Optimization of the Auxiliary Sources Method for 2D Arbitrary-Shaped Scattering Problems

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Abstract **─** The distribution of radiation centers plays an important role in the auxiliary sources method. It has a decisive influence on the accuracy of solutions, the convergence rate and the computation cost. The optimal selection of the MAS (method of auxiliary sources) parameters (auxiliary surface, position of radiation centers) is considered an open issue. This work presents a systematic optimization framework to achieve the optimal configuration of the MAS for scattering by an infinite arbitrary-shaped cylinder.

Index Terms **─** Auxiliary sources method, level set method, radar cross-section.

I. INTRODUCTION

The method of auxiliary sources presents an auspicious alternative to standard integral equation techniques (such as the moment method) to solve scattering problems with an optimal compromise between accuracy and computational resources. Previous researches [1,2] have shown that the rigorous location of the auxiliary sources (AS) is imperative to achieve efficiency of the MAS, the distribution of the AS strongly affects the convergence rate and the accuracy of the solution. It is shown that to guarantee the optimality of the MAS, the auxiliary surface should encircle the scattered field singularities as tightly as possible [3]. The standard allocation of the auxiliary sources is based on empirical rules and on the caustic hypothesis. Consequently, the appropriate distribution of the auxiliary sources, for a predesigned accuracy is achieved by try-and-error processes or by searching the corresponding caustic surfaces [4,5,6]. These approaches are applicable only for problems with simple canonical geometries. The obscurity associated with the location of the auxiliary sources for arbitrary-shaped objects is the prevailing breakdown point of the MAS.

The salient feature of the proposed technique is the replacement of laborious trial and error procedure of the standard MAS by a systematic and fully automated one. The outline of the paper is as follows. First we go over 2D scattering problem with perfectly conducting boundary conditions, we present the formulation of the MAS and we discuss the impact of the singularities localization on its efficiency. Secondly, we review the level set method and we adapt it to our purpose. Proceeding from this, we set up a numerical framework to automatically carry out the optimal auxiliary sources distribution. Finally, we report some numerical experiments to clearly demonstrate the accuracy and the robustness of the optimized MAS compared to the standard one.

II. AUXILIARY SOURCES METHOD FORMULATION

Let us consider a two-dimensional object having an arbitrary cross-section Ω as shown in Fig. 1. The object is assumed to be uniform in the z direction and a perfectly conducting boundary condition is held on its contour Γ. The polarization of the incident wave is assumed to be transverse magnetic (TM) with respect to the cylinder axis z. It is obvious that the scattered field is z directed too, which reduces the scattering problem to a bidirectional one.

On the boundary Γ the scattered electric field satisfies:

$$E_z^s = -E_z^i \quad \forall \vec{r} \in \Gamma. \tag{1}$$

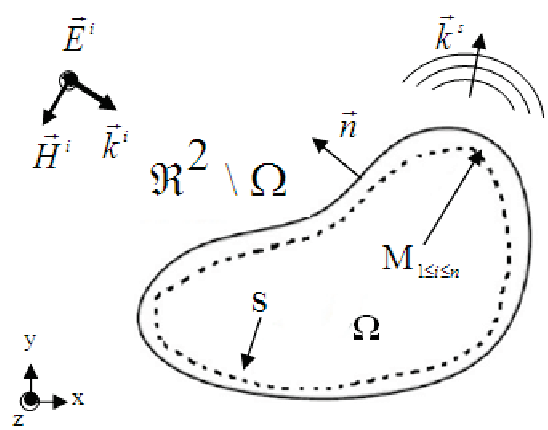

Fig. 1. MAS geometry.

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According to the auxiliary sources method, let S designates the auxiliary surface, $M_{1\le i\le n}$ the radiation centers located at points $\vec{r}_{1\le i\le n}$ as shown in Fig. 1, and $H_0^{(2)}(k|\vec{r}_i-\vec{r}|)_{1\le i\le n}$ the Hankel functions of zero order and second kind associated with elementary sources. Kupradze [2] proved that the set of functions $H_0^{(2)}(k|\vec{r}_i-\vec{r}|)_{1\le i\le \infty}$ is complete and linearly independent on the surface Γ. So, there are coefficients $a_{1\le i\le n}$ such that, using the n first functions of the aforementioned system, the scattered electric field can be approximated as follows:

$$E_z^s \approx \sum_{i=1}^{n} a_i H_0^{(2)}(k|\vec{r}_i-\vec{r}|). \quad (2)$$

This will approach exact solution as $n \mapsto \infty$ It has been proved that any scattered field that transfers energy to infinity must have areas of irregular sources within finite volume, otherwise the scattered field is null everywhere [7]. So, it is obvious that E_z^s certainly has irregular domains that could be in form of isolated points, lines or surfaces. S should enclose the singularities as tightly as possible. Ignoring this point leads to a weakening of convergence and even diverging of the equation (2) when n increases. By matching the boundary condition at m collocation points $\vec{r}_{1\le p\le m}$, the problem can be formulated as follows. Find $a_{1\le i\le n}$ such that:

$$\sum_{i=1}^{n} a_i H_0^{(2)}(k|\vec{r}_i-\vec{r}_p|) = -E_z^i(\vec{r}_p) \;\; 1\le p\le m. \quad (3)$$

The stability and the size of the obtained algebraic system depend on the proper choice of auxiliary parameters that are the shape of the auxiliary surface S and the distribution of the radiation centers $\vec{\mathbf{r}}_{1\le i\le n}$.

The necessary number of terms of the series (2) strongly depends on the relative distance between the real surface and the auxiliary surface S on which the auxiliary sources are placed. When the auxiliary surface moves away from the real one the number of terms in (2) decreases strongly and consequently, the computational cost decreases, but it should be noted that if the scattered field singularities appear outside the auxiliary surface, the computing process might diverge. Therefore a good description of singularities is an essential part of the method to carry out the optimal solution.

III. THE LEVEL SET METHOD

A. An overview of level set method

The level set method was introduced by Osher and Sethian [8] in the field of fluid dynamics, to trace interfaces between different phases of fluid flows. Later, it has been used for many different kinds of physical problems. The main idea behind this method is to represent the interface at each time t as the zero level set of a function $\varphi(\vec{r},t)$. Thus, given a contour S bounding an open region D in $\Re^2$, we wish to study its motion under a velocity field v. The level set idea consists in defining a smooth function to implicitly represent S as the set of points where $\varphi(\vec{r},t)$ vanishes. That is $S=\{\vec{r}\in\Re^2 / \varphi(\vec{r},t)=0\}$. The function $\varphi(\vec{r},t)$ is called the level set function, and it has the following properties:

$$\begin{aligned} &\varphi < 0 \; for \; \vec{r} \in D \\ &\varphi > 0 \; for \; \vec{r} \notin D \cdot \\ &\varphi = 0 \; for \; \vec{r} \in S \end{aligned} \quad (3)$$

This concept is illustrated by the Fig. 2.

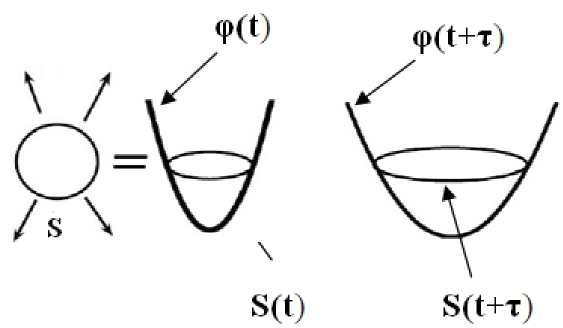

Fig. 2. Level set function.

The Fig. 2 shows a propagating contour and the accompanying function φ, S moves along its forwards pointing normal at a fixed speed v. We remark that the evolution of S is perfectly described by the zero level set of φ. The evolution of the implicit function φ can be described by the following partial differential equation, known as Hamilton-Jacobi equation [8]:

$$\frac{\partial\varphi}{\partial \mathrm{t}} + \mathrm{v}\|\nabla\varphi\| = 0, \; \varphi(\vec{\mathrm{r}},0)=\varphi_0, \quad (4)$$

where, $\frac{\partial}{\partial t}$ denotes a partial derivative to the temporal variable t and ∇ denotes the gradient operator. The function φ_0 embeds the initial position of the moving contour S.

B. The level set dictionary

Once the level set function φ is defined, most of the geometrical quantities of the contour S can be represented in terms of the function φ:

The normal vector is given by:

$$\vec{n} = \frac{\nabla\varphi}{\|\nabla\varphi\|}. \quad (5)$$

The mean curvature:

$$\kappa = \nabla\cdot\vec{n} = \nabla\cdot\frac{\nabla\varphi}{\|\nabla\varphi\|}. \quad (6)$$

The length of S:

$$L(\varphi) = \int_{\Omega} \delta(\varphi) \|\nabla \varphi\| dS, \quad (7)$$

where $\delta(\varphi)$ denotes the Dirac function:

$$\delta(\varphi) = \begin{cases} 1 \text{ if } \varphi = 0 \\ 0 \text{ if } \varphi \neq 0 \end{cases}. \quad (8)$$

Moreover, the integral of a function f along S can be writing in function of φ:

$$\int_S f(r) dS = \int_{\Omega} f(r) \delta(\varphi) \|\nabla \varphi\| dr. \quad (9)$$

IV. OPTIMIZATION METHOD

The algebraic system (3) can be written in the following form:

$$\int_S A(\vec{r}\,') H_0^{(2)}(k|\vec{r}\,' - \vec{r}_p|) dS = -E_z^i(\vec{r}_p) \quad 1 \leq p \leq m, \quad (10)$$

where $A(\vec{r})$ represents the distribution of the auxiliary sources over S. Let's consider D an open subset of Ω enclosed by the auxiliary surface S as shown in Fig. 3.

We define φ as level set function of S by:

$$\varphi(\vec{r}, t) = \begin{cases} -\text{distance}(\vec{r}, S) \; if \; \vec{r} \in D \\ \text{distance}(\vec{r}, S) \; if \; \vec{r} \notin D \end{cases}. \quad (11)$$

S divides the domain D into two parts, and then the level set function φ is negative inside and positive outside,

$$S = \{\vec{r} \in \Re^2 / \varphi(\vec{r}, t) = 0\}. \quad (12)$$

By using the property (9) of the level set method, the equation (10) can be writing as:

$$\int_{\Omega} A(\vec{r}\,') H_0^{(2)}(k|\vec{r}\,' - \vec{r}_p|) \|\nabla \varphi\| \delta(\varphi) dr' = -E_z^i(\vec{r}_p) \quad 1 \leq p \leq m. \quad (13)$$

The problem can be formulated as an optimization one:

$$(A^*, \varphi^*) = \underset{A,\varphi}{\operatorname{argmin}} J(A, \varphi). \quad (14)$$

Find distribution A^* and the level set function φ^* which minimize the cost function J:

$$J(A, \varphi) = \frac{1}{m} \sum_{p=1}^{m} \left\| \int_{\Omega} A(\vec{r}\,') H_0^{(2)}(k|\vec{r}\,' - \vec{r}_p|) \|\nabla \varphi\| \delta(\varphi) dr' + E_z^i(\vec{r}_p) \right\|^2 \quad (15)$$

The optimal distribution of the radiation centers strongly depends on the area of the auxiliary surface. By shifting the sources into the conducting body the scattered field function becomes smoother on the surface of the body and the fulfillment of the boundary conditions in the region between collocation points is improved. However, the shift of the auxiliary surface is restricted by the location of the scattered field singularities. So, the area of the auxiliary surface should be added to the cost functional J as regularization term. Therefore, we force the algorithm to search the best-suited auxiliary surface that encloses the singularities:

$$J(A, \varphi) = \frac{1}{m} \sum_{p=1}^{m} \left\| \int_{\Omega} A(\vec{r}\,') H_0^{(2)}(k|\vec{r}\,' - \vec{r}_p|) \|\nabla \varphi\| \delta(\varphi) dr' + E_z^i(\vec{r}_p) \right\|^2 + \beta L(\varphi) \quad (16)$$

where β is a real-valued regularization coefficient.

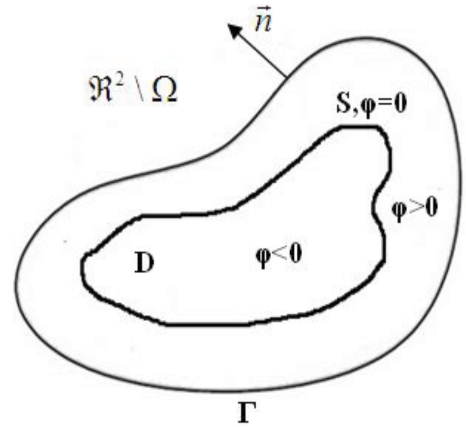

Fig. 3. Level-set representation of the auxiliary surface.

A. Localization of the auxiliary surface

The evolution of φ is described by the following Hamilton-Jacobi equation (4). In this differential form t stands for not actual time, but instead optimization steps. We want to choose an evolution law v such $\frac{\partial J}{\partial t} < 0$, J will decrease with the artificial time evolution during a sufficient small time interval, $[0, \tau]$ $\frac{\partial J}{\partial t}$ is given by: (by applying the chain's rule and the definition of Gateaux differential):

$$\frac{\partial \mathrm{J}}{\partial \mathrm{t}} = \int_{\mathrm{S}} \mathrm{v}\left[\alpha\kappa + \frac{\partial \alpha}{\partial \vec{\mathrm{n}}}\right] \mathrm{dS}, \quad (17)$$

where $\vec{n}$ denotes the unit normal to the auxiliary surface S and

$$\alpha = [\; \beta + \frac{1}{m} \sum_{p=1}^{m} real(< 2(\int_S A(\vec{r}\,') H_0^{(2)}(k|\vec{r}\,' - \vec{r}_p|) dS - E_Z(\vec{r}_p)), \; A(\vec{r}) H_0^{(2)}(k|\vec{r} - \vec{r}_p|) >) \quad (18)$$

$\langle , \rangle$ denotes the dot product. An obvious selection for v is:

$$\mathrm{v} = -\left(\beta\kappa + \frac{\partial \alpha}{\partial \vec{\mathrm{n}}}\right). \quad (19)$$

After substituting v into (4), the differential equation can be solved numerically using the upwind scheme.

B. Calculation of the radiation center positions

Suppose φ is perturbed by a small variation $\delta\varphi$ and let δr be the resulting variation of the point r as shown in Fig. 4. By taking the variations of the equation (4) between $t=0$ and $t=\tau$, we get:

$$\delta\varphi + \mathrm{v}\tau\|\nabla\varphi\| = 0 . \quad (20)$$

We have $\mathrm{v}\tau = \delta \mathrm{r}$. We find the relation between δr and $\delta\varphi$:

$$\delta r = -\frac{\delta\varphi}{\|\nabla\varphi\|} . \quad (21)$$

So, the radiation center positions $r_{1\le i\le n}$ are updated as follows:

$$r_i(t+\tau) = r_i(t) - \frac{\delta\varphi}{\|\nabla\varphi\|} . \quad (22)$$

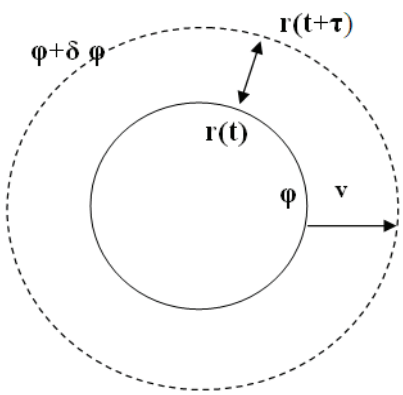

Fig. 4. Deformation of shapes by the level set formulation

C. Calculation of the auxiliary sources' amplitudes

To find the optimal auxiliary sources' amplitudes, we should update $a_{1\le i\le n}$ by following the descent direction of the cost function J. The descent direction is given by the negative derivative of J with respect to $a_{1\le i\le n}$. So, we just need to compute $\frac{\partial J}{\partial a_i}{}_{1\le i\le n}$ and updating a_i as follows. Choose the step size $\alpha > 0$:

$$a_i(t+\tau) = a_i(t) - \alpha\frac{\partial J}{\partial a_i} . \quad (23)$$

$\frac{\partial J}{\partial a_i}$ is given by: (by applying the chain's rule and the definition of Gateaux differential):

$$\frac{\partial J}{\partial a_i} = \frac{1}{m}\sum_{p=1}^{m} real(<2(\sum_{k=1}^{n} a_k H_0^{(2)}(k|\vec{r}_k - \vec{r}_p|) - E_Z^i(\vec{r}_p)),. \quad (24)$$
$$H_0^{(2)}(k|\vec{r}_i - \vec{r}_p|)>)$$

D. Numerical scheme

To summarize we can list the optimization steps as below:

1- Choose the initial level set function φ_0 that represents the initial auxiliary surface S_0.

2- Choose the initial positions and amplitudes of the radiation centers ($r_{1\le i\le n}, a_{1\le i\le n}$).

3- For $j \ge 0$:
  - Choose the regularization coefficient β and calculate v (equation (19)).
  - Determine the level set function φ^j by resolving the Hamilton-Jacobi equation in the time interval $[0,\tau]$ with the initial condition $\varphi = \varphi^{j-1}$ (equation (4)).
  - Update r_i^j (equation (22)).
  - Update a_i^j (equation (23)).
  - Go to the next iteration if not converged.

The error of the boundary condition is used for convergence criterion:

$$e_{1\le p\le m} = \frac{\left\| \int A(\vec{r}\,')H_0^{(2)}(k|\vec{r}\,'-\vec{r}_p|)dS - E_Z^i \right\|}{\|E_Z^i\|} .$$

Iterations continue until the stop criterion will be satisfied, typically when the errors $e_{1\le p\le m}$ exhibit, between two iterations, become smaller than a predefined threshold. The proposed optimization procedure provides three degrees of freedom (auxiliary surface, positions and amplitudes of the radiation centers) to achieve any boundary condition error, which is a great advantage over classical MAS implementations. Indeed, in the classical MAS implementations, the auxiliary surface and the radiation center positions are fixed beforehand. So, the accuracy is not automatically adjustable, the only degree of freedom is the auxiliary sources' amplitudes. Generally, the standard MAS is based on empirical rules and on the caustic concept leading to the following recommendations [9,10,11,12,13]. The distance d between the physical surface Γ and the auxiliary surface S should satisfy the condition d<Rmin, where Rmin is the minimal radius of positive curvature of the surface Γ. Several numerical methods have been proposed to overcome these constraints such as [6,14,15]. By using level set technique, the proposed method shows a great potential to determine the optimal MAS parameters that satisfy any accuracy.

V. NUMERICAL EXPERIMENTS

We present two experiments. The aim of the first one is showing the aptitude of the optimized method to trace scattered field singularities. The second numerical example is about RCS (radar cross-section) calculating, we compare the accuracy and the computational cost obtained by the proposed method and those obtained by the standard MAS.

A. Singularities localization

We consider an elliptical cylinder illuminated by an incident plane wave at 100 MHz. It is well-known that the singularities of the wave field in this problem are located at focal points [16]. The auxiliary surface evolution at 0, 23, 50 and 70 iterations is shown at Fig. 5. The angle α (Figs. 5 and 6) is used to indicate the positions of the radiation centers on the auxiliary surface.

When the boundary condition error does not exceed 1% (iteration 70), the distribution of the sources' amplitudes for different angle α is shown in Fig. 5. The obtained surface passes through the foci's region and for particular angles $\alpha \in \{0, \pi, 2\pi\}$ the sharpest amplitude is observed. These angles correspond to focal points F1 and F2, which is consistent with the analytic result.

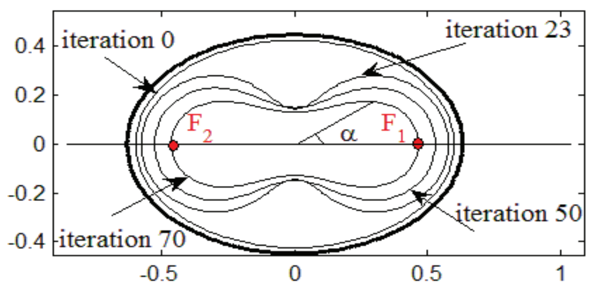

Fig. 5. Singularities location.

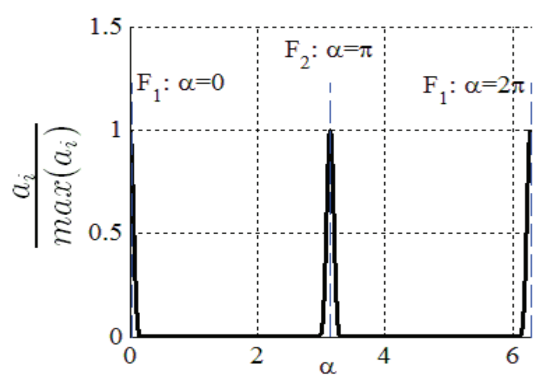

Fig. 6. The distribution of the auxiliary sources amplitudes.

B. Accuracy and computational cost evaluation

We present numerical experiments for some canonical geometries: infinite cylinders with square cross-section, circular cross-section, star-shaped cross-section and dumbbell-shaped cross-section. All these obstacles are illuminated by an incident plane wave at f = 300 Mhz. Figures 7, 8, 9 and 10 show a comparison between the bistatic RCS values obtained from the optimized MAS and those obtained by the standard implementation, the result from FEKO electromagnetic simulation software is taken as reference. Tables 1 and 2 show comparison between the optimized and the standard MAS method, the following criteria are taken into account: achieved accuracy (error on the boundary condition), number of auxiliary sources and number of collocation points.

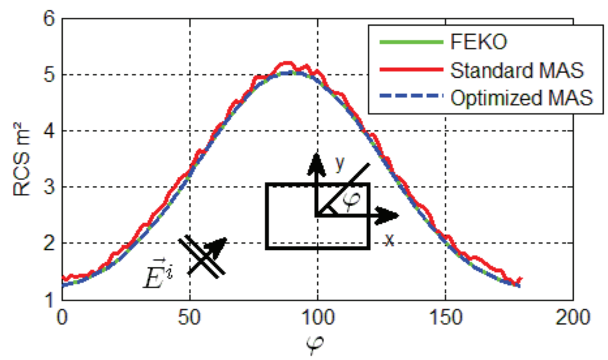

Fig. 7. RCS of an infinite length cylinder with square cross-section.

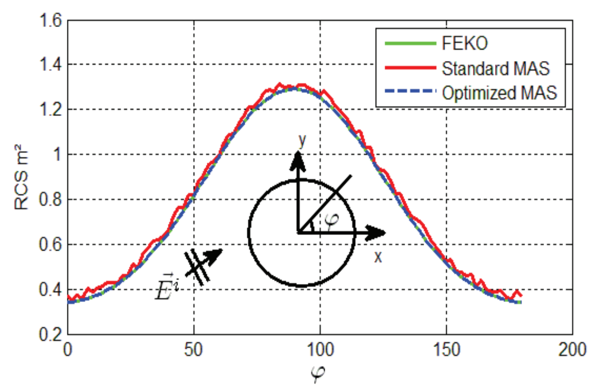

Fig. 8. RCS of an infinite length cylinder with circular cross-section.

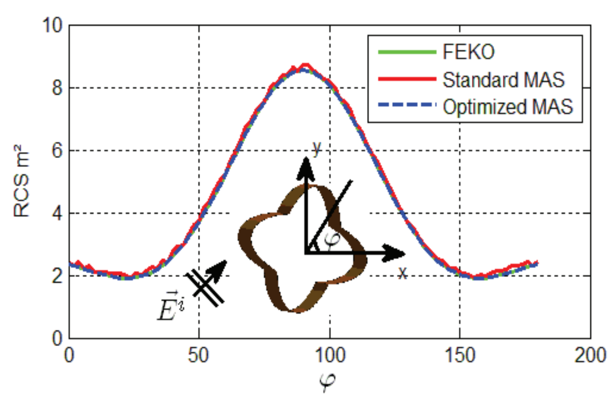

Fig. 9. RCS of an infinite length cylinder with star-shaped cross-section.

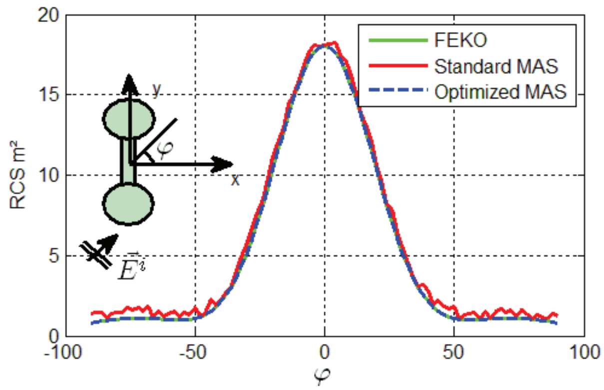

Fig. 10. RCS of an infinite length cylinder with dumbbell-shaped cross-section

Table 1: Results obtained by the standard MAS implementation

	Accuracy	Auxiliary Sources	Collocation Points
Circular cross-section	5%	300	300
Square cross-section	7%	400	400
Star-shaped cross-section	5%	400	400
Dumbbell shaped cross-section	6%	400	400

Table 2: Results obtained by the optimized MAS

	Accuracy	Auxiliary Sources	Collocation Points
Circular cross-section	0.1%	50	300
Square cross-section	0.2%	90	400
Star-shaped cross-section	0.1%	50	400
Dumbbell shaped cross-section	0.1%	50	400

As a conclusion of these numerical experiments, it appears clearly that the optimized method is able to achieve high accuracy with less implementation cost than the standard MAS implementation.

VI. CONCLUSION

We have reported a numerical scheme for determining the optimal MAS parameters for two-dimensional scattering problem by using the level set method. The comparison between RCS obtained from the proposed framework and those obtained from the standard MAS implementation shows that the proposed method can achieve high accuracy with less implementation cost. We have restricted our study to perfect electric cylinders, but the optimized method can be easily extended to study partially or fully penetrable ones.

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