Fast Solution of Low-Frequency Problems Using Efficient Form of MLACA with Loop-Tree Basis Functions

Zhaoneng Jiang [1], Xiaoyan Zhao [1], Ye Jiang [1*], Xuguang Qiao [1], and Quanquan Wang [2]

[1] Hefei University of Technology, Hefei 230009, jiangye@hfut.edu.cn

[2] Nanjing University of Posts and Telecommunications, Nanjing 210003, wangqq@njupt.edu.cn

Abstract **─** In this paper, an efficient scheme of numerical method is proposed to solve the low frequency (LF) problems, which combines the loop-tree basis functions with an efficient form of multilevel adaptive cross approximation (EFMLACA) algorithm. It utilizes the loop-tree basis functions to divide the vector part and scalar part of the impedance matrix. Meanwhile, the scalar part is frequency normalized. Through this operation, it can avoid the low frequency breakdown problem. In order to accelerate the matrix vector multiplication, the EFMLACA algorithm is applied. Meanwhile, the compressed block decomposition (CBD) preconditioner is applied to improve the condition number of poor convergence problems. The numerical results demonstrate that the memory requirement and computation time required for a matrix vector multiplication of EFMLACA algorithm is much less than that of MLACA and ACA-SVD. Moreover, the matrix vector multiplication of EFMLACA algorithm is also much more efficient than that of low-frequency multilevel fast multipole algorithm (LF-MLFMA).

Index Terms **─ C**ompressed block decomposition (CBD), efficient form of multilevel adaptive cross approximation (EFMLACA) algorithm, low frequency.

I. INTRODUCTION

The numerical simulations of acoustics, microwave filter design, interconnect modeling, and electromagnetic scattering require the solution of Helmholtz equation. One of the popular approaches of numerical solution is to convert them into integral equations [1-3], which are then discretized using appropriate quadrature formulae. This usually leads to large-scale systems of linear algebraic equations, which are in turn solved via appropriately chosen iterative schemes (such as generalized minimal residual (GMRES) [4-5]. Most iterative schemes require the application of the matrix to a sequence of recursively generated vectors. Applying a dense matrix to a vector is an order N^2 procedure, where N is the number of nodes in the discretization of domain. As a result, the whole process is at least of order N^2, which is prohibitive for many large-scale problems.

A further reduction in operations count can be achieved by applying the fast numerical methods [6-10], thus significantly reducing the computational complexity and the memory requirement. The multilevel fast multipole algorithm (MLFMA) is one such method, which can reduces the computational complexity to $O(N\log N)$. It is very efficient for analyzing the models contain few sub-wavelength geometrical detail. When the electric field integral equation (EFIE) is used to analyze the low frequency (LF) problems (when the electrical size of objects is much less than wavelength), a phenomenon called the low frequency (LF) breakdown will be encountered. The origin of LF breakdown for EFIE is not the reason of mathematical, which is caused by the inevitable numerical round-off error on a finite-precision computer. When the frequency tends to zero, the vector part of impedance matrix is much smaller than that of scalar part, resulting in the vector part to be omitted.

In order to overcome this problem, loop-tree basis functions are applied. It separates the vector part and the scalar part of impedance matrix, and frequency normalization is utilized to enhance the role of vector part. In [11], a new method called the low-frequency multilevel fast multipole algorithm (LF-MLFMA) is introduced. However, the matrix vector multiplication of LF-MLFMA is very slow. The low-rank decomposition method is one of the most popular techniques, which has been widely applied to solve electromagnetic problems [12-18]. It is a technology for finding sparse representations of large scale matrix, and is a powerful way to find potentially useful information from the matrix. The multilevel adaptive cross approximation (MLACA) algorithm is one of the fast low-rank methods. In [16], the structure of the MLACA representation is given in a multilevel recursion manner. To improve the efficiency of ACA, some SVD techniques are introduced in [15,18] to recompress the matrices of ACA. In this paper, an efficient form of multilevel adaptive cross approximation (EFMLACA) algorithm is introduced. Moreover, the compressed block decomposition (CBD) [19] preconditioner is applied to further accelerate the

Submitted On: November 2, 2017
Accepted On: December 29, 2018

matrix vector multiplication.

The Section 2 contains the theory of loop-tree method of moments. In Section 3, we discuss the procedure of efficient form of multilevel adaptive cross approximation (EFMLACA) algorithm. The performance of EFMLACA algorithm is illustrated in Section 4 with numerical results. Finally, Section 5 gives very brief conclusions.

II. LOOP-TREE METHOD OF MOMENTS

The traditional method of moments (MoM) utilizes Rao–Wilton–Glisson (RWG) basis function to solve the electromagnetic problem. When the frequency trends to zero, electric field integral equation (EFIE) will encounters the LF breakdown problem. The form of EFIE is given in the following:

$$\vec{n}\times[-j\omega\mu_0 A(r)+\frac{1}{j\omega\varepsilon_0}\nabla\phi(r)]=\vec{n}\times E(r)\,, \tag{1}$$

where $A(r)$ denotes the magnetic vector potential, $\phi(r)$ denotes the scalar potential,

$$A(r)=\int_{S'} G(r,r')\cdot J(r')dS'$$
$$\phi(r)=\int_{S'} G(r,r')\nabla\cdot J(r')dS' .$$

It can be observed from the formula (1) when the frequency tends to zero, the vector part $j\omega\mu_0 A(r)$ is much smaller than the scalar part $\nabla\phi(r)/j\omega\varepsilon_0$. Due to the inevitable numerical round-off error on a finite-precision computer, the information of vector potential will be omitted, and the rest of scalar potential is not accurate enough to describe the target surface current. Therefore, the solution is not stable. This problem can be improved by improving the accuracy of variables in the program, but if you want to solve this problem completely, the loop-tree basis function [20-21] is applied to modify the formula (1):

$$\begin{bmatrix}\omega Z_{LL} & \omega Z_{LC}\\ \omega Z_{CL} & \omega Z_{CC}+\omega^{-1}\tilde{Z}_{CC}\end{bmatrix}\begin{bmatrix}I_L\\ I_C\end{bmatrix}=\begin{bmatrix}V_L\\ V_C\end{bmatrix}, \tag{2}$$

where,

$$Z_{LL}=T_L^t\cdot Z_A\cdot T_L,\qquad Z_{LC}=T_L^t\cdot Z_A\cdot T_C$$
$$Z_{CL}=T_C^t\cdot Z_A\cdot T_L,\qquad Z_{CC}=T_C^t\cdot Z_A\cdot T_C$$
$$\tilde{Z}_{CC}=T_C^t\cdot Z_\phi\cdot T_C,\qquad V_L=T_L^t\cdot V,\qquad V_C=T_C^t\cdot V$$

where T represents the transformation matrix obtained from RWG basis functions to Loop-Tree basis functions [20-21]. T_C denotes the transformation matrix from RWG basis functions to surface-tree basis functions. Meanwhile, T_L denotes the transformation matrix from RWG basis functions to surface-loop basis functions. Z_A denotes the vector part of impedance matrix Z, and Z_ϕ denotes the scalar part of Z. Due to the huge amount of calculation needed for analyzing the LF problems with large number of unknowns, an efficient form of multilevel adaptive cross approximation (EFMLACA) algorithm is proposed.

III. EFMLACA ALGORITHM

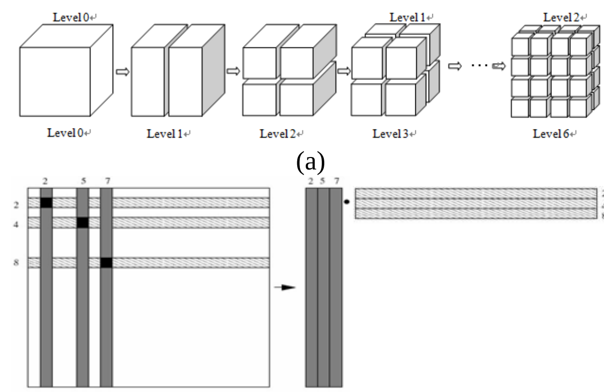
A. Multilevel adaptive cross approximation (MLACA)

Multilevel adaptive cross approximation (MLACA) algorithm [15-17] is based on tree structure to divide the object into many sub-domains shown in Fig. 1 (a), which is same as that of reference [15]. The impedance matrix Z can be expressed as:

$$[Z]=\sum_{l=3}^{L}[Z]_l\,,$$

where $[Z]_l$ denotes the matrix in level l. L denotes the total number of tree structure.

The MLACA algorithm decomposes the far-field matrix into relatively small sub-matrices multiplication. Only r rows and r columns (r denotes the rank of impedance matrix) of the matrix are required to fill, and the entire matrix is not required (shown in Fig. 1 (b)). Under the premise of ensuring the accuracy, as far as possible to extract the minimum rank value of each matrix, so as to effectively reduce memory consumption and computation time.



(b)

Fig. 1. (a) Subdivision of an object using tree structure in the MLACA algorithm, and (b) the scheme of selecting rows and columns in MLACA algorithm.

In the expression of ACA algorithm, the far-field matrix can be expressed in the following form:

$$[Z]^{m\times n}=[U]^{m\times r}[V]^{r\times n}, \tag{3}$$

where $[Z]^{m\times n}$ denotes the far-field matrix, r denotes the rank of far field matrix ($r\ll\min(m,n)$). The dimensions of two sub-matrices are both very small. For the large-scale problems, the computation efficiency of matrix multiplication of ACA algorithm is relatively low.

B. Efficient form of MLACA (EFMLACA) algorithm

In this section, a sparse expression of ACA algorithm is applied to accelerate the matrix vector multiplication. It utilizes transform matrix to obtain a compressed representation of the original matrix. The transform matrix is a block diagonal unitary matrix, which is shown in Fig. 2.

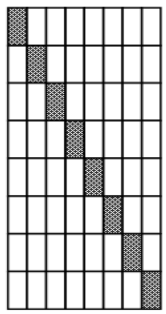


Fig. 2. The form of transform matrix.

The original matrix of equation (3) can be expressed as:

$$[Z]^{m\times n} = [W][\mathrm{Y}][W]^{\dagger}, \tag{4}$$

where $[W]$ is the transform matrix. $[W]^{\dagger}$ is the conjugate matrix of transform matrix $[W]$. The dimensions of $[Y]$ is very small, which requires little memory requirement.

The impedance matrix can be expressed as:

$$[Z]_{far} = \sum_{l=2}^{L}\sum_{i=1}^{M(l)}[W_{li}]\sum_{j=1}^{Far(l(i))}[Y_{lij}][W_{lj}]^{\dagger}, \tag{5}$$

where $M(l)$ is the number of nonempty groups at level l and, $Far(l(i))$ denotes the number of far interaction groups of the i-th nonempty group for each observer group $l(i)$ at level l. $[Z]_{far}$ denotes the far-field matrix. For each observation group, it needs to store the matrix $[W]$ only once. However, the $[U]$ and $[V]$ in equation (3) need to be stored many times for each observation group. The detailed solving process of transform matrix is given in the following.

(*i*) The filling process of matrix [*W*]

At the level l, extract the corresponding matrices $[V]_{ij}$ for each observation group j, which is formed by equation (3). Then concatenate all matrices $[V]_{ij}$ in a row and gain the transition matrix A, which is shown in Fig. 3.

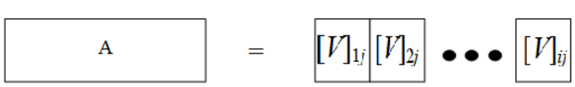


Fig. 3. The form of transition matrix *A*.

Then, the adaptive cross approximation-singular value decomposition (ACA-SVD) [18] is utilized to decompose the transition matrix A, which needs the whole information of matrix A:

$$[A]^{m\times n} \xrightarrow{ACA-SVD} [U']^{m\times k}[S']^{k\times k}\left\{[V']^{n\times k}\right\}^{*}, k \ll \min(m,n), \tag{6}$$

where n and m denotes dimensions of the matrix A. k denotes the rank of matrix A. $\left\{[V']^{n\times k}\right\}^{*}$ is the j-th diagonal block of $[W_l]$, which is a unitary matrix. Implement the procedure for all observation groups at level l as the above, can obtain $[W_l]$. Using the above procedures at all levels of octal tree structure, can gain the whole transform matrix of $[W]$.

(*ii*) The filling process of matrix [*Y*]

At the level l, the matrix $[W_l]$ is used to multiply the equation (3) on the both sides. The detail expression is given for a far-field sub-matrix in the following:

$$[Y_l] = [W]^{m\times r\dagger}[U']^{m\times r}[S']^{r\times r}\left\{[V']^{n\times r}\right\}^{*}[W_{lj}]^{n\times r}, \tag{7}$$

where r is much less than the dimensions of original matrix ($r \ll \min(m, n)$), so the dimension of matrix $[Y]$ is much smaller than that of original matrix. Using the equation (7) at all levels of tree structure, can obtain the whole matrix of $[Y]$.

The main idea of above procedure is to use transform matrix $[W]$ to recompress the far-field matrices of MLACA algorithm. The main memory requirement of EFMLACA algorithm is the storage of matrices $[W]$ and $[Y]$. The $[W]$ is a block diagonal matrix, which is very sparse. Meanwhile, the dimension of matrix $[Y]$ is very small. Therefore, the memory requirement of equation (4) is much less than that of equation (3).

C. Compressed block decomposition (CBD) preconditioner

In this paper, an efficient preconditioner is proposed to further accelerate the matrix vector multiplication. The amplitude difference of diagonal elements obtained by loop-tree basis functions is very large. Therefore, the impedance matrix should be diagonal normalized. In general, the inversion of whole impedance matrix are used as the preconditioner can get the best effect. However, computational complexity and memory consumption of its inverse process are too high, the good choice is to select an approximate matrix instead of the whole impedance matrix. The diagonal block matrices contain the strong interaction, which contain the main information of impedance matrix. Therefore, this paper applies the inversion of diagonal block matrices as the preconditioner to improve convergence problem. The compressed block decomposition (CBD) algorithm [19] is utilized to form the inversion of diagonal block matrices efficiently.

IV. NUMERICAL RESULTS

The low-frequency problems are analyzed in the following, and the computational efficiency of EFMLACA algorithm is compared with traditional MLACA. The truncating tolerance of both EFMLACA and MLACA is 10^{-3}. The restart number of generalized minimal residual (GMRES) is 30 and the stop precision for restarted GMRES is set to be 10^{-4}.

The first low-frequency problem is a sphere, shown in Fig. 4. The radius is 0.5 meters and the frequency is 3MHz. The electrical size of the sphere is 0.01 λ, which is a typical low-frequency problem. The number of tree structure is 3. The bistatic RCS of the structure is given in Fig. 5. It can be found that the results of EFMLACA algorithm are in agreement with that of Mie, which demonstrates the accuracy of EFMLACA algorithm. Figure 6 and Fig. 7 compare the efficiency of EFMLACA algorithm with that of MLACA and ACA-SVD [18]. The memory requirements and computation time required for a matrix vector multiplication (MVP) of EFMLACA and MLACA needed with the increase of the number of unknowns. It can be observed that the EFMLACA is much more efficient than that of traditional MLACA. The memory requirement and MVP time are only third of traditional MLACA. Through our simulation, it is found that the total number of layers affects the efficiency of the EFMLACA algorithm. Table 1 shows the best one of the total number of tree structure for EFMLACA algorithm varying with unknowns in this example. Table 2 also shows the computation time and steps needed for the iterative algorithm of CBD preconditioner and without preconditioner. It can be found that CBD preconditioner obviously improves the efficiency of matrix vector multiplication.

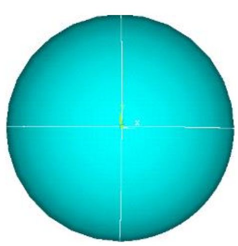


Fig. 4. The sphere model.

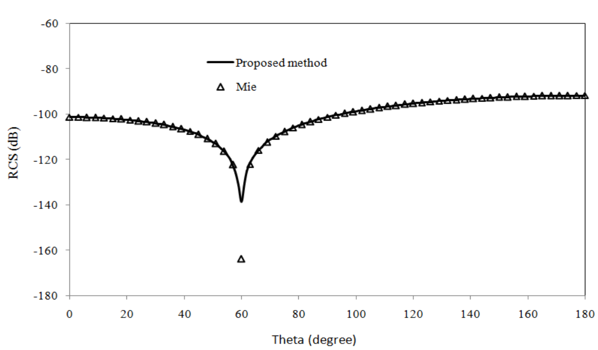


Fig. 5. The RCS results of the sphere structure.

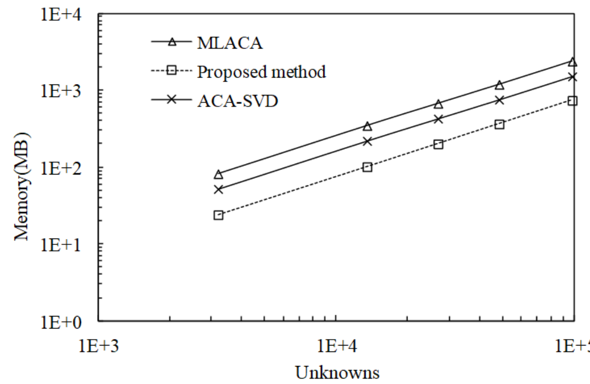


Fig. 6. The memory consumption of the far-field.

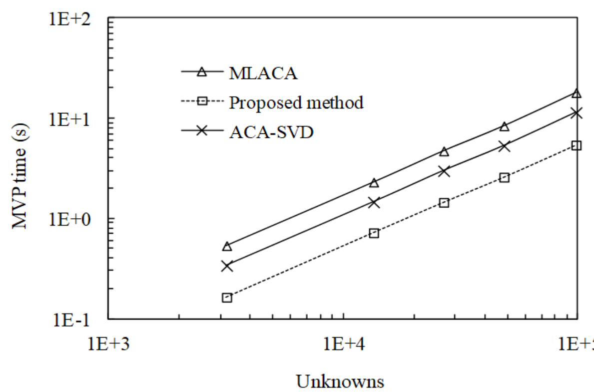


Fig. 7. The time needed of a matrix vector multiplication.

Table 1: The best one of the total number of tree structure for EFMLACA algorithm varying with unknowns in this example

Unknowns Number	The Best One
15000	1
30000	2
60000	3
80000	4
100000	5

Table 2: Computation time and steps needed for the iterative algorithm

Method	Step	Total Time (min)
Without Preconditioner	6741	167.1
CBD Preconditioner	103	2.57

The second low-frequency problem is a cylinder, shown in Fig. 8. The radius is 0.5 meters and the height is 1 meters. The frequency is 30MHz and the number of tree structure is 4. The electrical size of the cylinder is 0.1 λ. Figure 9 shows the results of bistatic RCS. According to the figure, the results of EFMLACA is nearly same as that of LF-MLFMA. The efficiency of EFMLACA algorithm is also analyzed in Fig. 10 and

Fig. 11. It can be found that the EFMLACA algorithm is much faster than that of LF-MLFMA. Meanwhile, the memory requirements of the EFMLACA algorithm is nearly same as that of LF-MLFMA. Table 3 shows the best one of the total number of tree structure for EFMLACA algorithm varying with unknowns in this example. Table 4 shows the computation time and steps needed for the iterative algorithm. The computation time and steps of CBD preconditioner are much less that of without preconditioner. The total solving times of EFMLACA and LF-MLFMA are given in Fig. 12. The CBD preconditioner is applied in the two methods. It can be found that EFMLACA is much more efficient than LF-MLFMA.

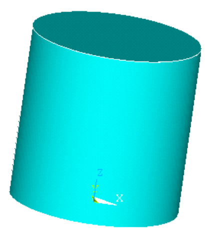


Fig. 8. The cylinder model.

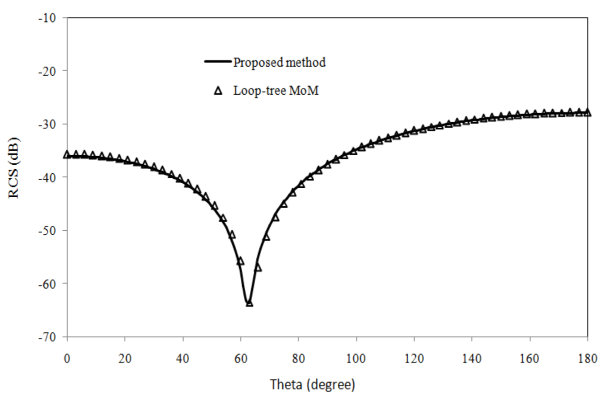


Fig. 9. The RCS results of the cylinder structure.

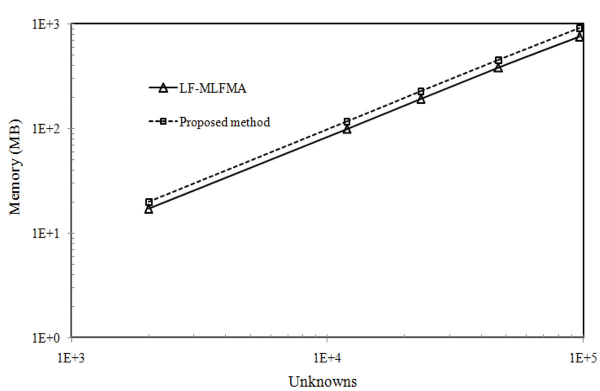


Fig. 10. The memory consumption of the far-field.

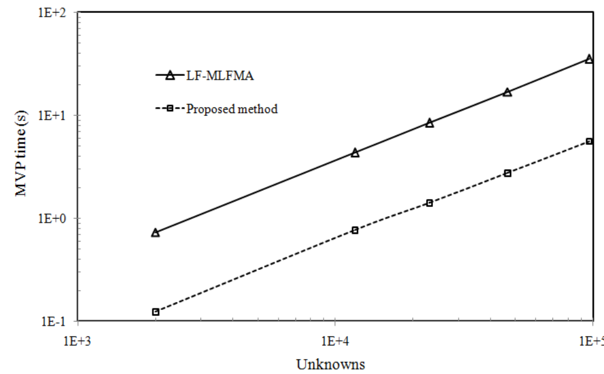


Fig. 11. The time needed of a matrix vector multiplication.

Table 3: The best one of the total number of tree structure for EFMLACA algorithm varying with unknowns in this example

Unknowns Number	The Best One
20000	1
40000	2
60000	3
80000	4
100000	5

Table 4: Computation time and steps needed for the iterative algorithm

Method	Step	Total Time (min)
Without Preconditioner	4987	117.2
CBD Preconditioner	86	2.01

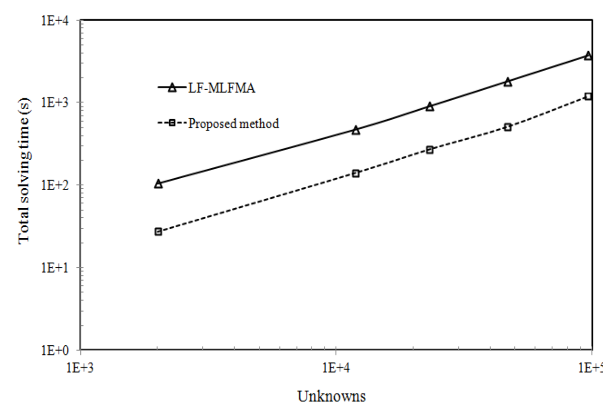


Fig. 12. The total solving times needed of EFMLACA and LF-MLFMA.

V. CONCLUSION

This paper introduces an efficient method to solve the electromagnetic scattering of low-frequency problem by using the EFMLACA algorithm combined with the CBD preconditioner. Numerical results show that EFMLACA algorithm can further improve the efficiency and save the memory requirements. Meanwhile, the matrix vector multiplication of EFMLACA algorithm is also much more efficient than that of LF-MLFMA and ACA-SVD.

ACKNOWLEDGMENT

This work was supported in part by National Natural Science Foundation of China under Contract Number 61501159, 61501252, HeFei University of Technology Teacher Foundation of JZ2014HGBZ0372, Open Research Program of State Key Laboratory of Millimeter Waves of Southeast University under Contract Number K201602, Natural Science Foundation of Anhui Province (1608085MF124).

REFERENCES

[1] M. S. Al Salameh and A. M. Olaimat, "Method of moments modelling of cylindrical microwave integrated circuits interconnections," *International Journal of Numerical Modelling: Electronic Networks, Devices and Fields*, vol. 17, no. 2, pp. 119-133, 2004.

[2] P. De Doncker, "A volume/surface potential formulation of the method of moments applied to electromagnetic scattering," *Engineering Analysis with Boundary Elements*, vol. 27, no. 4, pp. 325-331, 2003.

[3] C. Essid, M. B. B. Salah, and A. Samet, "Hybrid spatial-spectral MoM analysis of microstrip structures," *International Journal of Numerical Modelling: Electronic Networks, Devices and Fields*, vol. 29, no. 4, pp. 763-772, 2016.

[4] H. De Sterck, "Steepest descent preconditioning for nonlinear GMRES optimization," *Numerical Linear Algebra with Applications*, vol. 20, no. 3, pp. 453-471, 2013.

[5] M. K. Kim and I. Yun, "An efficient implementation of the generalized minimum residual algorithm with a new preconditioner for the boundary element method," *Engineering Analysis with Boundary Elements*, vol. 35, no. 11, pp. 1214-1224, 2011.

[6] Ö. Ergül, "Fast and accurate solutions of electromagnetics problems involving lossy dielectric objects with the multilevel fast multipole algorithm," *Engineering Analysis with Boundary Elements*, vol. 36, no. 3, pp. 423-432, 2012.

[7] W. C. Chew, J. M. Jin, E. Michielssen, and J. Song, *Fast Efficient Algorithms in Computational Electromagnetics*, Boston, MA: Artech House, 2001.

[8] K. Zhang and J.-M. Jin, "Parallelized multilevel fast multipole algorithm for scattering by objects with anisotropic impedance surfaces," *International Journal of Numerical Modelling: Electronic Networks, Devices and Fields*, vol. 28, no. 1 pp. 107-119, 2015.

[9] J. P. De, S. R. Gago, D. W. Kelly, O. C. Zienkiewicz, and I. Babuska, "A posteriori error analysis and adaptive processes in the finite element method: Part II—Adaptive mesh refinement," *International Journal for Numerical Methods in Engineering*, vol. 19, no. 11, pp. 1621-1656, Nov. 1983.

[10] M. Zhang, L. W. Li, and A. Y. Ma, "Analysis of scattering by a large array of waveguide-fed wide-slot millimeter wave antennas using precorrected-FFT algorithm," *IEEE Microwave and Wireless Components Letters*, vol. 15, no. 11, pp. 772-774, Nov. 2005.

[11] J. Aronsson, M. Shafieipour, and V. Okhmatovski, "Solution of large multiscale EMC problems with method of moments accelerated via low-frequency MLFMA," *Electromagnetic Compatibility (EMC), 2011 IEEE International Symposium on*, pp. 260-263, 2011.

[12] M. Bebendorf, *Hierarchical Matrices: A Means to Efficiently Solve Elliptic Boundary Value Problems*, Lecture Notes in Computational Science and Engineering, vol. 63, pp. 49-98, 2008.

[13] J. M. Rius, J. Parron, A. Heldring, J. M. Tamayo, and E. Ubeda, "Fast iterative solution of integral equations with method of moments and matrix decomposition algorithm - singular value decomposition," *IEEE Trans. Antennas Propag.*, vol. 56, no. 8, pp.2314-2324, Aug. 2008.

[14] L. Grasedyck and W. Hackbusch, "Construction and arithmetics of H-matrices," *Computing*, vol. 70, no. 4, pp. 295-344, 2003.

[15] A. Heldring, J. M. Tamayo, C. Simon, E. Ubeda, and J. M. Rius, "Sparsified adaptive cross approximation algorithm for accelerated method of moments computations," *IEEE Trans. Antennas Propag.*, vol. 61, no. 1, pp.240-246, Jan. 2013.

[16] J. M. Tamayo, A. Heldring, and J. M. Rius, "Multilevel adaptive cross approximation (MLACA)," *IEEE Transactions on Antennas and Propagation*, vol. 59, no. 12, pp. 4600-4608, Dec. 2011.

[17] D. Ding, S. Shen, Z. Jiang, and R. Chen, "The modified multilevel compressed block decomposition algorithms for analyzing the scattering of objects in half space," *International Journal of Numerical Modelling: Electronic Networks, Devices and Fields*, vol. 27, no. 1, pp. 153-163, Jan. 2014.

[18] M. Bebendorf and S. Kunis, "Recompression techniques for adaptive cross approximation," *J. Integral Equations Appl.*, vol. 21, no. 3, pp. 331-357, 2009.

[19] T. Wan and Z. Jiang, "Multilevel compressed block decomposition-based finite element domain decomposition method for the fast analysis of finite periodic structures," *International Journal of Numerical Modelling: Electronic Networks, Devices and Fields*, doi: 10.1002/jnm.2194, Sep. 2016.

[20] J.-S. Zhao and W. C. Chew, “Integral equation solution of Maxwell’s equations from zero frequency to microwave frequencies,” *IEEE Trans. Antennas Propag.*, vol. 48, no. 10, pp.1635-1645, Oct. 2000.

[21] M.-M. Li, H. Chen, Z. Jiang, and R. Chen, “Three dimensional low-frequency fast multipole method for analysis of electromagnetic scattering,” *Chinese Journal of Radio Science*, vol. 25, no. 1, pp. 127-131, Feb. 2010.