zx_{i,j}}^{n}), \tag{14}$$

$$f_{zx_{i,j}}^{n+1} = r_{0x_i}\cdot f_{zx_{i,j}}^{n} + r_{1x_i}\cdot\Gamma_x(H_{y_{i+1/2,j}}^{n}), \tag{15}$$

$$f_{zy_{i,j}}^{n+1} = r_{0y_j}\cdot f_{zy_{i,j}}^{n} + r_{1y_j}\cdot\Gamma_y(H_{x_{i,j+1/2}}^{n}), \tag{16}$$

$$g_{xy_{i,j+1/2}}^{n+1} = r_{0y_{j+1/2}}\cdot g_{xy_{i,j+1/2}}^{n} + r_{1y_{j+1/2}}\cdot\Gamma_y(E_{z_{i,j}}^{n}), \tag{17}$$

$$g_{yx_{i+1/2,j}}^{n+1} = r_{0x_{i+1/2}}\cdot g_{yx_{i+1/2,j}}^{n} + r_{1x_{i+1/2}}\cdot\Gamma_x(E_{z_{i,j}}^{n}). \tag{18}$$

The operator $\Gamma_\eta[*]$ denotes the difference form obtained by applying the CN method along direction η, for example:

$$\Gamma_y(E_{z_{i,j}}^{n}) = E_{z_{i,j+1}}^{n+1} - E_{z_{i,j}}^{n+1} + E_{z_{i,j+1}}^{n} - E_{z_{i,j}}^{n}. \tag{19}$$

Other operators take similar forms as (19). The corresponding coefficients in (12)-(18) are given by:

$$\Delta t_h = \Delta t/2,\ \chi_\eta = c\Delta t_h/(\Delta\eta),\ q_{\eta_k} = \sigma_{\eta_k}\Delta t_h/\varepsilon_0,$$

$$r_{0\eta_k} = (1-q_{\eta_k})/(1+q_{\eta_k}),$$

$$r_{1\eta_k} = [q_{\eta_k}/(1+q_{\eta_k})]\cdot(c/\Delta\eta),$$

where $\Delta\eta$ ($\eta=x,y$) is the space cell size, Δt is the time step, and k ($k=i,j$) is the inter-number indices of the computational cells.

It is noted that the discrete electric and magnetic field components are coupled, which leads to a huge sparse matrix to be solved expensively. One way to decouple the electric and magnetic fields is to insert (15)-(18) into (12)-(14) respectively, then substitute H_x^{n+1} and H_y^{n+1} into the expression of E_z^{n+1} to eliminate the implicit magnetic field components:

$$\begin{aligned}[1-(D_{2x}+D_{2y})]E_{z_{i,j}}^{n+1} &= [1+(D_{2x}+D_{2y})]E_{z_{i,j}}^{n} \\ &+2(\chi_x-\Delta t_h r_{1x_i})(H_{y_{i+1/2,j}}^{n} - H_{y_{i-1/2,j}}^{n}) - d_{x_i}^{+}g_{yx_{i+1/2,j}}^{n} \\ &-2(\chi_y-\Delta t_h r_{1y_j})(H_{x_{i,j+1/2}}^{n} - H_{x_{i,j-1/2}}^{n}) + d_{x_i}^{-}g_{yx_{i-1/2,j}}^{n} \\ &-d_{y_j}^{+}g_{xy_{i,j+1/2}}^{n} + d_{y_j}^{-}g_{xy_{i,j-1/2}}^{n} - \Delta t_h(1+r_{0x_i})f_{zx_{i,j}}^{n} \\ &+\Delta t_h(1+r_{0y_j})f_{zy_{i,j}}^{n},\end{aligned} \tag{20}$$

where D_{2x} and D_{2y} are defined as follows:

$$D_{2x}E_{z_{i,j}}^{n+1} = c_{x_i}^{+}E_{z_{i+1,j}}^{n+1} - (c_{x_i}^{+}+c_{x_i}^{-})E_{z_{i,j}}^{n+1} + c_{x_i}^{-}E_{z_{i-1,j}}^{n+1}, \tag{21}$$

$$D_{2y}E_{z_{i,j}}^{n+1} = c_{y_j}^{+}E_{z_{i,j+1}}^{n+1} - (c_{y_j}^{+}+c_{y_j}^{-})E_{z_{i,j}}^{n+1} + c_{y_j}^{-}E_{z_{i,j-1}}^{n+1}. \tag{22}$$

The coefficients of (20)-(22) are defined as:

$$c_{\eta_k}^{\pm} = (\chi_\eta - \Delta t_h r_{1\eta_k})(\chi_\eta - \Delta t_h r_{1\eta_{k\pm1/2}}),$$

$$d_{\eta_k}^{\pm} = (\chi_\eta - \Delta t_h r_{1\eta_k})\Delta t_h(1+r_{0\eta_{k\pm1/2}}).$$

Note that (20) leads to a block tri-diagonal matrix, which still requires very expensive matrix solution at each time step. For an efficient solution of E_z^{n+1}, the CNDG method proposed in [6] is introduced. By adding $D_{2x}D_{2y}E_z^{n+1}$ and $D_{2x}D_{2y}E_z^{n}$ to the left-hand-side (LHS) and right-hand-side (RHS) of (20) respectively, it can be factorized into:

$$(1-D_{2x})(1-D_{2y})E_{z_{i,j}}^{n+1} = (1+D_{2x})(1+D_{2y})E_{z_{i,j}}^{n} + \alpha_{i,j}, \tag{23}$$

where $\alpha_{i,j}$ denotes the other terms of the RHS of (20).

Then (23) can be solved with the following two-step update equations:

$$(1-D_{2x})E_{z_{i,j}}^{*} = (1+D_{2x}+2D_{2y})E_{z_{i,j}}^{n} + \alpha_{i,j}, \tag{24}$$

$$(1-D_{2y})E_{z_{i,j}}^{n+1} = E_{z_{i,j}}^{*} - D_{2y}E_{z_{i,j}}^{n}. \quad (25)$$

From (24) and (25), we can see that the updated equation for principal component E_z^{n+1} splits into two-step update equations by introducing an intermediate parameter E_z^{*}. However, the LHS of (24) and (25) form two tri-diagonal matrixes which can be solved easily. Once E_z^{n+1} is obtained, g_{xy}^{n+1}, g_{yx}^{n+1}, H_x^{n+1}, H_y^{n+1}, f_{zx}^{n+1} and f_{zy}^{n+1} can be updated explicitly.

III. NUMERICAL RESULT

To validate the effectiveness of the proposed formulation, we implemented the PML in a 2D domain. With 1 GHz of the bandwidth, a derivative gauss pulse is placed at the center of a 101×101 uniform mesh domain, which radiates into the free space as an electric-field source. The computational domain discretized with a space cell size of 3 mm in both x and y directions. All sides of the computational domain were terminated by PML [8, 2, 0.001%], as defined in [7].

To evaluate the reflection error of the proposed PML, this ADE CNDG-PML scheme under different CFL numbers (CFLN) are invoked for field computations. The term CFLN is defined as CFLN=$\Delta t/\Delta t_{max}^{FDTD}$, where Δt_{max}^{FDTD} is the maximum stability constraint of the conventional FDTD. For the sake of comparison, the results using the PML based on the conventional FDTD are also obtained. Figure 1 depicts the results obtained by using these two different approaches. The relative reflection error versus time is computed at an observation point in the corner of the computational domain and with one cell away from the interface between the PML and the computational domain using:

$$R_{dB} = 20\log_{10}\left[\frac{\left|E_z(t)-E_{zref}(t)\right|}{\left|E_{zref\,\max}\right|}\right], \quad (26)$$

where $E_z(t)$ represents the electric field computed using the test domain and $E_{zref}(t)$ is a reference solution based on an extended lattice with the size of 600 cells in both x and y directions and terminated by PML [128, 4, 0.0001%]. $E_{zref\max}$ is the maximum amplitude of the reference solution over the full time simulation.

The results are shown in Fig. 1. At early time to about 7 ns, the performance of the proposed PML degrade as the CFLN increasing. However, the maximum relative errors of the conventional SC-PML and the ADE CNDG-PML with different CFLN (CFLN=1, 2, 4) are -66.46 dB, -66.48 dB, -66.56 dB and -64.65 dB, respectively. Then it can be concluded that the proposed PML can almost maintain the same maximum relative error level with different CFLN.

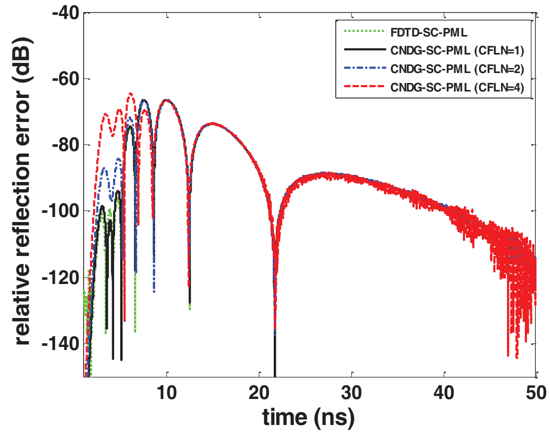

Fig.1. Relative reflection error at observation point in the computational domain terminated by PML [8, 2, 0.001%] for ADE CNDG-PML with various CFLN. The conventional FDTD SC-PML is also included.

IV. CONCLUSION

An efficient algorithm based upon the ADE method is presented in this paper for implementing the SC-PML formulations by making use of a CNDG scheme without the need of splitting the field components. Numerical results demonstrate that the ADE CNDG-PML can be used as a good absorbing boundary condition while the time step is beyond the CFL limit. The simulation time can be reduced by increasing the time step without decreasing of PML performance. Consequently, the computational process uses less time than the conventional SC-PML as the value of CFLN is larger than CFL limit.

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