Mathieu Functions of Complex Argument and Their Applications to the Scattering by Lossy Elliptic Cylinders

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Abstract

The aim of this paper is to outline the theory for calculating the angular and radial Mathieu functions of complex arguments. These functions are required for the computation of analytic solutions of electromagnetic scattering by lossy dielectric elliptic cylinders and waveguides. The backscattered echo width of a lossy dielectric elliptic cylinder is compared with the special case of lossy circular and weakly lossy elliptic cylinders and excellent agreement is obtained in all cases**.** Tabulated and plotted numerical results of typical Mathieu functions are presented.

1. Introduction

Many engineering and physics applications involve the solution of structures with elliptical geometries. Analytic solutions to such structures require the computation of angular and radial Mathieu functions in the elliptical coordinate system. Examples of such applications are the solution of elliptical waveguide problems [1-5], optical fibers [6], and elliptical loaded horn and corrugated elliptical horn antennas [2], field prediction inside lossy elliptic cylinders for biological body modeling [7], and scattering by mutilayered dielectric elliptic cylinders [8-12]. Furthermore, analytical solutions are generally needed to check the accuracy of numerical or approximate solutions.
Over the years many algorithms and programs have been developed to address the problem of computing Mathieu functions of integer and real arguments [4], [13-22]. However, none of these publications have addressed the more general problem of computing the angular and radial Mathieu functions of complex arguments. Caorsi et. al. have computed the solution of electromagnetic scattering by weakly lossy multilayer elliptic cylinders using first-order truncation of the Taylor expansion of each Mathieu function [23-24]. The same authors have presented the solution of the scattering of an electromagnetic wave by a lossy dielectric elliptic cylinder based on the exact Fourier series solution in terms of the Mathieu functions with complex argument [24]. Recently, Amendola outlined the theory for the computation of Mathieu functions of complex order by applying the Floquent theory [27]. Unfortunately, there are no numerical results for these functions given in the literature.
In this paper, the solution presented in [15] is extended to account for the characteristic values (eigenvalues) of Mathieu functions of complex argument. Once the characteristic values are computed, the complex Fourier coefficients of the Mathieu functions can be obtained. These coefficients are needed to compute the angular and radial Mathieu functions of complex arguments and their derivatives as will be shown.

2. Theoretical Background

The homogeneous wave equation (Helmholtz) in elliptical coordinates u, v, and z is given by

$$\frac{1}{F^2(\cosh^2 u-\cosh^2 v)}\left[\frac{\partial^2\psi}{\partial u^2}+\frac{\partial^2\psi}{\partial v^2}\right]+\left[\frac{\partial^2}{\partial z^2}+k^2\right]\psi=0 \quad (1)$$

The elliptical coordinate system (u,v,z) is defined in terms of the Cartesian coordinate system (x,y,z) by $x=F\cosh(u)\cos(v)$ and $y=F\sinh(u)\sin(v)$, where F is the semifocal length of the elliptical cross section. Assuming no variation of the function ψ along the *z-axis*, the z-derivative in (1) will drop out. The resulting differential equation can be separated into the following differential equations in terms of U(u) and V(v). The circumferential (ordinary) Mathieu's differential equation is

$$d^2V/dv^2+(a-2q\cos 2v)V=0 \quad (2)$$

and the radial (modified) Mathieu's differential equation is

$$d^2U/du^2-(a-2q\cosh 2u)U=0 \quad (3)$$

where $q=(kF)^2=q_R+jq_I$, $k=\omega\sqrt{\mu_1\varepsilon_1}$ and $\varepsilon_1=\varepsilon_1'-j\varepsilon_1''$.
It can be shown that there is a countable set of characteristic values of a, for which the solutions are π or 2π periodic.

One can easily distinguish between four kinds of characteristic values (eigenvalues) [15]:

$a = a_{2r}(q)$, even solution, period π

$a = a_{2r+1}(q)$, even solution, period 2π

$a = b_{2r+2}(q)$, odd solution, period π

$a = b_{2r+1}(q)$, odd solution, period 2π

$r = 0,1,2,...$

For every characteristic value, the solution of (2) can be given as a Fourier series:

$$Se_{2r}(v,q) = \sum_{m=0}^{\infty} A_{2m}^{2r} \cos 2mv \tag{4a}$$

$$Se_{2r+1}(v,q) = \sum_{m=0}^{\infty} A_{2m+1}^{2r+1} \cos(2m+1)v \tag{4b}$$

$$So_{2r+2}(v,q) = \sum_{m=0}^{\infty} B_{2m+2}^{2r+2} \sin(2m+2)v \tag{4c}$$

$$So_{2r+1}(v,q) = \sum_{m=0}^{\infty} B_{2m+1}^{2r+1} \sin(2m+1)v \tag{4d}$$

Where the A's and B's are complex Fourier coefficients. For computing the Fourier coefficients, one has to obtain first the characteristic values.

Substitution of the series (4a), (4b), (4c), and (4d) in the differential equation (2), yields four sets of equations for computing the Fourier coefficients:

$$\begin{bmatrix} a_{2r} & -q & 0 & & & \\ -2q & a_{2r}-4 & -q & & & \\ 0 & \ddots & \ddots & \ddots & & \\ & & \ddots & \ddots & -q & \\ & & & 0 & -q & a_{2r}-(2m)^2 \end{bmatrix} \begin{bmatrix} A_0^{2r} \\ A_2^{2r} \\ \vdots \\ \vdots \\ A_{2m}^{2r} \end{bmatrix} = 0 \tag{5a}$$

$$\begin{bmatrix} a_{2r+1}-1-q & -q & 0 & & & \\ -q & a_{2r+1}-9 & -q & & & \\ 0 & \ddots & \ddots & \ddots & & \\ & & \ddots & \ddots & -q & \\ & & & 0 & -q & a_{2r+1}-(2m+1)^2 \end{bmatrix} \begin{bmatrix} A_1^{2r+1} \\ A_3^{2r+1} \\ \vdots \\ \vdots \\ A_{2m+1}^{2r+1} \end{bmatrix} = 0 \tag{5b}$$

$$\begin{bmatrix} b_{2r+2}-4 & -q & 0 & & & \\ -q & b_{2r+2}-16 & -q & & & \\ 0 & \ddots & \ddots & \ddots & & \\ & & \ddots & \ddots & -q & \\ & & & 0 & -q & b_{2r+2}-(2m+2)^2 \end{bmatrix} \begin{bmatrix} B_2^{2r+2} \\ B_4^{2r+2} \\ \vdots \\ \vdots \\ B_{2m+2}^{2r+2} \end{bmatrix} = 0 \tag{5c}$$

$$\begin{bmatrix} b_{2r+1}-1-q & -q & 0 & & & \\ -q & b_{2r+1}-9 & -q & & & \\ 0 & \ddots & \ddots & \ddots & & \\ & & \ddots & \ddots & -q & \\ & & & 0 & -q & b_{2r+1}-(2m+2)^2 \end{bmatrix} \begin{bmatrix} B_1^{2r+1} \\ B_3^{2r+1} \\ \vdots \\ \vdots \\ B_{2m+1}^{2r+1} \end{bmatrix} = 0 \tag{5d}$$

We shall now show the computation of a_{2r+1} in (5b). All other characteristic values can be calculated in exactly the same way. It is clear from (5b) that a_{2r+1} can be seen as the eigenvalues of the infinite tridiagonal matrix,

$$\begin{bmatrix} 1+q & q & 0 & & \\ q & 9 & q & & \\ 0 & \ddots & \ddots & 0 & \\ & \ddots & \ddots & q & \\ & 0 & q & (2m+1)^2 \end{bmatrix} \tag{6}$$

In calculating the eigenvalues and Fourier coefficients, it is necessary to truncate the matrix after a finite number of columns and rows to obtain the required accuracy.

Once the coefficients A_{2m}^{2r}, A_{2m+1}^{2r+1}, B_{2m+2}^{2r+2}, B_{2m+1}^{2r+1} are determined using the routine DEVCCG from Microsoft IMSL Math Libraries, the angular Mathieu functions in equations (4) as well as their respective derivatives can be calculated. Next, we compute the radial Bessel functions of the first and second types using equation (3) along with the previously computed Fourier coefficients. The equations used for the computation of the even and odd functions of the first type are

$$J_{e2r}(u,q) = (-1)^r \sqrt{\pi/2} \sum_{m=0}^{M} (-1)^m A_{2m}^{2r} J_m(x_1) J_m(x_2) / A_0^{2r} \tag{7}$$

$$J_{e2r+1}(u,q) = (-1)^r \sqrt{\pi/2} \sum_{m=0}^{M} (-1)^m A_{2m+1}^{2r+1} [J_{m+1}(x_1) J_m(x_2) + J_m(x_1) J_{m+1}(x_2)] / A_1^{2r+1} \tag{8}$$

$$J_{o2r}(u,q) = (-1)^r \sqrt{\pi/2} \sum_{m=1}^{M} (-1)^m B_{2m}^{2r} [J_{m+1}(x_1) J_{m-1}(x_2) - J_{m+1}(x_1) J_{m-1}(x_2)] / B_2^{2r} \tag{9}$$

$$J_{o2r+1}(u,q) = (-1)^r \sqrt{\pi/2} \sum_{m=0}^{M} (-1)^m B_{2m+1}^{2r+1} [J_{m+1}(x_1) J_m(x_2) - J_m(x_1) J_{m+1}(x_2)] / B_1^{2r+1} \tag{10}$$

where $x_1 = \frac{\sqrt{q}}{2} e^{u}$, $x_2 = \frac{\sqrt{q}}{2} e^{-u}$. By replacing J_m by Y_m, we obtain the even and odd radial functions of the second type as follows

$$Y_{e2r}(u,q) = (-1)^r \sqrt{\pi/2} \sum_{m=0}^{M} (-1)^m A_{2m}^{2r} Y_m(x_1) J_m(x_2) / A_0^{2r} \tag{11}$$

$$Y_{e2r+1}(u,q) = (-1)^r \sqrt{\pi/2} \sum_{m=0}^{M} (-1)^m A_{2m+1}^{2r+1} [Y_{m+1}(x_1) J_m(x_2) + Y_m(x_1) J_{m+1}(x_2)] / A_1^{2r+1} \tag{12}$$

$$Y_{o2r}(u,q) = (-1)^r \sqrt{\pi/2} \sum_{m=1}^{M} (-1)^m B_{2m}^{2r} [Y_{m+1}(x_1) J_{m-1}(x_2) - J_{m+1}(x_1) Y_{m-1}(x_2)] / B_2^{2r} \tag{13}$$

$$Y_{o2r+1}(u,q) = (-1)^r \sqrt{\pi/2} \sum_{m=0}^{M} (-1)^m B_{2m+1}^{2r+1} [Y_{m+1}(x_1) J_m(x_2) - J_m(x_1) J_{m+1}(x_2)] / B_1^{2r+1} \tag{14}$$

Numerical results for the derivatives, with respect to q, of the even and odd angular and radial Mathieu functions are obtained using the equations given in [14-17].

3. Electromagnetic Scattering by a Lossy Dielectric Elliptic Cylinder

Consider the case of a linearly polarized electromagnetic plane wave incident on a lossy dielectric elliptic cylinder (of permeability μ , permitivity ε and conductivity σ) at an angle ϕ_i measured with respect to the x axis as shown in [25]. The electric field component of the TM polarized plane wave of amplitude E_0 is given by

$$E_z^i = E_0 e^{jk_o\rho\cos(\phi-\phi_i)} \tag{15}$$

where k_0 is the wave number in free space. The incident electric field may be expressed in terms of Mathieu functions as follows

$$E_z^i = \sum_{m=0}^{\infty} A_{em} R_{em}^{(1)}(c_0,\xi) S_{em}(c_0,\eta) + \sum_{m=1}^{\infty} A_{om} R_{om}^{(1)}(c_0,\xi) S_{om}(c_0,\eta) \tag{16}$$

$$A_{\substack{em\\om}} = E_0 j^m \frac{\sqrt{8\pi}}{N_{\substack{em\\om}}(c_0)} S_{\substack{em\\om}}(c_0,\cos\phi_i) \tag{17}$$

$$N_{\substack{em\\om}}(c) = \int_0^{2\pi} [S_{\substack{em\\om}}(c,\eta)]^2 \, dv \tag{18}$$

where $\xi = \cosh u$ and $\eta = \cos v$, $c_0 = k_0 F$, F is the semifocal length of the elliptical cross section, S_{em}, while S_{om} are the even and odd angular Mathieu functions of order *m*, respectively, $R_{em}^{(1)}$ and $R_{om}^{(1)}$ are the even and odd radial Mathieu functions of the first kind, and N_{em} and N_{om} are the even and odd normalized functions.

Electric field expansion inside and outside the lossy dielectric cylinder can be obtained by solving the Helmholtz equation in the elliptical coordinate system using the separation of variables technique. The scattered electric field outside the elliptic cylinder for ($\xi > \xi_1$) can be expressed in terms of a sum series of Mathieu and modified Mathieu functions as follows

$$E_z^s = \sum_{m=0}^{\infty} B_{em} R_{em}^{(4)}(c_0,\xi) S_{em}(c_0,\eta) + \sum_{m=1}^{\infty} B_{om} R_{om}^{(4)}(c_0,\xi) S_{om}(c_0,\eta) \tag{19}$$

where B_{em} and B_{om} are the unknown scattered field expansion coefficients, $R_{em}^{(4)}$ and $R_{om}^{(4)}$ are the even and odd modified Mathieu functions of the fourth kind. Similarly, the transmitted electric field inside the cylinder for $\xi < \xi_1$ can be written as

$$E_z^t = \sum_{m=0}^{\infty} C_{em} R_{em}^{(1)}(c_1,\xi) S_{em}(c_1,\eta) + \sum_{m=1}^{\infty} C_{om} R_{om}^{(1)}(c_1,\xi) S_{om}(c_1,\eta) \tag{20}$$

where $c_1 = k_1 F$ is complex, and $k_1 = \omega\sqrt{\mu_1\varepsilon_1}$. $\varepsilon_1 = \varepsilon_1' - j\varepsilon_1''$, C_{em} and C_{om} are the unknown transmitted field expansion coefficients. The magnetic field component inside and outside the cylinder can be obtained using Maxwell's equations, i.e.

$$H_u = \frac{-j}{\omega\mu_1 h}\frac{\partial E_z}{\partial v} \tag{21}$$

$$H_v = \frac{-j}{\omega\mu_1 h}\frac{\partial E_z}{\partial u} \tag{22}$$

$$h = F\sqrt{\cosh^2 u - \cos^2 v} \tag{23}$$

The unknown expansion coefficients can be obtained by imposing the continuity of the tangential field components at surface of the cylinder $\xi = \xi_1$, i.e.

$$\sum_{m=0}^{\infty} M_{enm}(c_1,c_0) B_{em}\left[\mu_r R_{em}^{(4)'}(c_0,\xi_1) - R_{em}^{(4)}(c_0,\xi_1)\frac{R_{en}^{(1)'}(c_1,\xi_1)}{R_{en}^{(1)}(c_1,\xi_1)}\right] = \sum_{m=0}^{\infty} M_{enm}(c_1,c_0) A_{em}\left[R_{em}^{(1)}(c_0,\xi_1)\frac{R_{en}^{(1)'}(c_1,\xi_1)}{R_{en}^{(1)}(c_1,\xi_1)} - \mu_r R_{em}^{(1)'}(c_0,\xi_1)\right] \tag{24}$$

$$\sum_{m=1}^{\infty} M_{onm}(c_1,c_0) B_{om}\left[\mu_r R_{om}^{(4)'}(c_0,\xi_1) - R_{om}^{(4)}(c_0,\xi_1)\frac{R_{on}^{(1)'}(c_1,\xi_1)}{R_{on}^{(1)}(c_1,\xi_1)}\right] = \sum_{m=0}^{\infty} M_{onm}(c_1,c_0) A_{om}\left[R_{om}^{(1)}(c_0,\xi_1)\frac{R_{on}^{(1)'}(c_1,\xi_1)}{R_{on}^{(1)}(c_1,\xi_1)} - \mu_r R_{om}^{(1)'}(c_0,\xi_1)\right] \tag{25}$$

$$M_{\substack{enm\\onm}}(c_0,c_1) = \int_0^{2\pi} S_{\substack{em\\om}}(c_1,\eta) S_{\substack{en\\on}}(c_0,\eta) dv \tag{26}$$

where μ_r is the relative permeability of the dielectric region. The TE scattering by a lossy elliptic cylinder is also presented. In this case a derivation dual to that of the TM case leads to the following solution [25]

$$\sum_{m=0}^{\infty} M_{enm}(c_1,c_0) B_{em}^{TE}\left[\varepsilon_r R_{em}^{(4)'}(c_0,\xi_1) - R_{em}^{(4)}(c_0,\xi_1)\frac{R_{en}^{(1)'}(c_1,\xi_1)}{R_{en}^{(1)}(c_1,\xi_1)}\right] = \sum_{m=0}^{\infty} M_{enm}(c_1,c_0) A_{em}\left[R_{em}^{(1)}(c_0,\xi_1)\frac{R_{en}^{(1)'}(c_1,\xi_1)}{R_{en}^{(1)}(c_1,\xi_1)} - \varepsilon_r R_{em}^{(1)'}(c_0,\xi_1)\right] \tag{27}$$

$$\sum_{m=1}^{\infty} M_{onm}(c_1,c_0) B_{om}^{TE}\left[\varepsilon_r R_{om}^{(4)'}(c_0,\xi_1) - R_{om}^{(4)}(c_0,\xi_1)\frac{R_{on}^{(1)'}(c_1,\xi_1)}{R_{on}^{(1)}(c_1,\xi_1)}\right] = \sum_{m=0}^{\infty} M_{onm}(c_1,c_0) A_{om}\left[R_{om}^{(1)}(c_0,\xi_1)\frac{R_{on}^{(1)'}(c_1,\xi_1)}{R_{on}^{(1)}(c_1,\xi_1)} - \varepsilon_r R_{om}^{(1)'}(c_0,\xi_1)\right] \tag{28}$$

The scattered near and far fields for the TM and TE cases can be calculated once the scattered fields expansion coefficients are known. The far scattered field expressions can be obtained as follows

$$E_z^s = \sqrt{\frac{j}{k_0\rho}} e^{-jk_0\rho} \sum_m j^m \left[B_{em}^{TM} S_{em}(c_0,\eta) + B_{om}^{TM} S_{om}(c_0,\eta) \right] \quad (29)$$

$$H_z^s = \sqrt{\frac{j}{k_0\rho}} e^{-jk_0\rho} \sum_m j^m \left[B_{em}^{TE} S_{em}(c_0,\eta) + B_{om}^{TE} S_{om}(c_0,\eta) \right] \quad (30)$$

Far Field data are usually expressed in terms of the scattering cross section per unit length, *i.e.*, the echo width. For the TM polarization case it is defined as

$$\sigma_{TM} = 2\pi\rho \lim_{\rho\to\infty} \frac{\left|E_z^s\right|^2}{\left|E_z^i\right|^2} \quad (31)$$

Eq. (30) can be put in simpler form for the purpose of computation as follows

$$\sqrt{\frac{\sigma_{TM}}{\lambda}} = \sum_m j^m \left[B_{em} S_{em}(c_0,\eta) + B_{om} S_{om}(c_0,\eta) \right] \quad (32)$$

4. Numerical Results

To determine the accuracy of the computer program, the Fourier expansion coefficients of Mathieu functions are computed. These results are in excellent agreement with those given in [16] for the case of real positive argument q, and the maximum difference was found to be 10^{-5}. Also**,** to verify the accuracy of the computer program for the case of complex argument q, we solved the problem of electromagnetic scattering by a lossy dielectric elliptic cylinder since no results are available in the literature. The computed numerical results were compared with the special case of scattering by a lossy circular dielectric cylinder, and with electromagnetic scattering by weakly lossy multilayer elliptic cylinder [23,26]. Fig. 1 shows the numerical results of the scattered normalized echo width σ of a lossy dielectric circular cylinder having a relative permittivity of $\varepsilon_r = 1 - j11.3$ and electrical radius $k_0 a = 3.33$ for TM and TE polarizations. These results compare very well with those presented in [26]. In this case the circular cylinder was approximated using an axial ratio close to 1, and was excited by an incident plane wave with $\phi_i = 180^o$.

Fig. 2 compares the back scattering normalized echo width $k_0\sigma(0)/4$ for a homogeneous elliptical cylinder with axial ratio $a/b = 2$ and $\varepsilon' = 4.0$. Two cases are studied; these are the lossless cylinder case with $\varepsilon'' = 0.0$ and a weak loss cylinder case where $\varepsilon'' = 0.6$. The calculated results are in very good agreement with those published in [23]. One can easily recognize the reduction in the back scattering normalized echo width due to the presence of a small loss. This can be attributed to the fact that the weak loss material absorbed part of the incident wave energy. Another clear observation is the shift in of resonance location. For a free loss material, resonance is observed around $k_0 a$ =3.6, while in the weak loss case the resonance has shifted to around $k_0 a$ =2.1. Fig. 3 illustrates the results obtained in [16] for the radial Mathieu function of the first kind and zero order with respect to u for different values of q, and presented here to validate the accuracy of the computer code. Finally, Figs. 4-13 present a collection of data for Mathieu functions (radial and angular) of complex arguments. The Fourier expansion coefficients of Mathieu functions for different arguments are tabulated in Tables 1-4. These results should be very helpful for researchers solving problems in the elliptic coordinate system.

5. Conclusions

The theory for calculating the angular and radial Mathieu functions of complex arguments has been outlined. The computed backscattered echo width of a lossy dielectric elliptic cylinder was compared with the special case of lossy circular and weakly lossy elliptic cylinders and the results are in complete agreement. Selected numerical results for Mathieu functions are plotted and tabulated for limited ranges due to limitation on space.

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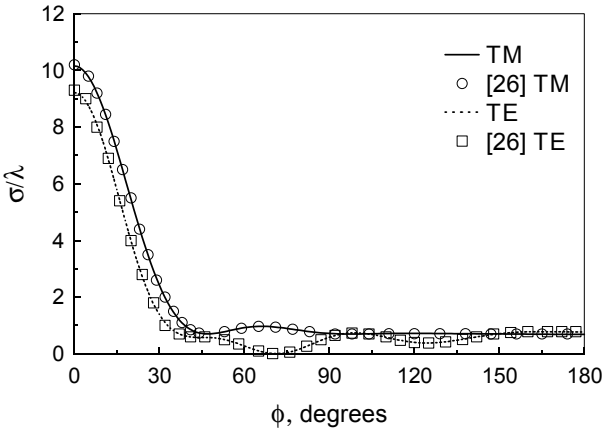

Figure 1: TM and TE Normalized backscattered echo width versus ϕ of a lossy circular cylinder with $k_0 a = 3.33$ and $\varepsilon_r = 1 - j11.3$.

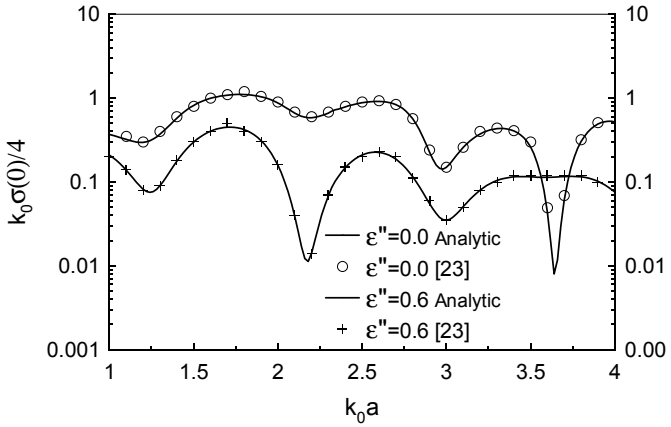

Figure 2: TM Normalized backscattered echo width versus ϕ of a lossy dieletric elliptic cylinder with $\varepsilon_r^{'} = 4$, a/b=2 and $\phi_i = 0^o$

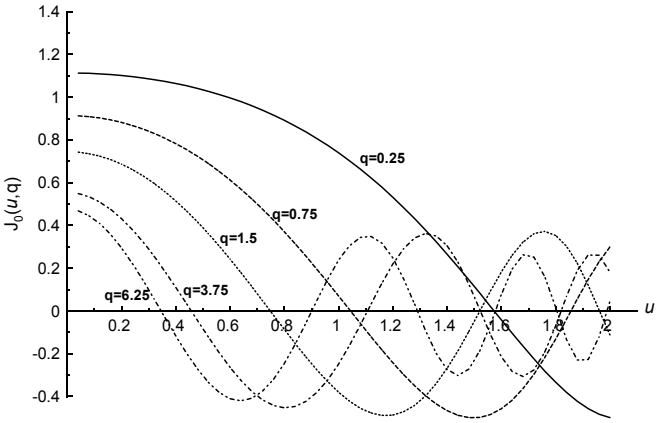

Figure 3: Radial Mathieu function of the first kind and zero order vs *u* for different values of *q*.

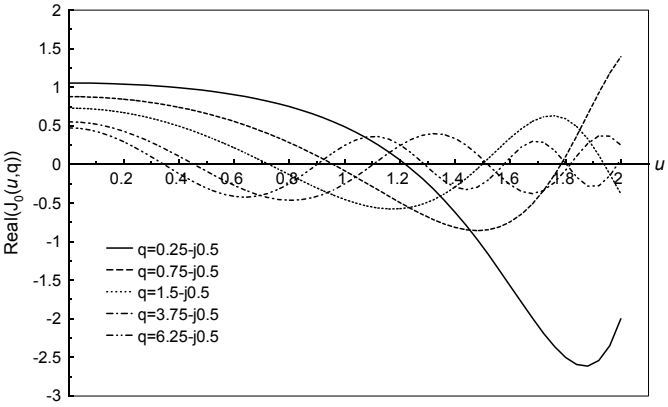

Figure 4: Real part of the radial Mathieu function of the first kind and zero order vs *u* for different values of complex *q*.

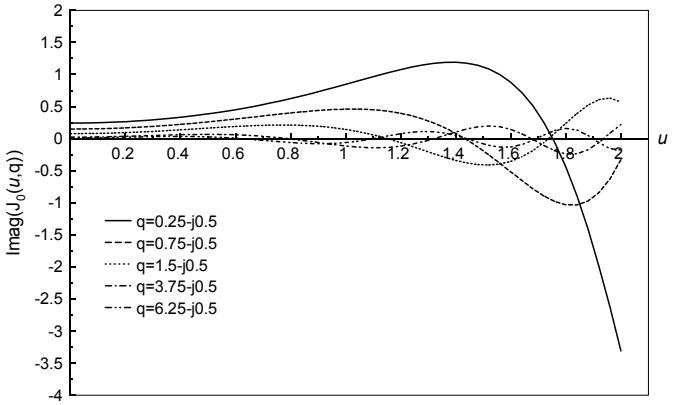

Figure 5: Imaginary part of radial Mathieu function of the first kind and zero order vs *u* for different values of *q*.

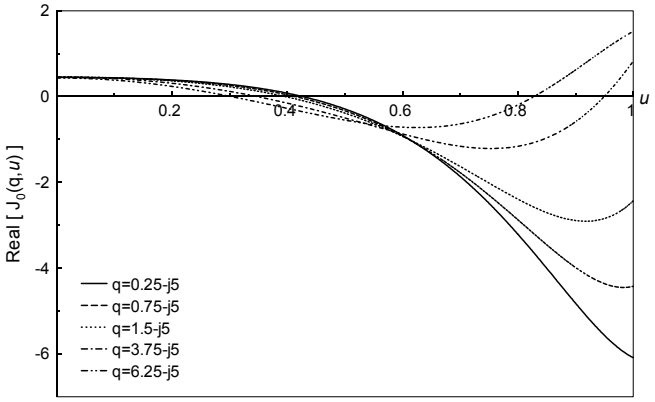

Figure 6: Real part of the radial Mathieu function of the first kind and zero order vs *u* for different values of *q*.

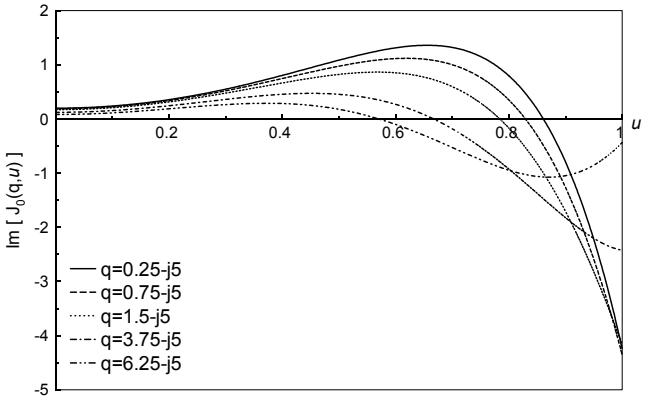

Figure 7: Imaginary part of the radial Mathieu function of the first kind and zero order vs *u* for different values of complex *q*.

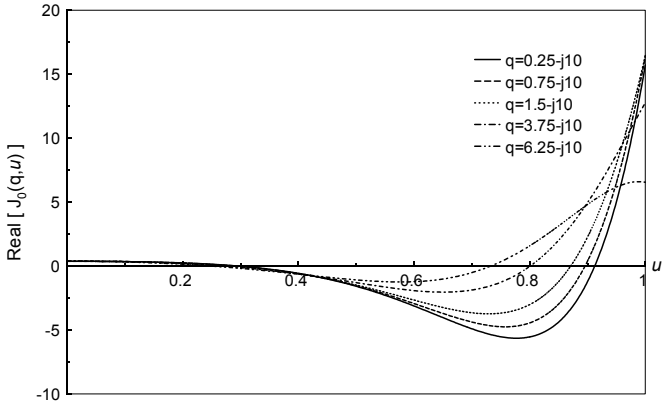

Figure 8: Real part of the radial Mathieu function of the first kind and zero order vs u for different values of q.

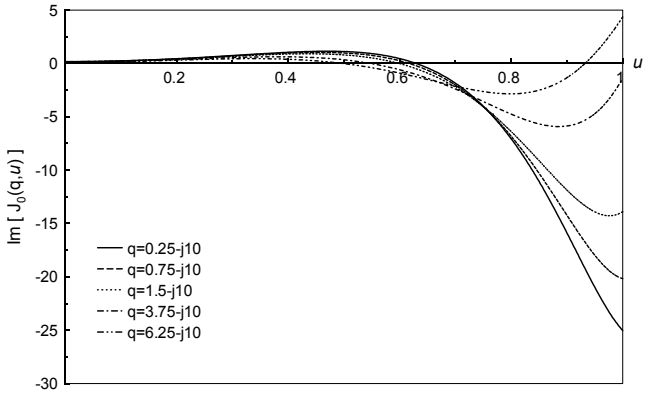

Figure 9: Imaginary part of the radial Mathieu function of the first kind and zero order vs u for different values of q.

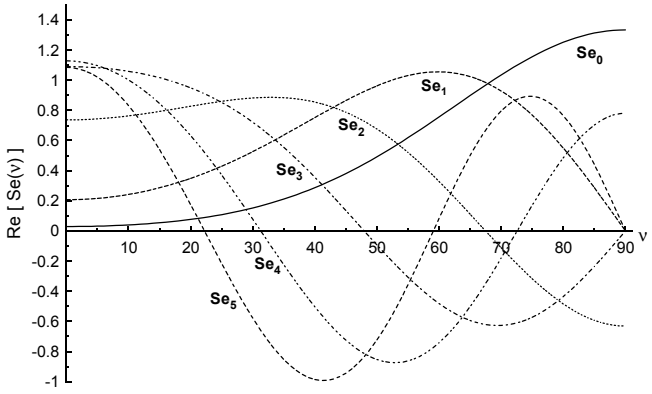

Figure 10: Real part of even periodic Mathieu function with orders 0-5 and $q = 5 - j2$ vs v.

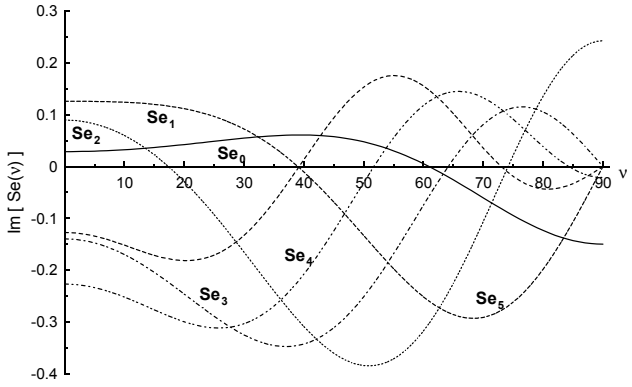

Figure 11: Imaginary part of even periodic Mathieu function with orders 0-5 and $q = 5 - j2$ vs v.

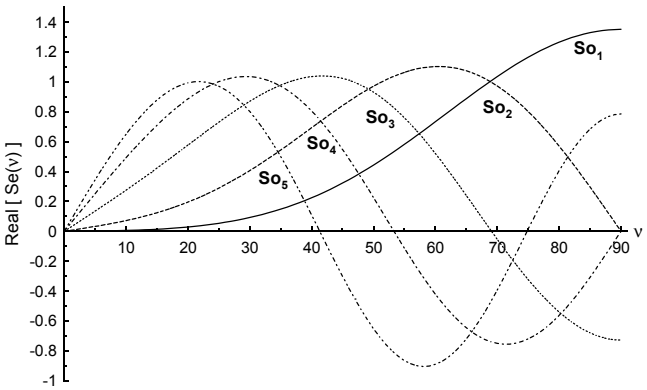

Figure 12: Real part of odd periodic Mathieu function with orders 1-5 and $q = 5 - j5$ vs v.

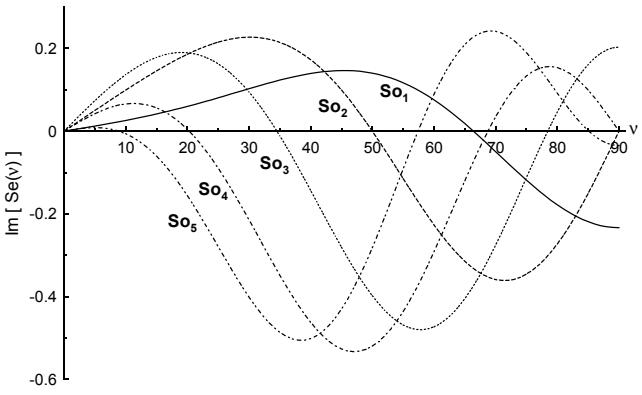

Figure 13: Imaginary part of odd periodic Mathieu function with orders 1-5 and $q = 5 - j5$ vs v.

Table 1: Complex Fourier expansion coefficients of even-even Mathieu function (A_{2m}^{2r})

A_{2m}^{2r}

	$m \setminus r$	0		2	
$q = 5 + j0$		*Real*	*Imag.*	*Real*	*Imag.*
	0	0.540612446	0	0.438737166	0
	2	-0.627115413	0	0.65364026	0
	4	0.14792709	0	-0.426578935	0
	6	-0.017848061	0	0.075885673	0
$q = 5 + j5$					
	0	0.49810151	0	0.314442843	0.171487341
	2	-0.662750285	-0.130176082	0.604942355	0
	4	0.168770876	0.133382581	0.026159729	-0.59020582
	6	-0.010394981	-0.034782531	-0.140969223	0.089859201
$q = 10 + j10$					
	0	-0.44951258	0.056260383	-0.294924019	0.029961136
	2	0.694591697	0	-0.059497099	0.255064311
	4	-0.296630326	-0.105862211	0.746844887	0
	6	0.058531831	0.064924524	-0.323904602	-0.281747705

Table 2: Complex Fourier expansion coefficients of even-odd Mathieu function (A_{2m+1}^{2r+1})

A_{2m+1}^{2r+1}

	$m \setminus r$	1		3	
$q = 5 + j0$		*Real*	*Imag.*	*Real*	*Imag.*
	1	0.762463687	0	0.077685798	0
	3	-0.63159632	0	0.30375103	0
	5	0.139684806	0	0.927728396	0
	7	-0.014915596	0	-0.201706148	0
$q = 5 + j5$					
	1	-0.517275706	0.286011205	-0.041658559	0.131675807
	3	0.765785524	0	0.239854654	0.290697818
	5	-0.197812838	-0.153451695	0.878673205	0
	7	0.007517743	0.03807304	-0.166961766	-0.194677341
$q = 10 + j10$					
	1	-0.433393417	0.130050198	0.666818493	0
	3	0.775016392	0	0.494260893	-0.299695541
	5	-0.373034841	-0.201569455	-0.01872188	-0.426395143
	7	0.054220182	0.106990887	-0.184358865	0.054869005

Table 3: Complex Fourier expansion coefficients of odd-even Mathieu function (B_{2m}^{2r})

B_{2m}^{2r}

	$m \setminus r$	2		10	
$q = 5 + j0$		*Real*	*Imag.*	*Real*	*Imag.*
	2	0.933429442	0	0.000033444	0
	4	-0.354803915	0	0.000642976	0
	6	0.052963729	0	0.010784806	0
	8	-0.004295885	0	0.13767512	0
$q = 5 + j5$					
	2	0.870294564	0	-0.000130329	-0.000002685
	4	-0.396630117	-0.272694975	-0.001280155	0.001222013
	6	0.028211436	0.099835145	-0.000421242	0.021019319
	8	0.0053059	-0.01083967	0.132565552	0.136516812
$q = 10 + j10$					
	2	0.734255554	0	-0.001900098	-0.000156527
	4	-0.580164625	-0.25100546	-0.009959144	0.008264692
	6	0.137051442	0.19916337	-0.006136282	0.076604876
	8	0.002015162	-0.052591626	0.231256474	0.260037365

Table 4: Complex Fourier expansion coefficients of odd-odd Mathieu function (B_{2m+1}^{2r+1})

B_{2m+1}^{2r+1}

	$m \setminus r$	1		3	
$q = 5 + j0$		*Real*	*Imag.*	*Real*	*Imag.*
	1	0.940019022	0	0.050382462	0
	3	-0.336541963	0	0.297365513	0
	5	0.055477529	0	0.931566996	0
	7	-0.005089553	0	-0.202193639	0
$q = 5 + j5$					
	1	0.904373155	0	0.016113831	0.092448918
	3	-0.389707951	-0.145292103	0.268912553	0.288134908
	5	0.057923929	0.074976514	0.87655541	0
	7	-0.000091975	-0.012224892	-0.16648902	-0.196783024
$q = 10 + j10$					
	1	0.852110863	0	-0.043283898	0.203330605
	3	-0.460992769	-0.18202088	0.234907973	0.510680316
	5	0.083408677	0.142179121	0.741126923	0
	7	0.009774806	-0.031241645	-0.081180316	-0.285889059

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