New Alternating Direction Implicit Finite-Difference Time-Domain Method with Higher Efficiency

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Abstract **─** This letter presents a new unconditionally stable three-dimensional alternating direction implicit finite-difference time-domain (ADI-FDTD) method. The implicit differences of the method along different directions are irrelevant to each other, which results in a new updating equation with much simpler and more concise right-hand sides. This leads to substantial reduction in the number of arithmetic operations required for their computations. The unconditional stability of the proposed method is presented analytically, and the numerical performance of the method over the conventional ADI-FDTD method is demonstrated through numerical example.

Index Terms - ADI-FDTD scheme, CFL condition, implicit difference, unconditional stability

I. INTRODUCTION

To overcome the Courant limit on the time step size of the finite-difference time-domain (FDTD) method, unconditionally stable methods such as the alternating-direction implicit (ADI) FDTD scheme have been studied extensively [1-8]. This method has been demonstrated to be useful for the problems where fine scale structures are involved. However, from the implementation point of view, the ADI-FDTD algorithm is rather complicated. This is because not only there are tridiagonal systems that need to be dealt with, but the right-hand sides of their updating equations also contain numerous terms that call for considerable arithmetic operations.

In this letter, a new unconditionally stable three-dimensional ADI-FDTD method is presented. The implicit differences of the method along different directions are irrelevant to each other, which results in a new updating equation. In this method, four time steps (n, n+1/3, n+2/3 and n+1) are used for defining the field components and three sub-iterations are required for field advancement from n to n+1. It must solve six tridiagonal matrices and six explicit updates for one full update cycle, which is same as the conventional ADI-FDTD method. However, for that the new algorithm is with updating equations whose right-hand sides are much simpler and more concise than those in the conventional implementation, the number of arithmetic operations required is reduced substantially. Thus, the new ADI-FDTD method is with higher computational efficiency than the conventional one. The formulations of the new ADI-FDTD method are given, and the unconditional stability of the method is presented analytically. The numerical performance of the new method over the conventional ADI-FDTD method is demonstrated through numerical example.

II. FORMULATION

In the new ADI-FDTD method, the calculation for one discrete time step is performed using three procedures. The electromagnetic field components are arranged on the Yee's cells in the same way as that using the conventional FDTD method. The numerical formulations of the new ADI-FDTD method for a full three-dimensional wave are presented in (1)-(3).

A) First procedure from n to $n+1/3$

$$E_x^{n+1/3} = E_x^n - a D_z \left(H_y^n + H_y^{n+1/3} \right) \tag{1.1}$$

$$H_y^{n+1/3} = H_y^n - b D_z \left(E_x^n + E_x^{n+1/3} \right) \tag{1.2}$$

$$E_y^{n+1/3} = E_y^n + a D_z \left(H_x^n + H_x^{n+1/3} \right) \tag{1.3}$$

$$H_x^{n+1/3} = H_x^n + b D_z \left(E_y^n + E_y^{n+1/3} \right) \tag{1.4}$$

$$E_z^{n+1/3} = E_z^n \tag{1.5}$$

$$H_z^{n+1/3} = H_z^n \tag{1.6}$$

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B).Second procedure from $n+1/3$ *to* $n+2/3$

$$E_y^{n+2/3} = E_y^{n+1/3} - a D_x \left(H_z^{n+1/3} + H_z^{n+2/3} \right) \quad (2.1)$$

$$H_z^{n+2/3} = H_z^{n+1/3} - b D_x \left(E_y^{n+1/3} + E_y^{n+2/3} \right) \quad (2.2)$$

$$E_z^{n+2/3} = E_z^{n+1/3} + a D_x \left(H_y^{n+1/3} + H_y^{n+2/3} \right) \quad (2.3)$$

$$H_y^{n+2/3} = H_y^{n+1/3} + b D_x \left(E_z^{n+1/3} + E_z^{n+2/3} \right) \quad (2.4)$$

$$E_x^{n+2/3} = E_x^{n+1/3} \quad (2.5)$$

$$H_x^{n+2/3} = H_x^{n+1/3} \quad (2.6)$$

C).Third procedure from $n+2/3$ *to* $n+1$

$$E_z^{n+1} = E_z^{n+2/3} - a D_y \left(H_x^{n+1} + H_x^{n+2/3} \right) \quad (3.1)$$

$$H_x^{n+1} = H_x^{n+2/3} - b D_y \left(E_z^{n+1} + E_z^{n+2/3} \right) \quad (3.2)$$

$$E_x^{n+1} = E_x^{n+2/3} + a D_y \left(H_z^{n+1} + H_z^{n+2/3} \right) \quad (3.3)$$

$$H_z^{n+1} = H_z^{n+2/3} + b D_y \left(E_x^{n+1} + E_x^{n+2/3} \right) \quad (3.4)$$

$$E_y^{n+1} = E_y^{n+2/3} \quad (3.5)$$

$$H_y^{n+1} = H_y^{n+2/3} \quad (3.6)$$

where, $a = \Delta t/2\varepsilon$, $b = \Delta t/2\mu$; $D_w = \partial/\partial w$ ($w = x, y, z$) represents the first derivative with respect to w; n and Δt are the index and size of time-step; ε and μ are the permittivity and permeability of the surrounding media, respectively.

It can be seen from eqs. (1.1)-(1.6) that, only the implicit difference along the z-directions is applied in the first procedure.. Updating of $E_x^{n+1/3}$ component, as shown in eq. (1.1), needs the unknown $H_y^{n+1/3}$ component at the same time; thus the $E_x^{n+1/3}$ component has to be updated implicitly. Substituting eq. (1.2) into eq. (1.1), the equation for $E_x^{n+1/3}$ field is given as,

$$\begin{aligned}
&(1 + 2S_1) E_x^{n+1/3}(i + 1/2, j, k) \\
&-S_1 \begin{bmatrix} E_x^{n+1/3}(i + 1/2, j, k - 1) \\ +E_x^{n+1/3}(i + 1/2, j, k + 1) \end{bmatrix} \\
&= (1 - 2S_1) E_x^{n}(i + 1/2, j, k) \\
&+S_1 \begin{bmatrix} E_x^{n}(i + 1/2, j, k - 1) \\ +E_x^{n}(i + 1/2, j, k + 1) \end{bmatrix} \\
&-S_2 \begin{bmatrix} H_y^{n}(i + 1/2, j, k + 1/2) \\ -H_y^{n}(i + 1/2, j, k - 1/2) \end{bmatrix}
\end{aligned} \quad (4)$$

with, $S_1 = \Delta t^2 / (4\varepsilon\mu\Delta z^2)$, $S_2 = \Delta t / (\varepsilon \Delta z)$.

The updating equation for the electronic field component $E_y^{n+1/3}$ can be written down similarly as eq.(4), then, the magnetic field components $H_y^{n+1/3}$ and $H_x^{n+1/3}$ are explicitly updated straightforward.

In the second and third procedure, the implicit difference along the x and y directions are applied respectively. The flow chart of the new ADI-FDTD method is shown in Fig. 1. It is obvious that, the implicit differences of proposed method along different directions are irrelevant to each other. Thus, at each time step, the new ADI-FDTD method requires the solution of six tridiagonal

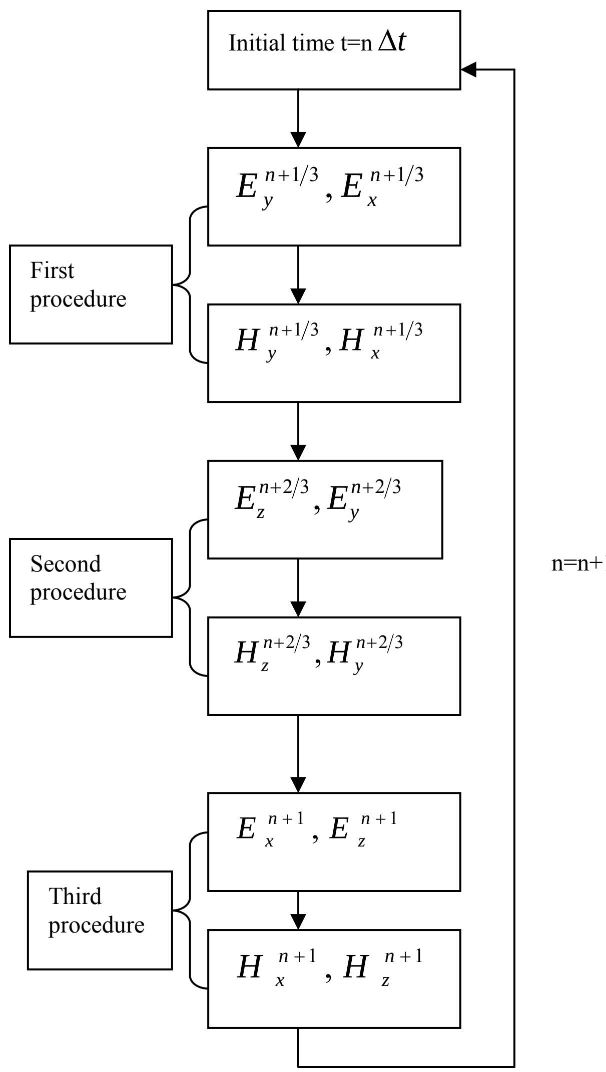

Fig. 1. The flow chart of the new ADI-FDTD method.

matrices and six explicit updates, which is same as the conventional ADI-FDTD method. However, the right side of the updating equation of the new ADI-FDTD method takes much simpler and more concise form compared to the conventional one.

Table 1: Flops count for 3-D ADI-FDTD algorithms

Scheme	Algorithm	Implicit	Explicit	Total
ADI-FDTD	M/D	18	12	102
	A/S	48	24	
New ADI-FDTD	M/D	18	6	72
	A/S	24	24	

To clarify this point further, we recall the updating equation of the conventional ADI-FDTD method. The implicit updating for the E_x component of the conventional ADI-FDTD method is as follows,

$$\begin{aligned}
&(1+2S_1)E_x^{n+1/2}(i+1/2,j,k)\\
&-S_1\begin{bmatrix} E_x^{n+1/2}(i+1/2,j,k-1)\\ +E_x^{n+1/2}(i+1/2,j,k+1)\end{bmatrix}\\
&=E_x^n(i+1/2,j,k)\\
&-S_3\begin{bmatrix} E_z^n(i+1,j,k+1/2)-E_z^n(i,j,k+1/2)\\ -E_z^n(i+1,j,k-1/2)+E_z^n(i,j,k-1/2)\end{bmatrix}\\
&+S_4\begin{bmatrix} H_z^n(i+1/2,j+1/2,k)\\ -H_z^n(i+1/2,j-1/2,k)\end{bmatrix}\\
&-S_5\begin{bmatrix} H_y^n(i+1/2,j,k+1/2)\\ -H_y^n(i+1/2,j,k-1/2)\end{bmatrix}
\end{aligned} \tag{5}$$

$S_3=\Delta t^2/(4\varepsilon\mu\Delta x\Delta z)$, $S_4=\Delta t/(2\varepsilon\Delta y)$, $S_5=\Delta t/(2\varepsilon\Delta z)$

To verify the new algorithm, it is desirable to provide detailed assessment and comparison with regard to the computation efficiency of these two methods. The floating point operations (flops) counts taking into account the number of multiplications/divisions (M/D) and additions/subtractions (A/S) required for one complete time step for both conventional and present algorithms are listed in Table 1, based on the right-hand sides of their respective updating equations such as (4) and (5), etc. For simplicity, the number of electric and magnetic field components in all directions has been taken to be the same and assume that all multiplicative factors have been precomputed and stored. From the table, it is clear that the total flops count (M/D +A/S) has been reduced substantially from 102 to 72 in the present implementation. This corresponds to an efficiency gain of 1.42 in flops count reduction for the right-hand sides of updating equations.

III. STABILITY

In this section, the unconditional stability of the new ADI-FDTD method is derived by following a similar procedure described in [9].

The relations between the field components of eq.(1) in the new ADI-FDTD method can be represented in matrix form as,

$$[A]U^{n+1/3}=[B]U^n \tag{6}$$

with,

$$[A]=\begin{bmatrix} 1&0&0&0&aD_z&0\\ 0&1&0&-aD_z&0&0\\ 0&0&1&0&0&0\\ 0&-bD_z&0&1&0&0\\ bD_z&0&0&0&1&0\\ 0&0&0&0&0&1\end{bmatrix}$$

$$[B]=\begin{bmatrix} 1&0&0&0&-aD_z&0\\ 0&1&0&aD_z&0&0\\ 0&0&1&0&0&0\\ 0&bD_z&0&1&0&0\\ -bD_z&0&0&0&1&0\\ 0&0&0&0&0&1\end{bmatrix}$$

For simplicity, we define,

$$\begin{aligned} U^{n+1}&=\zeta_y U^{n+\frac{2}{3}}=\zeta_y\left[\zeta_x U^{n+\frac{1}{3}}\right]\\ &=\zeta_y\left[\zeta_x\left(\zeta_z U^n\right)\right]=\zeta U^n\end{aligned} \tag{7}$$

here, ζ_x, ζ_y and ζ_z denote the growth factor along the x, y and z directions respectively; ζ is the total growth factor from the time step n to $n+1$. In such a case, we have,

$$\left([A]\zeta_z-[B]\right)U^n=0 \tag{8}$$

For a nontrivial solution of eq. (8), the determinant of the coefficient matrix in eq. (8) should be zero, namely,

$$\left| [A]\zeta_z - [B] \right| = 0 \tag{9}$$

By solving eq. (9), the growth factor ξ_z is obtained:

$$\zeta_{z\,1,2} = 1 \tag{10}$$

$$\zeta_{z\,3,4} = \zeta_{z\,5,6} = \frac{1 + abD_z^2 \pm \sqrt{4abD_z^2}}{1 - abD_z^2} \tag{11}$$

Approximating each derivative in space by centered second-order finite differences, it obtains [9],

$$D_z = \frac{2\hat{j}\,\sin\left(\dfrac{k_z \Delta z}{2}\right)}{\Delta z} \tag{12}$$

Then, we have,

$$\left|\zeta_z\right| = 1 \tag{13}$$

With same manipulation, we can get,

$$\left|\zeta_x\right| = \left|\zeta_y\right| = 1 \tag{14}$$

Thus, equation $|\zeta| = |\zeta_y||\zeta_x||\zeta_z| = 1$ is unconditionally satisfied. It means that the new ADI-FDTD scheme is unconditionally stable.

IV. NUMERICAL VALIDATION

To demonstrate the computational efficiency of the proposed ADI-FDTD method, a simple numerical example is presented here. An air-filled cavity with dimensions $50\text{cm} \times 50\text{cm} \times 50\text{cm}$, is excited by a current source J_z with Gaussian pulse waveform $\left[\exp\left(-4\pi(t-t_0)^2/t_1^2\right)\right]$, $t_0 = t_1 = 1ns$. The cell size is chosen as $\Delta x = \Delta y = \Delta z = 1\text{cm}$, so that the computational domain is $50 \times 50 \times 50$ cells. The source and observation point is set at the cells (5,5,5) and (45,45,45), respectively. Figure 2 plots the time-domain E_z component recorded at the observation point. The simulations have been carried out using Yee's FDTD method, conventional ADI-FDTD method and new ADI-FDTD method, with same time-step size $\Delta t_{CFL} = 19.2\text{ps}$, which is the maximum time-step size to satisfy the stability condition of the FDTD algorithm. It can be seen from this figure that, both the result of the conventional ADI-FDTD method and the new ADI-FDTD method agree well with that of the FDTD method, which demonstrates that the new ADI-FDTD method is with same accuracy as that of the conventional one.

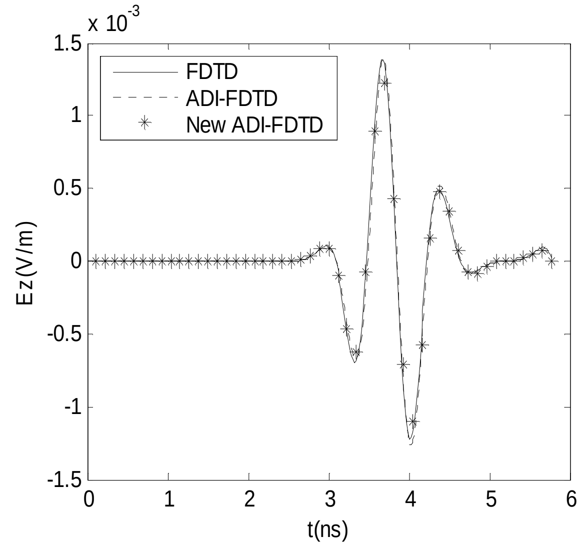

Fig. 2. Time-Domain electric field recorded at the observation point calculated by the FDTD, conventional ADI-FDTD and new ADI-FDTD methods.

The computation efficiency gain of these two ADI-FDTD methods with various $CFLN = \Delta t / \Delta t_{CFL}$ is plotted in Fig. 3. T_0 is the simulation time of the standard FDTD method,

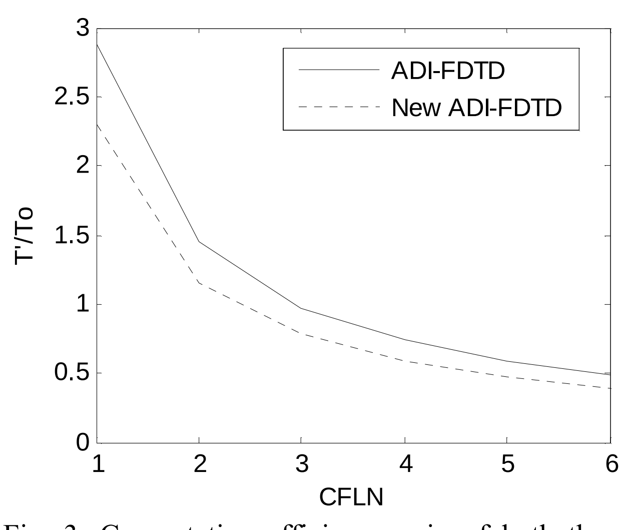

Fig. 3. Computation efficiency gain of both the conventional and present ADI-FDTD methods.

and T' denotes the computational time of the two ADI-FDTD methods respectively. It can be seen from Fig. 3 that, the simulation times for both the conventional and present ADI-FDTD method are reduced by increasing the time step size, and the computation efficiency of the new ADI-FDTD method is higher than that of conventional one obviously.

V. CONCLUSION

This letter has presented a new efficient algorithm for the unconditionally stable ADI-FDTD method. The algorithm applies the implicit difference along different directions independently, which results in updating equations whose right-hand sides are much simpler and more concise than those in the conventional implementation. This leads to substantial reduction in the number of arithmetic operations required for their computations. Compared with the conventional ADI-FDTD method, the new ADI-FDTD method is with same accuracy and higher computational efficiency, which is demonstrated by numerical example.

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