

# Non-physical Impedance Matching

William D. May  
 Raytheon/Applied Signal Technology  
 william.d.may@raytheon.com

**Abstract**—Non-physical matching (NPM) is a method for transforming the impedance of a CEM modeled device into a target impedance using algebraic methods. NPM is a useful tool for replicating existing devices in CEM models.

**Index Terms**—commutative algebra; EM modeling; Gröbner basis; impedance matching.

## I. INTRODUCTION

Although rarely mentioned in the CEM literature, it is not unusual to incorporate models of existing devices (such as antennas) into larger computational EM models. Most of the modeling challenges are the same as for developing new devices, but some are quite different.

Perhaps the most interesting difference is that the model need not resemble the actual device. It only needs to replicate the measured electrical characteristics of the original. If, for example, a simple wire dipole can be made to match the important characteristics in simulation then there is no need to develop a more accurate model.

Non-physical matching (NPM) is a method for transforming the impedance of a modeled device into a target impedance. The target might be the measured impedance of a physical device, or it might be a predetermined value, such as 50 ohms. The name comes from the fact that the method is algebraic, not physical.

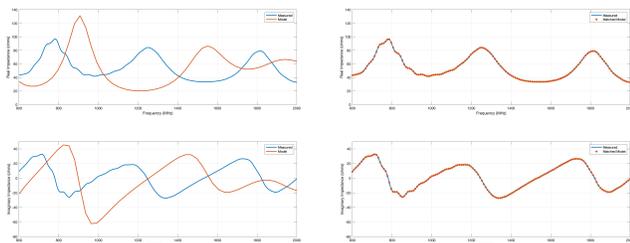


Fig. 1. Before and after of non-physical matching. The simulated impedance is clearly different from measurements in the left figure, it is indistinguishable from measurements in the right figure.

The data plotted in Fig. 1 illustrates NPM. The left-hand plots show the measured real and imaginary measured impedance for a commercial UHF antenna (plotted in blue) versus simulated impedance (plotted in red, simulations were performed using FEKO [1]). Measured and simulated values are clearly different.

The results after applying NPM are shown on the right-hand side. Simulation and measurements are now indistinguishable.

## II. THE SINGLE ANTENNA SOLUTION

The NPM solution for a single antenna is a very useful result in its own right, and the derivation is a template for developing a general solution.

The measured (target) impedance,  $Z_m$ , and simulated impedance,  $Z_s$ , are known. These are complex scalar values with strictly positive real parts.

The NPM solution will be a 2-port network (impedance matrix  $Z$ ) that transforms  $Z_s$  into  $Z_m$ . For matching to work for both transmit and receive the matrix  $Z$  must be symmetric. The network should also be lossless, so  $Z$  is pure imaginary.

The starting point is the 2-port impedance relationship [2]

$$\begin{pmatrix} V_m \\ V_s \end{pmatrix} = i \begin{pmatrix} z_{11} & z_{12} \\ z_{12} & z_{22} \end{pmatrix} \begin{pmatrix} I_m \\ -I_s \end{pmatrix},$$

where  $V_m$  and  $V_s$  are measured and simulated voltage respectively,  $I_m$  and  $I_s$  are measured and simulated current, and the  $z_{ij}$  are unknowns (real).

The NPM solution follows after using  $Z_m = V_m/I_m$  and  $Z_s = V_s/I_s$  to eliminate voltage and current, substituting  $Z_m = x + iy$ ,  $Z_s = u + iv$  (with  $x > 0$  and  $u > 0$ ), then separating into real and imaginary parts. The result is 2 equations in 3 unknowns, so it is a family of solutions with a single parameter.

If  $z_{22}$  is chosen to be the parameter, then the NPM solution is:

$$Z = i \begin{pmatrix} y + \frac{x(v+z_{22})}{u} & \sqrt{\frac{x(u^2+(v+z_{22})^2)}{u}} \\ \sqrt{\frac{x(u^2+(v+z_{22})^2)}{u}} & -v \end{pmatrix}.$$

The results shown in Fig. 1 were obtained using this solution.

It is possible for  $Z$  to be poorly conditioned, especially for electrically small antennas (where  $x$  and/or  $u$  are very close to 0). A search over the range of  $z_{22}$  will locate well-conditioned  $Z$  matrices, so this is not a problem in practice.

## III. THE GENERAL SOLUTION

The solution to the general  $k$ -antenna problem follows in a similar manner to the single antenna solution, with scalars replaced by square matrices.  $Z_m$  and  $Z_s$  are known  $k \times k$  complex matrices, with the real parts of diagonal elements strictly greater than 0. The impedance relationship is

$$\begin{pmatrix} V_m \\ V_s \end{pmatrix} = i \begin{pmatrix} Z_{11} & Z_{12} \\ Z_{12} & Z_{22} \end{pmatrix} \begin{pmatrix} I_m \\ -I_s \end{pmatrix},$$

where  $V_m, V_s, I_m, I_s$  are  $k \times 1$  complex vectors and the  $Z_{mn}$  are real  $k \times k$  matrices. Since  $Z$  is symmetric both  $Z_{11}$  and  $Z_{22}$  are symmetric.

Substituting  $V_m = Z_m I_m$ ,  $V_s = Z_s I_s$ ,  $Z_m = X + iY$ , and  $Z_s = U + iV$ , separating into real and imaginary parts, the result is:

$$\begin{aligned} X &= -Z_{12}(U^2 + (V + iZ_{22})^2)^{-1}U Z'_{12} \\ Y &= iZ_{11} + Z_{12}(U^2 + (V + Z_{22})^2)^{-1}(V + Z_{22})Z'_{12}. \end{aligned}$$

This can be simplified by letting  $Z_{11} = Y$  and  $Z_{22} = -V$  to eliminate the equation in  $Y$ :

$$X = Z_{12}U^{-1}Z'_{12}.$$

This simple expression is deceptive. Solving it is challenging, even for two antennas.

#### IV. THE TWO ANTENNA SOLUTION

The two-antenna solution begins with the general solution:

$$\Re(Z_m) = Z_{12}\Re(Z_s)^{-1}Z'_{12},$$

expanded to

$$\begin{aligned} \begin{pmatrix} x_{11} & x_{12} \\ x_{12} & x_{22} \end{pmatrix} &= \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} -u_{22} & u_{12} \\ u_{12} & -u_{11} \end{pmatrix} \begin{pmatrix} a & c \\ b & d \end{pmatrix} / K \\ &= \begin{pmatrix} a^2 u_{22} - 2abu_{12} + b^2 u_{11} & bdu_{11} - (bc + ad)u_{12} + acu_{22} \\ bdu_{11} - (bc + ad)u_{12} + acu_{22} & u_{22}c^2 - 2du_{12}c + d^2 u_{11} \end{pmatrix} / K, \end{aligned}$$

where  $K = u_{11}u_{22} - u_{12}^2$ , and  $a$ ,  $b$ ,  $c$  and  $d$  are the unknowns (real). This corresponds to a system of quadratic equations:

$$\begin{aligned} a^2 u_{22} - 2abu_{12} + b^2 u_{11} - Kx_{11} &= 0 \\ bdu_{11} - (bc + ad)u_{12} + acu_{22} - Kx_{12} &= 0 \\ c^2 u_{22} - 2cd u_{12} + d^2 u_{11} - Kx_{22} &= 0. \end{aligned}$$

With 3 equations and 4 unknowns one variable will be a parameter. By letting  $d$  be the parameter the third equation can be solved for  $c$ . The problem is to find  $a$  and  $b$  in terms of  $d$ .

Solving systems of polynomial equations falls within commutative algebra [3]. A major development in computational commutative algebra was the introduction of Gröbner bases in 1965. Gröbner bases are used to solve systems of polynomial equations in a manner similar to solving linear systems using Gaussian elimination.

The Gröbner basis for the two equations in  $a$  and  $b$  ( $c$  and  $d$  are treated as knowns) can be calculated using Mathematica [4]:

$$\begin{aligned} g_1 &= b^2 K(c^2 u_{22} - 2cd u_{12} + d^2 u_{11}) - 2bdK^2 x_{12} \\ &\quad + K(Ku_{22}x_{12}^2 - x_{11}(du_{12} - cu_{22})^2) \\ g_2 &= -acu_{22} + adu_{12} + bcu_{12} - bdu_{11} + Kx_{12} \\ g_3 &= abu_{22}(du_{11} - cu_{12}) - K(au_{22}x_{12} - cu_{22}x_{11} + du_{12}x_{11}) \\ &\quad + b^2(-cu_{11}u_{22} - 2cu_{12}^2 + du_{11}u_{12}) + 2bKu_{12}x_{12} \\ g_4 &= abcKu_{22} - Ku_{12}(au_{22}x_{12} - cu_{22}x_{11} + du_{12}x_{11}) \\ &\quad + b^2 K(du_{11} - 2cu_{12}) + bKx_{12}(2u_{12}^2 - u_{11}u_{22}) \\ g_5 &= a^2 u_{22} - 2abu_{12} + b^2 u_{11} - Kx_{11}. \end{aligned}$$

It isn't unusual for a Gröbner basis to have more elements than the polynomial system (one of the many ways polynomial systems differ from linear systems).

The solution follows by solving the first basis element for  $b$ :

$$b = \frac{dKx_{12} - \frac{1}{2}\sqrt{4(c^2 u_{22} - 2cd u_{12} + d^2 u_{11})(x_{11}(du_{12} - cu_{22})^2 - Ku_{22}x_{12}^2) + 4d^2 K^2 x_{12}^2}}{c^2 u_{22} - 2cd u_{12} + d^2 u_{11}}$$

and the fifth basis element for  $a$ :

$$a = \frac{bu_{12} - \sqrt{K(u_{22}x_{11} - b^2)}}{u_{22}}. \quad (1)$$

It is essential to know the possible values for  $d$ . This is easily determined:

$$\begin{aligned} d &= x_{12} \sqrt{\frac{u_{22}}{x_{11}}} & G < 0 \wedge x_{12} \neq 0 \\ -\sqrt{u_{22}x_{22}} \leq d &\leq \sqrt{u_{22}x_{22}} & G \geq 0 \vee x_{12} = 0, \end{aligned}$$

where  $G = x_{11}x_{22} - x_{12}^2$ .

The end result,  $Z$ , is then

$$Z = i \begin{pmatrix} y_{11} & y_{12} & a & b \\ y_{12} & y_{22} & c & d \\ a & c & v_{11} & v_{12} \\ b & d & v_{12} & v_{22} \end{pmatrix}.$$

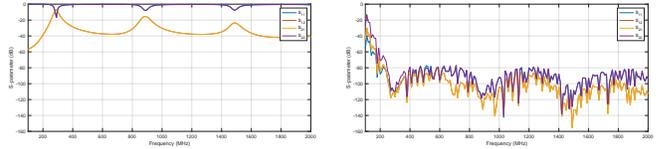


Fig. 2. Before and after of applying NPM to 2 dipole antennas. The vertical scale is 0 to -100 dB for both plots.

As an example, the plot on the left side of Fig. 2 shows the S-parameters ( $S_{11}$ ,  $S_{22}$  and  $S_{12}$  in dB) of two dipole antennas referenced to a 50-ohm impedance. The antennas perform well in narrow frequency bands around resonant frequencies, and there is coupling at all frequencies. The right-hand plot shows the S-parameters after applying NPM. The antennas now perform extremely well over a wide frequency range and coupling is negligible.

#### V. CONCLUSIONS

NPM is an easily used method to match simulated to measured impedance in CEM models with one or two antennas. It would be quite difficult to precisely match simulation to measurements, such as shown in Fig. 1, using traditional matching methods.

#### REFERENCES

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