A Redundant Loop Basis for Closed Structures with Application to MR Basis

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Abstract **─** A redundant loop basis is proposed and applied as the solenoidal part of the recently developed multiresolution (MR) basis for closed surfaces at low frequencies. By keeping all loop basis functions, the "symmetry" of the MR solenoidal basis can be maintained for closed surfaces. As a consequence, the convergence of iterative solvers for the expanded MR basis can be effectively improved by using the redundant loop basis without disturbing the accuracy of results. Since the expanded MR basis functions are linear combinations of standard Rao-Wilton-Glisson (RWG) functions, it can be applied to the existing MoM codes easily. The positive behavior of redundant loop basis on MR basis for closed surfaces is analyzed and discussed in detail in this paper. Numerical results demonstrate that the expanded MR basis performs better than the original MR basis and has significant advantages over the traditional loop-tree basis for 3D electromagnetic scattering of closed structures in the low frequency range.

Index Terms **─** Electromagnetic scattering, low frequency, method of moments (MoM), multiresolution techniques.

I. INTRODUCTION

The method of moments (MoM) solution of the electric field integral equation (EFIE) is always preferred for analysis of 3D electromagnetic scattering problems [1]. However, the EFIE suffers from the low-frequency breakdown problem when using the well known Rao - Wilton–Glisson (RWG) basis [2], which is associated with the poorly-conditioned MoM matrix when the frequency tends to zero. As a consequence, the MoM matrix is hard to get convergence and even not solvable with iterative solvers. The solution to prevent the low-frequency breakdown problem is to extract the solenoidal part of the current [3-10], thus the loop-tree/star basis is proposed for this purpose. The multiresolution (MR) basis developed in recent years provides a more effective basis than the loop-tree/star basis [11-14]. More recently, an alternative MR basis was proposed for analysis of low-frequency problems [15, 16]. Compared with the MR basis in [11-14], the MR basis in [15, 16] can be constructed much easier and provides more direct physical meanings.

A simple and direct way of generating the solenoidal MR basis is taking the loop basis in the loop-tree/star basis as the solenoidal MR basis [12]. In the loop basis, each loop basis function corresponds to an interior vertex of surfaces. However, the loop basis is "asymmetry" for closed surfaces, since one loop associated to one arbitrary vertex must be eliminated. As pointed in [14], this "asymmetry" of loop basis for closed surfaces will cause the bad conditioning of the corresponding MoM matrix. To remedy the shortcoming of the loop basis, a novel, symmetrical solenoidal basis was proposed in [14]. The proposed solenoidal basis is generated by applying the singular value decomposition (SVD) to local small "charge" matrices and it generates a well-conditioned MoM matrix. Alternatively, we propose a simple way to improve the conditioning of the loop basis for closed surfaces, i.e. taking a redundant loop basis as the solenoidal part of the MR basis. By keeping all loop basis functions, the "symmetry" of loop basis is kept. Also, as pointed in [18] that the redundant loop basis gives more freedom for the solution of flux to converge. Therefore, the application of the redundant loop basis to MR basis gives better convergence.

II. MULTIRESOLUTION BASIS

Due to the fact that MR basis possesses some degree of Fourier spectral resolution, the condition number of the corresponding MoM matrix can be drastically reduced with a diagonal preconditioning [17]. Therefore, MR basis has significant advantage than the classical loop-tree/star basis for analysis of low-frequency problems. Since the improvement of the MR basis in [15, 16] for closed surfaces is mainly concerned in this paper, the essential concepts of the MR basis are briefly described in this section.

A. Generalized mesh and generalized RWG basis

As proposed in [12], the generalized mesh and generalized RWG (gRWG) basis is the two basic concepts in generating the MR basis. The MR basis functions are constructed on the hierarchical generalized meshes and generated as the linear combinations of the gRWG basis functions. The hierarchical generalized meshes are generated via a grouping algorithm (e.g. a sophisticated algorithm in [13]) and starts from the level-0 mesh, i.e. the input triangular mesh. In the subsequent procedure of the grouping algorithm, each cell of level-l ($l\geq1$) mesh is constructed by grouping about four near cells of level-(l-1) mesh. Finally, the mesh of the highest level is decided by the size of its cells, i.e. the size of the cells should be smaller than the wavelength of the incident EM wave. An example of four levels of hierarchical generalized meshes generated by the grouping algorithm is depicted in Fig. 1. The gRWG basis functions of each level are defined on the mesh of the corresponding level. Similar to the RWG basis function, each gRWG basis function of level-l mesh is defined on a pair of cells of level-l. Denoting a gRWG basis function of level-l with $\vec{R}_i^l(\vec{r})$, the divergence of the gRWG basis function is given as

$$\nabla_s \cdot \vec{R}_i^l(\vec{r}) = \begin{cases} \ell_i^l / A_{+,i}^l & \vec{r} \in C_{+,i}^l \\ -\ell_i^l / A_{-,i}^l & \vec{r} \in C_{-,i}^l \\ 0 & \text{otherwise,} \end{cases} \tag{1}$$

where $A_{+,i}^l$ and $A_{-,i}^l$ are the area of the two adjacent cells $C_{+,i}^l$ and $C_{-,i}^l$, and ℓ_i^l is the common side of the two cells.

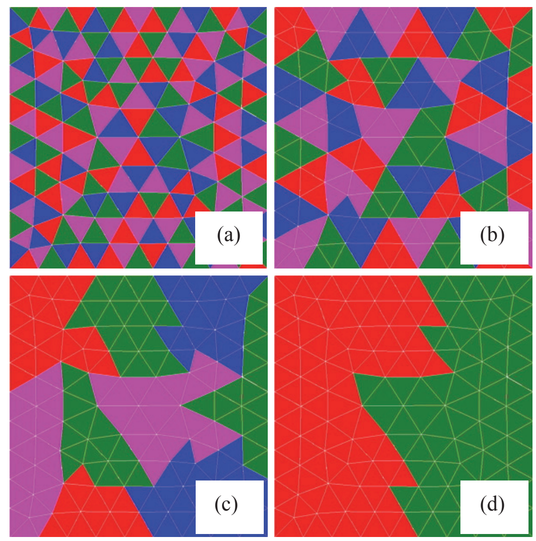

Fig. 1. An example of four levels hierarchical generalized meshes. (a) Level-0 mesh, (b) level-1 mesh, (c) level-2 mesh, (d) level-3 mesh.

B. Generation of MR basis

The MR basis is split into the solenoidal and nonsolenoidal parts. The solenoidal and nonsolenoidal functions of the MR basis span the same space as for the loop-tree/star basis. For a general 3-D surface, the number of solenoidal functions N_S and the number of nonsolenoidal functions N_X of the MR basis are given by [12, 16]

$$N_S = V_{\text{int}} + N_\Gamma - 1, \tag{2}$$

$$N_X = F - 1, \tag{3}$$

where V_{int}, N_Γ, F denote the number of internal vertices, separated boundary contours, and triangular faces, respectively. Obviously, the number of the solenoidal functions equals the number of the vertices minus one for closed surfaces.

1). Solenoidal Basis

A simple way of constructing the solenoidal part of the MR basis is taking the traditional loop basis as the solenoidal basis, since a non-hierarchical loop basis suffices to obtain well conditioned MoM matrices for low-frequency problems [12]. However, one loop basis function should be eliminated for closed surfaces according to (2).

2). Nonsolenoidal Basis

The nonsolenoidal part of MR basis is a hierarchical basis constructed on the hierarchical meshes. The nonsolenoidal basis first proposed in [12] is constructed via a rank-revealing QR decomposition. However, this approach requires the cells of the structure to be finally grouped into a single big cell. As a consequence, the regularizing property of the MR basis may be destroyed. To remedy this problem, a different approach is proposed in [13, 14], where the nonsolenoidal basis is constructed using SVD on small charge matrices and the cells of the highest level are smaller than the working wavelength. Nevertheless, the above approaches in generating of the MR basis rely on the indirect mathematical operations. Alternatively, a nonsolenoidal basis can be much easier to construct and comprehend in theory as proposed in [15, 16].

The generation of the nonsolenoidal functions of the highest level (level-L) is different from the functions of the other levels. Similar to the generation of the tree basis in the classical loop-tree basis, the nonsolenoidal functions of level-L are generated by connecting all the cells of level-L mesh in a tree. An example of the level-L nonsolenoidal functions is shown in Fig. 2 (a), in which the nonsolenoidal functions are depicted with bold black lines. The nonsolenoidal functions of level-l ($l<L$) are the union of the functions constructed on the trees in all the level-(l+1) cells. An example of the nonsolenoidal functions of level-l is shown in Fig. 2 (b).

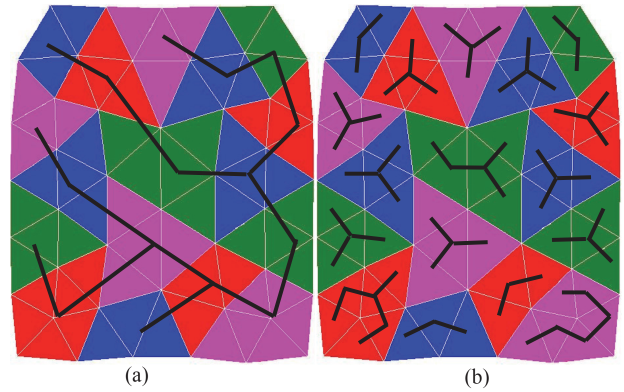

Fig. 2. Examples of the nonsolenoidal functions. (a) the level-L nonsolenoidal functions, (b) the level-l ($l<L$) nonsolenoidal functions.

III. REDUNDANT LOOP BASIS

Although the MR basis in [12, 15] is very effective for analysis of low-frequency problems, the "asymmetry" is still remained in the solenoidal part of the MR basis for closed surfaces. To maintain the "symmetry" of the solenoidal MR basis, all loop basis functions are proposed to be kept in the solenoidal MR basis, i.e. no solenoidal MR basis function associated to the interior vertex of closed surfaces need to be eliminated. As a result, the conditioning of the MoM matrix of the expanded MR basis can be improved. The redundant loop basis' property and its application to MR basis are investigated in this section.

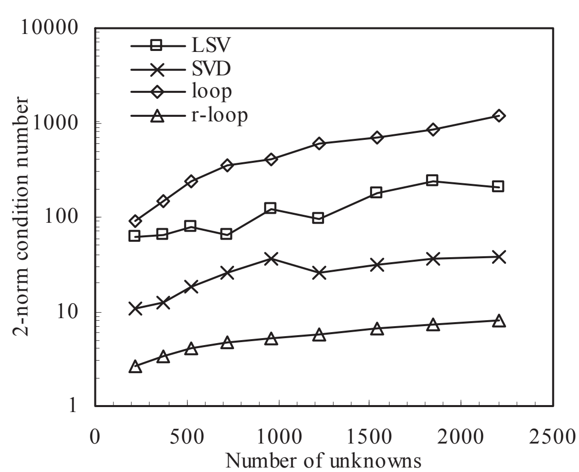

Fig. 3. The 2-norm condition number of MoM submatrices of a metallic sphere (radius = 1 m) discretized with different number of unknowns using different solenoidal bases at 1 MHz.

A. Property of redundant loop basis

Firstly, the 2-norm condition number of MoM solenoidal submatrix of different solenoidal bases with different discretization density is analyzed. As an example, the 2-norm condition number of MoM submatrices (after diagonal preconditioning) of a metallic sphere with a radius of 1 m discretized with the number of unknowns from about 200 to 2200 at the frequency of 1MHz is shown in Fig. 3. The LSV and SVD depicted in Fig. 3 represent the solenoidal bases generated from local SVD and SVD operations on charge matrices, respectively [14]. The 2-norm condition number of the redundant loop basis (denoted with r-loop in Fig. 3) is stated as the ratio of the largest singular value to the smallest nonzero singular value here. It can be found from Fig. 3 that the 2-norm condition number increases very fast as the number of unknowns increases for loop basis, while it performs more stable for the other solenoidal bases. The same phenomenon of the above solenoidal bases (except the redundant loop basis) is reported in [14], where the worse behavior of loop basis is explained. It is very interesting to be observed from Fig. 3 that the 2-norm condition number of the redundant loop basis is even smaller than that of SVD solenoidal basis. It is reasonable, since SVD solenoidal basis functions are not really orthogonal to each other although their divergences do.

To give a more direct illustration of the performance for the above solenoidal bases, Fig. 4 gives the eigenvalue distribution of MoM solenoidal submatrices (after diagonal preconditioning) in the case of the sphere discretized with 216 unknowns. It can be found from Fig. 4 (a) that the eigenvalues of SVD solenoidal basis are more closely clustered than that of LSV solenoidal basis, which indicates the MoM submatrix of SVD solenoidal basis has better conditioning than that of LSV solenoidal basis. By comparing Fig. 4 (a) and (b), it can be found that the eigenvalues of loop basis are more closely clustered than that of SVD solenoidal basis except there is an eigenvalue very close to zero. The eigenvalue of loop basis closest to zero is supposed to be the origin of the worse condition number for the matrix of loop basis compared with that of SVD solenoidal basis. Luckily, the redundant loop basis removes this eigenvalue to zero. As a result, the conditioning of the MoM solenoidal submatrix can be greatly improved. An explanation of the relation between the eigenvalue distribution of a matrix and the convergence of iterative solvers in solving the matrix can be found in [19].

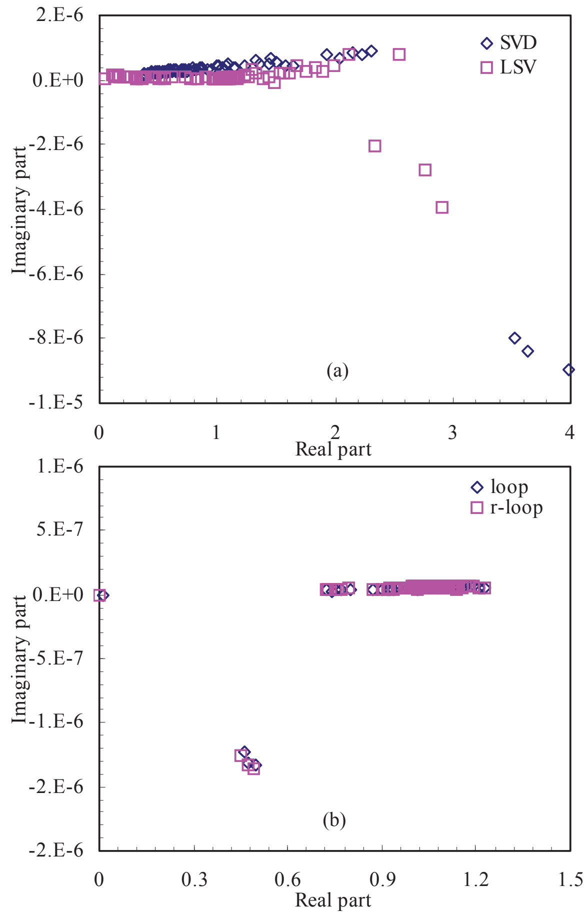

Fig. 4. The eigenvalue distribution of MoM solenoidal submatrices of the sphere discretized with 216 unknowns using different solenoidal bases at 1 MHz. (a) SVD and LSV solenoidal bases, (b) loop and redundant loop bases.

From the above discussion for the property of the redundant loop basis, it can be concluded that the redundant loop basis can remove the eigenvalue of its corresponding MoM submatrix closest to zero to zero and hence to form a well-conditioned MoM submatrix.

B. Application of redundant loop basis to MR basis

It has been discussed in Section II-B that the MR basis can take loop basis as its solenoidal part.

In order to improve the conditioning of the MoM matrix of the MR basis for closed structures, the redundant loop basis is proposed to be taken as the solenoidal part of the MR basis. To demonstrate the performance of the redundant loop basis in MR basis, it is compared with the MR bases which take the other solenoidal bases as their solenoidal part. Also, the application of the redundant loop basis in the MR basis is compared with its application in loop-tree basis.

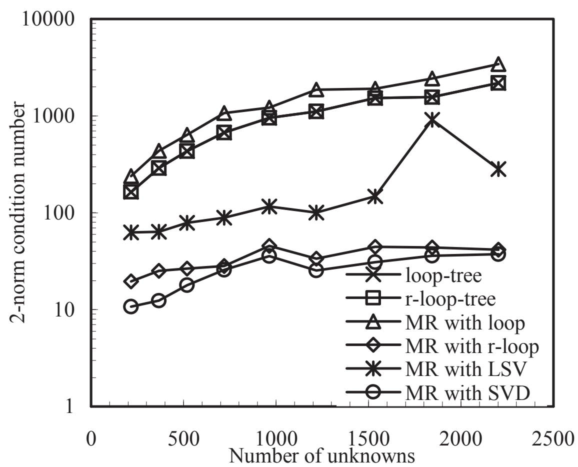

Fig. 5. The 2-norm condition number of MoM matrices of the sphere discretized with a different number of unknowns using different bases at 1 MHz.

Firstly, the 2-norm condition number of the MoM matrices of MR bases with different solenoidal parts and the loop-tree bases with loop or redundant loop solenoidal part is analyzed. The 2-norm condition number of the MoM matrices of these bases in the example of the sphere described above is depicted in Fig. 5. It can be observed from Fig. 5 that the 2-norm condition number of the MR basis with redundant loop solenoidal part is much lower than that of the MR basis with loop solenoidal part, while the 2-norm condition number of the loop-tree basis with loop solenoidal part (denoted with loop-tree in Fig. 5) is almost equal to that of the loop-tree basis with redundant loop solenoidal part (denoted with r-loop-tree in Fig. 5). It can also be observed from Fig. 5 that the 2-norm condition number of the MR basis with LSV solenoidal part is lower than the loop-tree basis and the MR basis with SVD solenoidal part is lower than the MR basis with LSV solenoidal part. However, the SVD solenoidal basis is generated by the SVD operations on the large charge matrix formed from all unknowns, which is prohibitive for a large number of unknowns due to the huge computational cost.

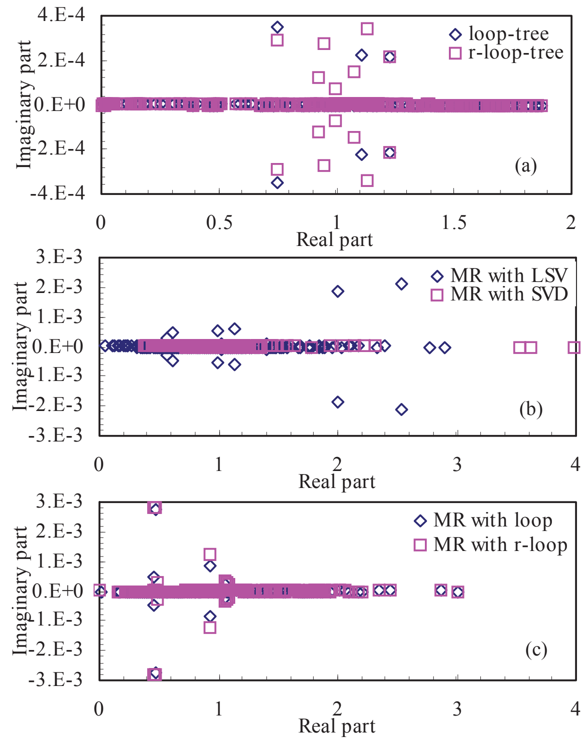

Fig. 6. The eigenvalue distribution of MoM matrices of the sphere discretized with 216 unknowns using different bases at 1 MHz. (a) Loop-tree bases with loop or redundant loop solenoidal part, (b) MR bases with LSV or SVD solenoidal part, (c) MR bases with loop or redundant loop solenoidal part.

Since the convergence behavior of iterative solvers in solving a MoM matrix is mainly determined by the eigenvalue distribution of the MoM matrix, the eigenvalue distribution of MoM matrices of the above bases are investigated. An example of the eigenvalue distribution of MoM matrices (after diagonal preconditioning) of the sphere discretized with 216 unknowns using different bases at the frequency of 1 MHz is shown in Fig. 6. It can be observed from Fig. 6 (a) (see also Fig. 9 (a) for a block) that there are a lot of eigenvalues of the loop-tree bases close to zero, which explains why it has trivial effect on the conditioning of the corresponding MoM matrix by taking the redundant loop basis as the solenoidal part of the loop-tree basis. By comparing Fig. 6 (a) with Fig. 6 (b) and (c), it can be found that the number of eigenvalues of the loop-tree bases nearby zero is much larger than that of the MR bases, which explains why the conditioning of the MoM matrices of the loop-tree bases is worse than that of MR bases. It can be observed from Fig. 6 (b) that the eigenvalues of the MR basis with SVD solenoidal part is more closely clustered than that of the MR basis with LSV solenoidal part, which explains why the MoM matrix of the MR basis with SVD solenoidal part has a well condition number. It can be observed from Fig. 6 (c) (see also Fig. 9 (b) for a block) that the eigenvalues of the MR bases with loop or redundant loop solenoidal part are well distributed except there is a eigenvalue of the MR basis with loop solenoidal part very close to zero. Also, the smallest eigenvalue whose absolute value is about 1.3×10^{-2} is removed to zero by taking the redundant loop basis as the solenoidal part of the MR basis. Namely, the smallest eigenvalue is replaced by the previous second smallest eigenvalue whose value is about 0.15 by keeping the redundant loop basis in MR basis. As a result, the ratio of the absolute values of the largest eigenvalue to the smallest non-zero eigenvalue is improved which explains why the redundant loop basis can improve the conditioning of the MoM matrix of the MR basis effectively.

From the discussion given above, it can be concluded that the application of redundant loop basis in MR basis can improve conditioning of the MoM matrix of MR basis effectively, while it has trivial effect on the conditioning of the MoM matrix of loop-tree basis.

C. Solvability of expanded MR basis with iterative solvers

A similar phenomenon has been reported in [18], where a redundant volume loop basis is used to speed up the convergence of solutions in solving the volume integral equation and a proof is given to validate that the MoM matrix of the redundant volume loop basis can still be solved with an iterative solver without losing accuracy. The proof of the solvability of the MoM matrix of the expanded MR basis, which taking the redundant loop basis as its solenoidal part, can also be proved similarly. Alternatively, it can be proved easily by using a theorem in [19].

The expanded MR basis functions can be represented by the RWG basis functions via a basis-changing matrix $[T]$

$$\left[\vec{f}_{MR}\right]=\left[T\right]^{T}\left[\vec{R}^{0}\right], \tag{4}$$

where the number of RWG basis functions and MR basis functions is N and N+1 respectively, and the matrix $[T]$ is a row full-rank matrix. Then the corresponding MoM matrix equation can be written as

$$\left[Z_{MR}\right]\cdot\left[I_{MR}\right]=\left[b_{MR}\right], \tag{5}$$

where $\left[b_{MR}\right]=\left[T\right]^{T}\cdot\left[b\right]$, $\left[I\right]=\left[T\right]\cdot\left[I_{MR}\right]$, and $\left[Z_{MR}\right]=\left[T\right]^{T}\cdot\left[Z\right]\cdot\left[T\right]$ in which $\left[Z\right]$ is a full rank matrix formed from the RWG basis. It can be concluded that $\left[b_{MR}\right]$ belongs to the range space of $\left[Z_{MR}\right]$, since the rank of matrix $\left[Z_{MR}\right]$ and $\left[Z\right]$ is equal and $[T]$ is a row full-rank matrix. It has been pointed out in [19] that a square linear system $Ax = b$ has a Krylov solution if and only if b belongs to the range space of A. Therefore, the MoM matrix of the expanded MR basis can still be solved by Krylov iterative solvers. The bistatic RCS of a sphere (radius = 1 m) discretized with 3009 unknowns at 1 MHz is taken as an example and shown in Fig. 7. It can be found from Fig. 7 that a good agreement is obtained between the exact Mie series and the expanded MR basis.

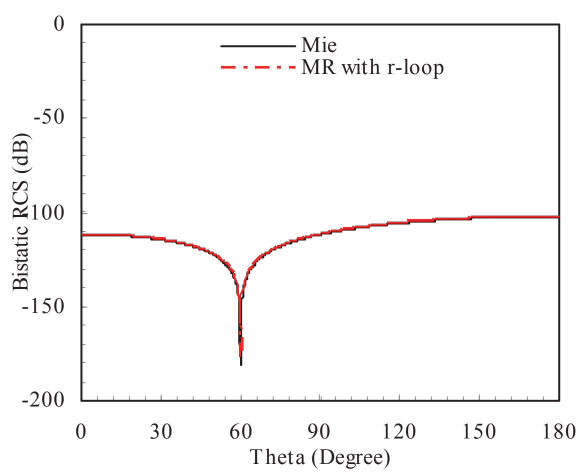

Fig. 7. Bistatic RCS of a sphere (radius = 1 m) discretized with 3009 unknowns at 1 MHz with the exact Mie series and the expanded MR basis with redundant loop solenoidal part.

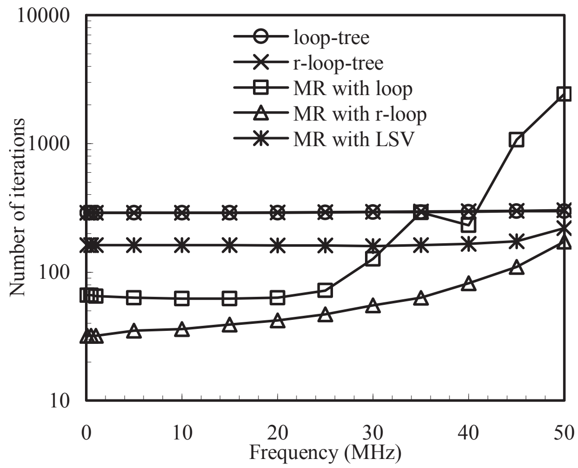

Fig. 8. The convergence behavior of GMRES(30) as a function of frequency for a block using the loop-tree bases and MR bases.

IV. NUMERICAL RESULTS

To validate the performance of the redundant loop basis applied in the MR basis, two examples will be analyzed in this section. The restarted GMRES(30) algorithm was chosen as the iterative solver. The examples were simulated in double precision with a relative residual of 10^{-5}. All the results with different bases were obtained after applying a diagonal preconditioning to the MoM matrix.

The first example is a metallic block (1m×1m×1m) with 1998 unknowns. The convergence behavior of GMRES(30) for the block using the loop-tree and MR bases over a frequency range of 0.1-50 MHz is shown in Fig. 8. It can be found from Fig. 8 that the redundant loop basis can further improve the performance of the MR basis while it has trivial effect on the loop-tree basis. Also, the LSV solenoidal basis performs more stable than the loop basis for the MR basis. However, the LSV solenoidal basis doesn't perform as well as the redundant loop basis as shown in Fig. 8. The eigenvalue distribution of MoM matrices of the block using the above bases at 20 MHz is shown in Fig. 9. It can also be observed from Fig. 9 that the eigenvalue closest to zero is removed to zero by taking the redundant loop basis as the solenoidal part for both loop-tree basis and MR basis. However, there is only one eigenvalue very close to zero for MR basis which explains why the redundant loop basis can improve conditioning of the MoM matrix of the MR basis effectively.

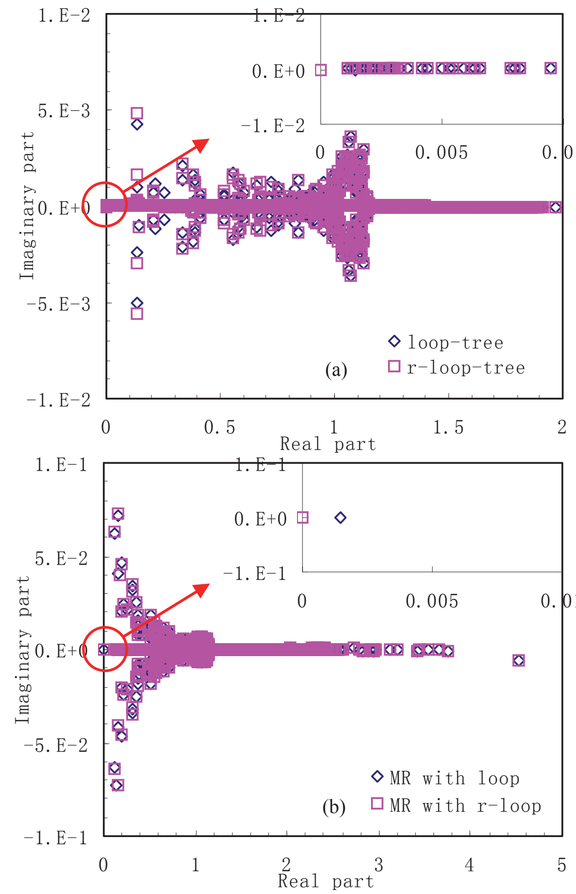

Fig. 9. The eigenvalue distribution of MoM matrices of a block using different bases at 20 MHz. (a) Loop-tree bases with loop or redundant loop solenoidal part, (b) MR bases with loop or redundant loop solenoidal part.

The second example is a more complex metallic plane model, whose length, width, and height is given by 9.3 m, 12.1 m, and 2.2 m, respectively. As shown in Fig. 10, the airplane model is nonuniformly discretized with 5034 unknowns. The number of iterations of the GMRES(30) using the above bases over a frequency range of 0.1-16 MHz is shown in Fig. 10. Obviously, the results of Fig. 10 also indicates that the redundant loop basis has significant advantage over the loop basis for MR basis. Also, the LSV solenoidal basis performs much better than the loop basis for the MR basis.

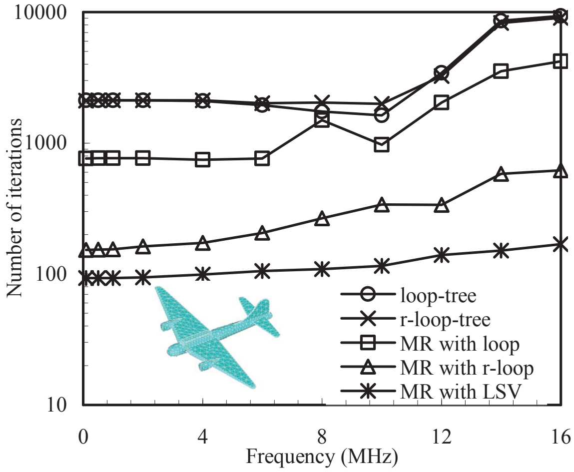

Fig. 10. The convergence behavior of GMRES(30) as a function of frequency for a metallic airplane model using the loop-tree bases and MR bases.

V. CONCLUSION

A redundant loop basis generated by simply keeping all loop functions is applied in MR basis to improve the convergence of MR basis for closed structures. The properties of the MoM matrices using the loop-tree and MR bases which take the loop or redundant loop basis as their solenoidal part are analyzed. It is found that the eigenvalue of the MoM matrix of the MR basis closest to zero is removed to zero by using the redundant loop basis, i.e. keeping the redundant loop basis moves the smallest eigenvalue to zero. As a consequence, the expanded MR basis taking the redundant loop basis as its solenoidal part has significant advantage than the MR basis with the loop basis in convergence for iterative solvers for closed surfaces at the low frequency range.

REFERENCES

[1] R. F. Harrington, *Field Computations by Moment Methods*, MacMillan, New York, 1968.

[2] S. Rao, D. Wilton, and A. Glisson, "Electromagnetic Scattering by Surfaces of Arbitrary Shape," *IEEE Trans. Antennas Propagat.*, vol. 30, pp. 409–418, May 1982.

[3] D. R. Wilton and A. W. Glisson, "On Improving the Electric Field Integral Equation at Low Frequencies," in *Proc. URSI Radio Science Meeting Dig.*, Los Angeles, CA, p. 24, June 1981.

[4] J. Mautz and R. F. Hanington, "An E-Field Solution for a Conducting Surface Small or Comparable to the Wavelength," *IEEE Trans. Antennas Propagat.*, vol. AP-32, no. 4, pp. 330-339, April 1984.

[5] M. Burton and S. Kashyap, "A Study of a Recent Moment-Method Algorithm that is Accurate to Very Low Frequencies," *Applied Computational Electromagnetic Society (ACES) Journal*, vol. 10, no. 3, pp. 58–68, Nov. 1995.

[6] W. L. Wu, A. Glisson, and D. Kajfez, "A Study of Two Numerical Solution Procedures for the Electric Field Integral Equation at Low Frequency," *Applied Computational Electromagnetic Society (ACES) Journal*, vol. 10, no. 3, pp. 69–80, Nov. 1995.

[7] G. Vecchi, "Loop-Star Decomposition of Basis Functions in the Discretization of the EFIE," *IEEE Trans. Antennas Propagat.*, vol. 47, no. 2, pp. 339–346, Feb. 1999.

[8] J. S. Zhao and W. C. Chew, "Integral Equation Solution of Maxwell's Equations from Zero Frequency to Microwave Frequency," *IEEE Trans. Antennas Propagat.*, vol. 48, pp. 1635–1645, Oct. 2000.

[9] J. F. Lee, R. Lee, and R. J. Burkholder, "Loop Star Basis Functions and a Robust Precodnitioner for EFIE Scattering Problems," *IEEE Trans. Antennas Propagat.*, vol. 51, pp. 1855–1863, Aug. 2003.

[10] T. F. Eibert, "Iterative-Solver Convergence for Loop-Star and Loop-Tree Decomposition in Method-of-Moments Solutions of the Electric-Field Integral Equation," *IEEE Antennas Propag. Mag.*, vol. 46, pp. 80–85, Jun. 2004.

[11] F. P. Andriulli, F. Vipiana, and G. Vecchi, "Enhanced Multiresolution Basis for the MoM Analysis of 3D Structures," in *Proc. IEEE Int. Symp. Antennas Propagat.*, pp. 5612–5615, Honolulu, HI, Jun. 2007.

[12] F. P. Andriulli, F. Vipiana, and G. Vecchi, "Hierarchical Bases for Non-Hierarchic 3-D Triangular Meshes," *IEEE Trans. Antennas Propagat.*, vol. 56, pp. 2288–2297, Aug. 2008.

[13] F. Vipiana, F. P. Andriulli, and G. Vecchi, "Two-Tier Non-Simplex Grid Hierarchic Basis for General 3D Meshes," *Waves in Random and Complex Media*, vol. 19, no. 1, pp. 126–146, Feb. 2009.

[14] F. Vipiana, and G. Vecchi, "A Novel, Symmetrical Solenoidal Basis for the MoM

Analysis of Closed Surfaces," *IEEE Trans. Antennas Propagat.*, vol. 57, no. 4, pp. 1294-1299, April 2009.

[15] J. J. Ding, R. S. Chen, J. Zhu, Z. H. Fan, and D. Z. Ding, "A Multiresolution Basis for Analysis of Scattering Problems," in *2010 International Conference on Microwave and Millimeter Wave Technology*, Chengdu, pp. 607-609, May 2010.

[16] J. J. Ding, J. Zhu, R. S. Chen, Z. H. Fan, and K. W. Leung, "An Alternative Multiresolution Basis in EFIE for Analysis of Low-Frequency Problems," *Applied Computational Electromagnetic Society (ACES) Journal*, vol. 26, no. 1, pp. 26-36, Jan. 2011.

[17] F. Vipiana, P. Pirinoli, and G. Vecchi, "Spectral Properties of the EFIE-MoM Matrix for Dense Meshes with Different Types of Bases," *IEEE Trans. Antennas Propagat.*, vol. 55, no. 11, pp. 3229–3238, Nov. 2007.

[18] M. K. Li and W. C. Chew, "Applying Divergence-Free Condition in Solving the Volume Integral Equation," *Progress In Electromagnetics Research*, PIER 57, pp. 311-333, 2006.

[19] I. C. F. Ipsen and C. D. Meyer, "The Idea Behind Krylov Methods," *Amer. Math. Monthly*, vol. 105, no. 10, pp. 889–899, Dec. 1998.