A Second-Order Symplectic Partitioned Runge-Kutta Scheme for Maxwell's Equations

Huang Zhi-Xiang and Wu Xian-Liang
Key Lab of Intelligent Computing & Signal Processing,
Anhui University, Ministry of Education
Hefei 230039, China

Abstract: In this paper, we construct a new scheme for approximating the solution to infinite dimensional non-separable Hamiltonian systems of Maxwell's equations using the symplectic partitioned Runge-Kutta (PRK) method. The scheme is obtained by discretizing the Maxwell's equations in the time direction based on symplectic PRK method, and then evaluating the equation in the spatial direction with a suitable finite difference approximation. The scheme preserves the symplectic structure in the time direction and shows substantial benefits in numerical computation for Hamiltonian system, especially in long-term simulations. Also several numerical examples are presented to verify the efficiency of the scheme.

I. INTRODUCTION

Symplectic schemes include a variety of different time discretization schemes designed to preserve the global symplectic structure of the phase space for a Hamiltonian system. They show substantial benefits in numerical computation for Hamiltonian system, especially in long-term simulations. Since the Maxwell's equations can be written as a system of infinite-dimensional Hamiltonian equations, the proper solution should be obtained using the symplectic schemes, which preserve the symplectic structure in the time direction. The conservation of symplecticness must be considered for solving Maxwell's equations. Recently, the symplectic schemes have been adapted in computational electromagnetic (CEM). The advantages of the symplectic schemes have been verified in [1]-[6]. These schemes are almost constructed under the assumption that the Hamiltonian system of Maxwell's equations is separable [1,4,6]. In fact, when the scattering objects are presented the corresponding Hamiltonian system is non-separable [7]. Thus the assumption limits the application of the symplectic in the area of CEM.

In this paper, we will explore the application of the symplectic scheme for non-separable Hamiltonian system of Maxwell's equations, i.e. the scattering object is presented, using a symplectic PRK scheme [7–8] for the first time. For convenience we will discuss details of the scheme only for second-order explicit method, however, the high order explicit scheme could be obtained using similar symplectic PRK scheme for infinite dimensional non-separable Hamiltonian system of Maxwell's Equations. We will also present several numerical examples to confirm the accuracy of our scheme.

II. HAMILTONION SYSTEM AND SYMPLECTICSCHEMES

Maxwell's Equations as Hamiltonian System

Within linear isotropic material, the basic equations can be written as

$$\frac{\partial \mathbf{B}}{\partial t} = -\nabla \times \mathbf{E} \tag{1}$$

$$\frac{\partial \mathbf{E}}{\partial t} = \frac{1}{\mu\varepsilon}\nabla \times \mathbf{B} - \frac{1}{\varepsilon}\mathbf{J} \tag{2}$$

where **B**, **E**, **J** and μ, ε are magnetic flux density, electric flux density, current density and permeability, permittivity, respectively. In this paper, μ and ε are assumed to be constant.

Under the Hamiltonian framework, (1) and (2) can be rewritten in a form of an infinite dimensional Hamiltonian system. By introducing two temporary variables **Y** and **A** such that

$$\mathbf{Y} = -\mathbf{E} \tag{3}$$

$$\mathbf{B} = \nabla \times \mathbf{A} \tag{4}$$

We now can write Maxwell's Equations with (3) and (4) into the following infinite dimensional Hamiltonian system

$$\frac{\partial \mathbf{A}}{\partial t} = \frac{\partial \mathrm{H}}{\partial \mathbf{Y}} \tag{5}$$

$$\frac{\partial \mathbf{Y}}{\partial t} = -\frac{\partial \mathrm{H}}{\partial \mathbf{A}} \tag{6}$$

where H is a Hamiltonian energy function given by

$$\mathrm{H}(\mathbf{A},\mathbf{Y}) = \int\left(\frac{1}{2}|\mathbf{Y}|^2 + \frac{1}{2\mu\varepsilon}|\nabla\times\mathbf{A}|^2 - \frac{1}{\varepsilon}\mathbf{J}\cdot\mathbf{A}\right)dV \tag{7}$$

For simplicity we will focus our discussion on the Maxwell's equations in two- dimensional (2-D) TM case, where **A** and **- Y** denote the z-component of the vector potential and the electric displacement, respectively [9]. Thus combined with eqn. (7), eqn. (5) and eqn. (6) can be rewritten as follows

$$\frac{\partial A_z}{\partial t} = Y_z \tag{8}$$

$$\frac{\partial Y_z}{\partial t} = \frac{\nabla^2 A_z}{\mu\varepsilon} + \frac{J_z}{\varepsilon} \tag{9}$$

field components are derived from A_z and Y_z as follows:

$$E_z = -Y_z \tag{10}$$

$$H_x = \frac{1}{\mu}\frac{\partial A_z}{\partial y} \tag{11}$$

$$H_y = -\frac{1}{\mu}\frac{\partial A_z}{\partial x}. \tag{12}$$

Symplectic Schemes for Hamiltonian System

Here we assume that all Hamiltonian systems considered are autonomous, i.e. they are time-independent. As for time-dependent the schemes are similar [6].

Often the case that J_z acts as independent sources of **E**-field, i.e. $J_z = J_{source}$, the corresponding Hamiltonian system (7) is called separable [1,7]. There exists little difficulty in solving eqn. (8) and eqn. (9) using explicit symplectic schemes [10–11].

When allowing for general cases where materials with electric losses that attenuate **E**-field, this yields:

$$J_z = J_{source} + \sigma E_z \tag{13}$$

where σ is the electric conductivity. The Hamiltonian system (7) is non-separable, how to handle this situation, to the authors knowledge, has not given rise to a thoroughly answer up to now. Fortunately, in this case we can also obtain high order explicit symplectic schemes for eqn. (8) and eqn. (9) with composite symplectic partitioned Runge-Kutta (PRK) method [7–8].

In this paper particular, we only consider the 2-stag symplectic PRK Lobatto Ⅲ A- Ⅲ B method of second-order with the temporal error of $O(dt^3)$ (see [7]). When applied in eqn. (8) and eqn. (9) with $J_z = \sigma E_z$, the scheme has the following forms,

$$A_z^1(i,j) = A_z^n(i,j) + \frac{dt}{2} Y_z^n(i,j) \tag{14}$$

$$Y_z^{n+1}(i,j) = \frac{2\varepsilon - dt\sigma}{2\varepsilon + dt\sigma} Y_z^n(i,j) + \frac{2\varepsilon dt}{\mu\varepsilon\cdot(2\varepsilon + dt\sigma)} L[A_z^1(i,j)] \tag{15}$$

$$A_z^{n+1}(i,j) = A_z^1(i,j) + \frac{dt}{2} Y_z^{n+1}(i,j) \tag{16}$$

where $A_z^n(i,j)$ and $Y_z^n(i,j)$ are respectively for the discrete value of A_z and Y_z at mesh point (i,j) and the n-th time step, $A_z^1(i,j)$ are the intermediate value, dt is the time increment, L is a difference operator approximating the ∇^2 operator and it allows flexibility during the selection of the spatial discretizations. Here we select the most commonly used central discretizations to approximate ∇^2 operator in our examples. Given appropriate absorbing boundary conditions (ABC) in computation domain, we can solve Maxwell's equations using the symplectic PRK scheme.

III. NUMERICAL RESULTS

A TEM wave propagation in one dimension

We first consider a one-dimensional TEM wave propagation problem within a finite domain [0, 2π] along the x-axis. We discretize the problem using a uniform grid with *N = 200* subintervals and choose the time step $dt = 0.1dx$. We set both μ and ε be one and take $E_Y(x,0)$ =cos$x$, $H_z(x,0)$ =cos$x$ as the initial conditions. In addition, the boundary conditions are the periodic boundary conditions. Compared with exact solution and the second-order symplectic PRK scheme (S-PRK2o), the x-axis variation of the electric flux density E_y at 10,000 and 15,000 time steps is displayed in figure 1. The electric flux density profile propagates without any changes in the profile. The results clearly show that the present scheme is pretty good for long-term simulations.

Wave propagation in two dimensions

Next we consider a two-dimensional TM case involving a sinusoidal source of frequency 30GHz. The source is generated in the middle of the problem domain. We use Mur's ABC [12] to truncate the computational domain [0,1] × [0,1]. We also discretized the problem on the domain with $N_x = N_y = 100$ grid points in each direction and with $dt = \frac{dx}{2\sqrt{\mu\varepsilon}}$. Figure 2 demonstrates a simulation for the electric flux density

E_z after the 5,000 time steps. As comparison, we simulate the problem using standard FDTD under the same conditions. The results show the efficiency of the present scheme.

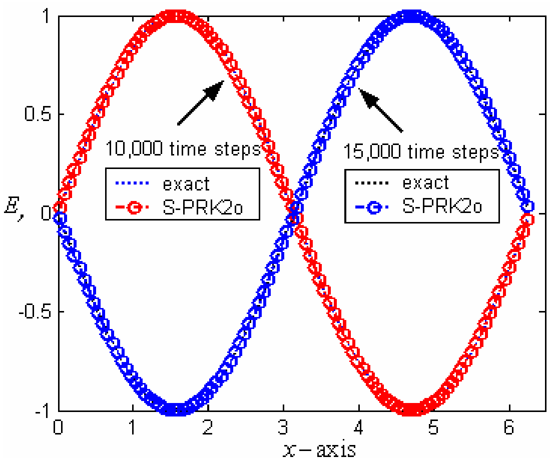
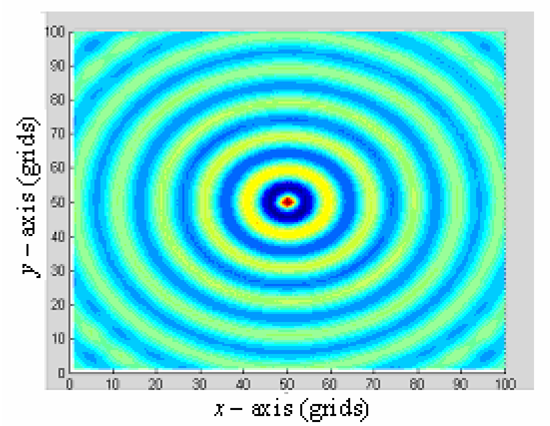
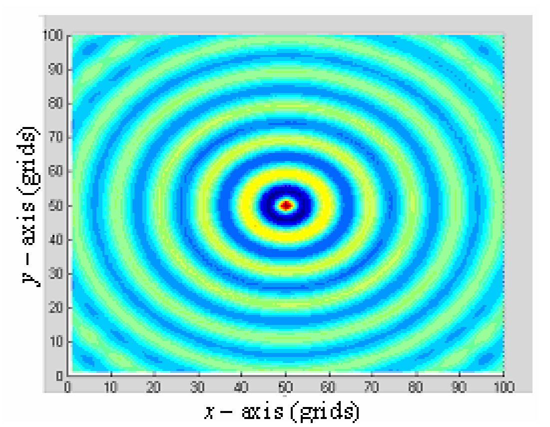


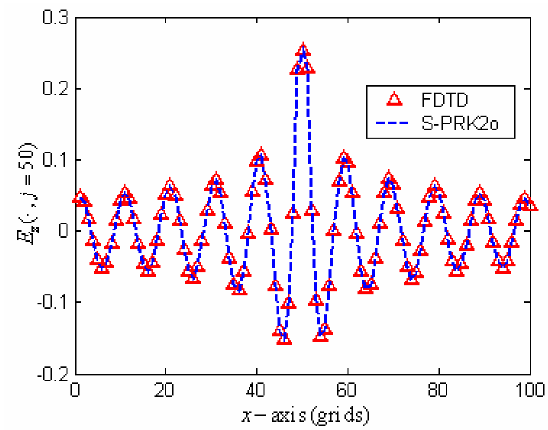
Fig. 1. Comparison of E_y calculated by S-PRK2o and exact solution after 10,000 and 15,000 time steps.



(a) Symplectic PRK scheme (S-PRK2o)



(b) Standard FDTD

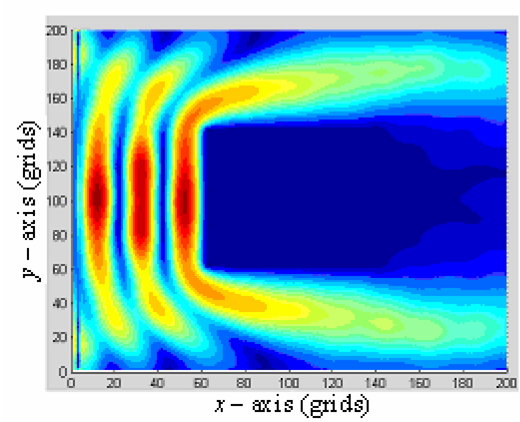


(c) The electric flux density E_z at grid j = 50

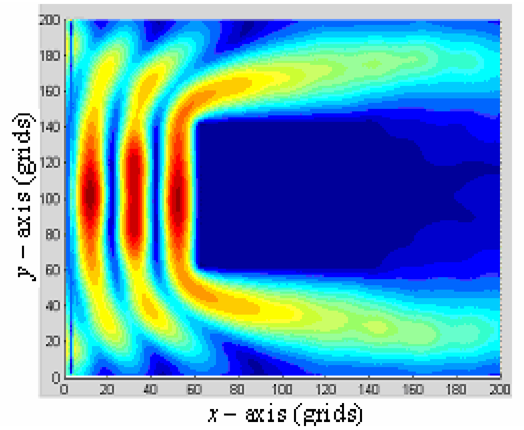
Fig. 2. The drawings of electric flux density E_z after 5,000 time steps. The source is sinusoidal source and generated in the middle of the problem domain. The absorbing boundary conditions are the Mur's ABC. (a), (b) The phase of E_z in the x-y plane. (c) The amplitude of E_z at y-grid $j = 50$.

Plane wave impinging on a infinite square cylinder

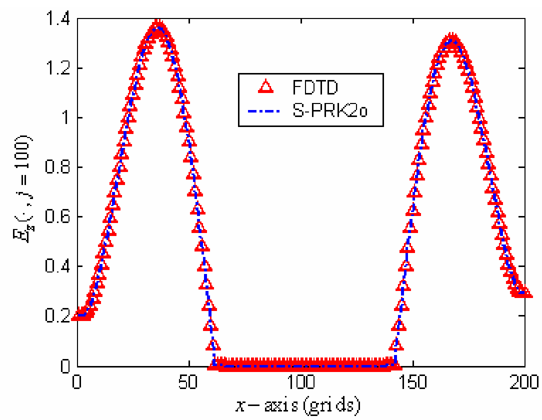
In this example, we consider the scattering of a plane wave impinging on a infinite square cylinder with side length a = 2λ (λ = 1×10^{-2} m), where λ is wavelength. The incident plane wave is a TM case and propagates from the left. We discretize the problem on the domain with $N_x = N_y = 100$ grid points in each direction and with $dx = dy = \lambda/40$, $dt = dx/2(\mu\varepsilon)^{1/2}$ independently. Here we also use Mur's ABC to truncate the computational domain. The numerical solution after 1,000 time steps using present scheme and the standard FDTD method under the same conditions are presented in Fig. 3. Figure 3 (a) and (b) demonstrate the distributions of the electric flux density E_z after the



(a) Symplectic PRK scheme (S-PRK2o)



(b) Standard FDTD (FDTD)



(c) The electric flux density E_z at grid $j = 100$

Fig. 3. The distributions of the electric flux density E_z after the 1,000 time steps. The incident plane wave is a TM case and propagates from the left. The absorbing boundary conditions are the Mur's ABC. (a), (b) The amplitude of the E_z in the x - y plane. (c) The amplitude of E_z at y- grid $j = 100.$

1,000 time steps. Figure 3 (c) shows the electric flux density E_z at point j = 100 grid. The results indicate that the performance of symplectic PRK scheme (S-PRK2o) are as at least efficient as the standard FDTD.

IV. CONCLUSION

We construct and present a symplectic PRK scheme (S-PRK2o) for the non-separable Hamiltonian system of Maxwell's Equations. The scheme is second-order explicit and has the temporal error of $O(dt^3)$. Our numerical examples demonstrate that the scheme is very effective in computing different types of wave propagations and scattering for the Maxwell's Equations.

Although the scheme is second-order explicit method, the high order explicit scheme could be obtained using similar symplectic PRK scheme for non-separable Hamiltonian system of Maxwell's Equations. Our numerical tests are running on the regular domain using square mesh, but the scheme could be adopted to compute the problem on any irregular domain.

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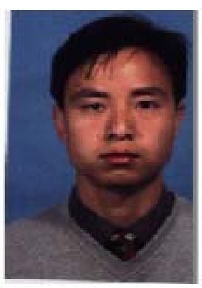

Huang Zhi-Xiang was born in Anhui, China, on December 23,1979. He is currently pursuing a Ph.D. degree in Anhui University. His research interests include high frequency method and symplectic scheme for time domain electromagnetics.

Wu Xian-Liang was born in 1958. He is currently a professor of Anhui University, doctor supervisor. His research interests include signal processing, target tracking and numerical method for electromagnetics.