The Computational Expressions of Spheroidal Eigenvalues

Wan-xian Wang
Space Astronomy Laboratory, University of Florida
Gainesville, FL 32609

ABSTRACT

The higher order terms of eigenvalues in spheroidal differential equation are developed by using power-series expansion and asymptotic ones for both prolate and oblate wave functions, these important multipole expansions greatly facilitate and improve the computations of the electromagnetic scattering by different kinds of spheroids with various size parameters, refractive indices, and aspect ratios.

I. INTRODUCTION

The prolate and oblate spheroidal eigenvalues λ_{mn} are usually calculated following Bouwkamp's method[1] while restricted to the case of small value of size parameter c ($= \kappa(a^2-b^2)^{\frac{1}{2}}$, where κ is wave number, a is semi-major axis, and b is semi-minor axis) and large number n. For large value of c (say > 15) and/or small number n, the asymptotic expansions must be employed; J. Meixner[2,3] had performed the asymptotic developments of prolate and oblate spheroidal eigenvalues up to c^{-5}, respectively. However, for moderate value of c and the intermediate number n, there appears a gap between Bouwkamp's and the asymptotic expansions because of the orders of the included terms being not high enough for these expansions.

The author has pushed the power-series expansion forward to c^{16} term, and the prolate and oblate asymptotic developments till c^{-6}, respectively. Thus the correct arrangement of the eigenvalues λ_{mn} from small value through large value of c and from small number through large number n is formed in increasing order by correspondingly selecting one of these two expansions. The developments of the analytical expressions of the spheroidal eigenvalues, together with further improvements on calculating the spheroidal radial functions, have made it feasible to compute the scattering coefficients for different kinds of spheroids within very wide range.

II. POWER-SERIES EXPANSION OF THE SPHEROIDAL EIGENVALUES

The angular differential equation of the spheroidal wave functions can be written in the form

$$(1-\eta^2)\frac{d^2S_{mn}}{d\eta^2} - 2\eta\frac{dS_{mn}}{d\eta} + (\lambda_{mn} - c^2\eta^2 - \frac{m^2}{1-\eta^2})S_{mn} = 0 \tag{1}$$

where η is angular coordinate in the spheroidal system, $-1 \leqslant \eta \leqslant 1$;

S_{mn} are the spheroidal angle functions of order m and degree n;

λ_{mn} are the eigenvalues of the spheroidal differential equation;

m and n are positive integers with $n \geqslant m$.

This equation is for prolate spheroidal wave functions. By replacing c by -ic in Eq. (1) we would have the equation for oblate one. For small value of c, the following expression of the eigenvalues λ_{mn} in the form of continued fraction can be obtained by using three-term recursion relation of the expansion coefficients, $d_r^{mn}(c)$, of the spheroidal angle functions S_{mn} with respect to the associated Legendre functions P_{m+r}^m (where r is the summation index):[4]

$$\lambda_{mn} = \gamma_{n-m}^m - \frac{\beta_{n-m}^m}{\gamma_{n-m-2}^m-\lambda_{mn}-}\,\frac{\beta_{n-m-2}^m}{\gamma_{n-m-4}^m-\lambda_{mn}-}\cdots - \frac{\beta_{n-m+2}^m}{\gamma_{n-m+2}^m-\lambda_{mn}-}\,\frac{\beta_{n-m+4}^m}{\gamma_{n-m+4}^m-\lambda_{mn}-}\cdots \tag{2}$$

where $$\gamma_r^m = (m+r)(m+r+1) + \frac{1}{2}c^2\left[1 - \frac{4m^2-1}{(2m+2r-1)(2m+2r+3)}\right] \qquad (r \geqslant 0) \tag{2-1}$$

$$\beta_r^m = \frac{r(r-1)(2m+r)(2m+r-1)c^4}{(2m+2r-1)^2(2m+2r-3)(2m+2r+1)} \qquad (r \geqslant 2) \tag{2-2}$$

Substituting in Eq. (2) the power-series expansion

$$\lambda_{mn} = \sum_{k=0}^{\infty} \ell_{2k}^{mn} c^{2k} \tag{3}$$

and then developing the continued fraction by raising consecutively each partial denominator up to the associated numerator with the use of binomial expansion, we can find the coefficients ℓ_{2k}^{mn} by equating the power of c^{2k}. The coefficients in the book by C. Flammer[5] were given till ℓ_{10}^{mn}; it might be remarked that he obtained the coefficient

ℓ_{10}^{mn} as given below, but apart from a sign error in the third term of the second part (the numerator 6n−25 instead of the correct 6n+25).[6] In most cases, the coefficients up to ℓ_{12}^{mn} are often used, the coefficients ℓ_{14}^{mn} and ℓ_{16}^{mn} are just applied to some certain size parameters of moderate value of c and intermediate numbers n. To save space the coefficients derived by author are not fully listed; ℓ_{14}^{mn} and ℓ_{16}^{mn} can be found from author's paper.[7]

The power-series expansion of the spheroidal eigenvalues λ_{mn} is as follows:

$$\lambda_{mn} = \sum_{k=0}^{\infty} \ell_{2k}^{mn} c^{2k} \tag{4}$$

where

$$\ell_0^{mn} = n(n+1) \tag{4-1}$$

$$\ell_2^{mn} = \frac{1}{2}\left[1 - \frac{4m^2-1}{(2n-1)(2n+3)}\right] \tag{4-2}$$

$$\ell_4^{mn} = \frac{\beta_0}{2(2n-1)} - \frac{\beta_2}{2(2n+3)} \tag{4-3}$$

$$\ell_6^{mn} = -\frac{\beta_0(4m^2-1)}{4(2n-1)^2}\left[\frac{4}{(2n-5)(2n-1)(2n+3)}\right] + \frac{\beta_2(4m^2-1)}{4(2n+3)^2}\left[\frac{4}{(2n+7)(2n+3)(2n-1)}\right] \tag{4-4}$$

$$\begin{aligned}\ell_8^{mn} = &-\frac{\beta_0(4m^2-1)}{4(2n-1)^2}\left[\frac{\ell_4^{mn}}{(4m^2-1)} - \frac{8(4m^2-1)}{(2n-5)^2(2n-1)^3(2n+3)^2} - \frac{\beta_{-2}}{4(4m^2-1)(2n-3)}\right]\\ &-\frac{\beta_2(4m^2-1)}{4(2n+3)^2}\left[\frac{\ell_4^{mn}}{(4m^2-1)} + \frac{8(4m^2-1)}{(2n+7)^2(2n+3)^3(2n-1)^2} + \frac{\beta_4}{4(4m^2-1)(2n+5)}\right]\end{aligned} \tag{4-5}$$

$$\begin{aligned}\ell_{10}^{mn} = &-\frac{\beta_0(4m^2-1)}{4(2n-1)^2}\left[\frac{\ell_6^{mn}}{(4m^2-1)} - \frac{4\ell_4^{mn}}{(2n-5)(2n-1)^2(2n+3)} + \frac{16(4m^2-1)^2}{(2n-5)^3(2n-1)^5(2n+3)^3}\right.\\ &\left.+\frac{\beta_{-2}(6n-19)}{2(2n-9)(2n-5)(2n-3)(2n-1)^2(2n+3)}\right]\\ &-\frac{\beta_2(4m^2-1)}{4(2n+3)^2}\left[\frac{\ell_6^{mn}}{(4m^2-1)} - \frac{4\ell_4^{mn}}{(2n+7)(2n+3)^2(2n-1)} - \frac{16(4m^2-1)^2}{(2n+7)^3(2n+3)^5(2n-1)^3}\right.\\ &\left.-\frac{\beta_4(6n+25)}{2(2n+11)(2n+7)(2n+5)(2n+3)^2(2n-1)}\right]\end{aligned} \tag{4-6}$$

$$
\begin{aligned}
\ell_{12}^{mn} = -\frac{\beta_0(4m^2-1)}{4(2n-1)^2}\Bigg\{ & \frac{\ell_8^{mn}}{(4m^2-1)} - \frac{(\ell_4^{mn})^2}{2(4m^2-1)(2n-1)} - \frac{4\ell_6^{mn}}{(2n-5)(2n-1)^2(2n+3)} + \frac{12(4m^2-1)\ell_4^{mn}}{(2n-5)^2(2n-1)^4(2n+3)^2} \\
& - \frac{32(4m^2-1)^3}{(2n-5)^4(2n-1)^7(2n+3)^4} - \frac{\beta_{-2}}{(4m^2-1)(2n-3)}\Bigg[\frac{(4m^2-1)^2 d_1}{(2n-9)^2(2n-5)^2(2n-1)^4(2n+3)^2} \\
& - \frac{(10n-13)\ell_4^{mn}}{16(2n-3)(2n-1)} + \frac{\beta_{-2}}{32(2n-3)(2n-1)} + \frac{\beta_{-4}}{96(2n-5)(2n-3)}\Bigg]\Bigg\} \\
- \frac{\beta_2(4m^2-1)}{4(2n+3)^2}\Bigg\{ & \frac{\ell_8^{mn}}{(4m^2-1)} + \frac{(\ell_4^{mn})^2}{2(4m^2-1)(2n+3)} - \frac{4\ell_6^{mn}}{(2n+7)(2n+3)^2(2n-1)} + \frac{12(4m^2-1)\ell_4^{mn}}{(2n+7)^2(2n+3)^4(2n-1)^2} \\
& + \frac{32(4m^2-1)^3}{(2n+7)^4(2n+3)^7(2n-1)^4} + \frac{\beta_4}{(4m^2-1)(2n+5)}\Bigg[\frac{(4m^2-1)^2 d_2}{(2n+11)^2(2n+7)^2(2n+3)^4(2n-1)^2} \\
& + \frac{(10n+23)\ell_4^{mn}}{16(2n+5)(2n+3)} + \frac{\beta_4}{32(2n+5)(2n+3)} + \frac{\beta_6}{96(2n+7)(2n+5)}\Bigg]\Bigg\}
\end{aligned} \tag{4-7}
$$

where $\beta_{-4} = \beta_{n-m-4}/c^4$, $\quad\beta_{-2} = \beta_{n-m-2}/c^4$, $\quad\beta_0 = \beta_{n-m}/c^4$,

$\beta_2 = \beta_{n-m+2}/c^4$, $\quad\beta_4 = \beta_{n-m+4}/c^4$, $\quad\beta_6 = \beta_{n-m+6}/c^4$.

and $\quad d_1 = (2n-1)^2 + 2(2n-1)(2n-9) + 3(2n-9)^2$

$\quad d_2 = (2n+3)^2 + 2(2n+3)(2n+11) + 3(2n+11)^2$

The power-series expansion of the oblate eigenvalues λ_{mn} is obtained from Eq. (4) by simply replacing c^2 by $-c^2$.

III. ASYMPTOTIC DEVELOPMENT OF THE PROLATE SPHEROIDAL EIGENVALUES

Let us set

$$S_{mn} = (1 - \eta^2)^{\frac{m}{2}}\, u_{mn} \tag{5}$$

substitute this expression in Eq. (1), and make the transformation

$$\eta = (2c)^{-\frac{1}{2}}x \tag{6}$$

there results

$$(1 - \frac{x^2}{2c})\frac{d^2u_{mn}}{dx^2} - \frac{(m+1)}{c}x\,\frac{du_{mn}}{dx} + (K - \frac{x^2}{4})u_{mn} = 0 \tag{7}$$

where constant $K = \frac{\lambda_{mn} - m(m+1)}{2c}$ (7-1)

First of all, developing u_{mn} and K in the asymptotic forms of size parameter c:

$$u_{mn} = u_0 + u_1 c^{-1} + u_2 c^{-2} + u_3 c^{-3} + \cdots + u_k c^{-k} \quad (8\text{-}1)$$

$$K = \alpha_0 + \alpha_1 c^{-1} + \alpha_2 c^{-2} + \alpha_3 c^{-3} + \cdots + \alpha_k c^{-k} \quad (8\text{-}2)$$

and substituting them in Eq. (7), we can determine the terms u_0, u_1, u_2, $\cdots$ of eigenfunctions u_{mn} by a series of differential equations of the second order, namely —

$$u_0'' + (\alpha_0 - \frac{x^2}{4})u_0 = 0 \quad (9\text{-}1)$$

$$u_1'' + (\alpha_0 - \frac{x^2}{4})u_1 = -\alpha_1 u_0 + \frac{1}{2}\left[x^2 u_0'' + 2(m+1)xu_0'\right] \quad (9\text{-}2)$$

$$u_2'' + (\alpha_0 - \frac{x^2}{4})u_2 = -\alpha_1 u_1 - \alpha_2 u_0 + \frac{1}{2}\left[x^2 u_1'' + 2(m+1)xu_1'\right] \quad (9\text{-}3)$$

......

$$u_k'' + (\alpha_0 - \frac{x^2}{4})u_k = -\alpha_1 u_{k-1} - \alpha_2 u_{k-2} - \cdots - \alpha_k u_0 + \frac{1}{2}\left[x^2 u_{k-1}'' + 2(m+1)xu_{k-1}'\right] \quad (9\text{-}k\text{+}1)$$

Let

$$\alpha_0 = p + \frac{1}{2} \quad (10)$$

where p is a positive integer or zero. The solution of Eq. (9-1) is

$$u_0 = D_p(x) \quad (11)$$

where $D_p(x)$ is parabolic cylinder function. For simplicity, we just denote it by D_p.

In the first approximation of K, we have

$$K = \alpha_0 = p + \frac{1}{2} \quad (12)$$

It implies that

$$\lambda_{mn} = (2p + 1)c \qquad \text{while } c \to \infty \quad (13)$$

According to the asymptotic property of the eigenvalues λ_{mn}, we find that

$$\lambda_{mn} = [2(n - m) + 1]c \qquad \text{while } c \to \infty \quad (14)$$

The foregoing suggests that

$$p = n - m \quad (15)$$

Next we define the operator[8]

$$\nabla D_p = x^2 D_p'' + 2(m+1)xD_p' \tag{16}$$

Utilizing the recursion relation of the parabolic cylinder functions:

$$D_p' + \frac{x}{2}D_p - pD_{p-1} = 0 \tag{17}$$

we obtain

$$\nabla D_p = C_{1,p}D_{p+4} + C_{2,p}D_{p+2} + C_{3,p}D_p + C_{4,p}D_{p-2} + C_{5,p}D_{p-4} \tag{18}$$

where

$$C_{1,p} = \frac{1}{4} \tag{18-1}$$

$$C_{2,p} = -m \tag{18-2}$$

$$C_{3,p} = -\frac{2p^2+2p+3}{4} - m \tag{18-3}$$

$$C_{4,p} = m|p|_2 \tag{18-4}$$

$$C_{5,p} = \frac{|p|_4}{4} \tag{18-5}$$

with $|p|_2 = p(p-1)$, and $|p|_4 = p(p-1)(p-2)(p-3)$.

Again, we define the operator

$$\Delta u_r = u_r'' + (p + \frac{1}{2} - \frac{x^2}{4})u_r \tag{19}$$

where $r = 1, 2, \cdots k$.

Remembering that

$$\Delta D_p = 0 \tag{20}$$

we find that D_p term will not appear in the solution u_r of Eqs. (9-2), (9-3), $\cdots$.

Furthermore we set

$$u_r = \sum_{\ell \neq 0} A_{2\ell}D_{p+2\ell} \tag{21}$$

then

$$\Delta u_r = \sum_{\ell \neq 0} (-2\ell)A_{2\ell}D_{p+2\ell} = \sum_{\ell \neq 0} B_{2\ell}D_{p+2\ell} \tag{22}$$

where we have expressed the right-hand sides of Eqs. (9-2), (9-3), $\cdots$ in the forms of $\sum_{\ell \neq 0} B_{2\ell}D_{p+2\ell}$.

Hence the coefficients $A_{2\ell}$ are found to be

$$A_{2\ell} = -\frac{B_{2\ell}}{2\ell} \tag{23}$$

Now we can solve Eqs. (9-2), (9-3), $\cdots$.

From Eq. (9-2), we get

$$\alpha_1 = \frac{C_{3,p}}{2} = -\frac{2p^2+2p+3}{8} - \frac{m}{2} \tag{24}$$

By assuming

$$q = 2p + 1 \tag{25}$$

we have

$$\alpha_1 = -\frac{q^2+5+8m}{16} \tag{26}$$

The term u_1 of the eigenfunctions u_{mn} is found in the form

$$u_1 = \frac{1}{2}\left[f_1 D_{p+4} + f_2 D_{p+2} + f_3 D_{p-2} + f_4 D_{p-4}\right] \tag{27}$$

where

$$f_1 = -\frac{1}{16} \tag{27-1}$$

$$f_2 = \frac{m}{2} \tag{27-2}$$

$$f_3 = \frac{m}{2}\,|p|_2 \tag{27-3}$$

$$f_4 = \frac{1}{16}\,|p|_4 \tag{27-4}$$

Similarly we can obtain the other coefficients α_r of the eigenvalues λ_{mn} and the terms u_r of the eigenfunctions u_{mn} by the successive substitutions. The expression of terms u_r will be lengthy and lengthy while r increases; therefore I only list the coefficients α_r.

$$\alpha_2 = -\frac{q(q^2+11-32m^2)}{2^7} \tag{28}$$

$$\alpha_3 = -\frac{5(q^4+26q^2+21)-384m^2(q^2+1)}{2^{11}} \tag{29}$$

$$\alpha_4 = -\frac{(33q^5+1594q^3+5621q)-128m^2(37q^3+167q)+2048m^4q}{2^{15}} \tag{30}$$

$$\alpha_5 = -\frac{(63q^6+4940q^4+43327q^2+22470)-128m^2(115q^4+1310q^2+735)+24576m^4(q^2+1)}{2^{17}} \tag{31}$$

$$\alpha_6 = -\Big[(527q^7+61529q^5+1043961q^3+2241599q)-32m^2(5739q^5+127550q^3+298951q)+ 2048m^4(355q^3+1505q)-65536m^6q\Big]2^{-21} \tag{32}$$

$$\alpha_7 = -\Big[(9387q^8+1536556q^6+43711178q^4+230937084q^2+93110115)-1536m^2(2989q^6+112020q^4+ 648461q^2+270690)+196608m^4(175q^4+1814q^2+939)-12582912m^6(q^2+1)\Big]2^{-26} \tag{33}$$

Therefore the eigenvalues λ_{mn} are in the form:

$$\lambda_{mn} = qc - \frac{1}{8}(q^2+5-8m^2) + \sum_{r=2}^{7} 2\alpha_r c^{-r+1} + 0(c^{-7}) \tag{34}$$

Correspondingly the eigenfunctions u_{mn} are in the form:

$$u_{mn} = \sum_{r=0}^{6} u_r c^{-r} + 0(c^{-7}) \tag{35}$$

As E.L.Ince said: "If anyone had the courage to push the development on a stage or two further he would greatly enhance the value of an important expansion. But any reader who attempts to verify the results given above will realize that the work involved would be tremendous".[9]

The expression of the eigenvalues λ_{mn} of the prolate spheroidal differential equation can be converted to that of the eigenvalue Λ of the Mathieu differential equation, as follows:

$$\Lambda = -2h^2 + 2qh - \frac{1}{8}(q^2+1) - (q^3+3q)2^{-7}h^{-1} - (5q^4+34q^2+9)2^{-12}h^{-2} - (33q^5+410q^3+405q)2^{-17}h^{-3} - (63q^6+1260q^4+2943q^2+486)2^{-20}h^{-4} - (527q^7+15617q^5+69001q^3+41607q)2^{-25}h^{-5} - (9387q^8+388780q^6+2845898q^4+4021884q^2+506979)2^{-31}h^{-6} + 0(h^{-7}) \tag{36}$$

where

$$h = \frac{c}{2} \tag{36-1}$$

$$\Lambda = \lambda_{mn} + \frac{1}{4} - \frac{1}{2}c^2 \tag{36-2}$$

herein the author has developed one more high order term h^{-6}.

IV. ASYMPTOTIC DEVELOPMENT OF THE OBLATE SPHEROIDAL EIGENVALUES

The oblate spheroidal differential equation for angle functions S_{mn} is expressed as

$$(1-\eta^2)\frac{d^2S_{mn}}{d\eta^2} - 2\eta\frac{dS_{mn}}{d\eta} + (\lambda_{mn} + c^2\eta^2 - \frac{m^2}{1-\eta^2})S_{mn} = 0 \tag{37}$$

Referring to C. Flammer's book, and using three-term recursion relation of the expansion coefficients, A_s^{mn}, of the oblate spheroidal angle functions S_{mn} with respect to the Laguerre functions $L_{\nu+s}^{(m)}$ (where $\nu = \frac{1}{2}(n-m)$ if $(n-m)$ is even, and $\nu = \frac{1}{2}(n-m-1)$ if $(n-m)$ is odd, and s is the summation index), we have the following expression of the eigenvalues Λ_{mn} in the form of transcendental equation:

$$\Lambda_{mn} = \frac{Q_0^2}{\Lambda_{mn} + P_{-1} -} \frac{Q_{-1}^2}{\Lambda_{mn} + P_{-2} -} \cdots + \frac{Q_1^2}{\Lambda_{mn} + P_1 -} \frac{Q_2^2}{\Lambda_{mn} + P_2 -} \cdots \tag{38}$$

where

$$Q_s = (s+\nu)(s+\nu+m) \tag{38-1}$$

$$P_s = 2s(2\nu+m+1-2c+s) \tag{38-2}$$

The eigenvalues Λ_{mn} are related to the eigenvalues λ_{mn} by

$$\lambda_{mn} = -c^2 + 2qc - \frac{1}{2}(q^2+1-m^2) + \Lambda_{mn} \tag{39}$$

where

$$q = n + 1 \qquad \text{while } (n-m) \text{ even} \tag{39-1}$$

$$q = n \qquad \text{while } (n-m) \text{ odd} \tag{39-2}$$

Substituting in Eq. (38) the inverse power-series with respect to size parameter c:

$$\Lambda_{mn} = \sum_{\ell=1}^{\infty} \ell_i c^{-i} \tag{40}$$

and then expanding the continued fraction by doing same procedure as power-series expansion discussed in Section II, we can get the coefficients ℓ_i. The eigenvalues λ_{mn} of the oblate spheroidal differential equation would be:

$$\begin{aligned}\lambda_{mn} = {} & -c^2 + 2qc - \frac{1}{2}(q^2+1-m^2) - q(q^2+1-m^2)2^{-3}c^{-1} - [(5q^4+10q^2+1)-2m^2(3q^2+1)+m^4]2^{-6}c^{-2} \\ & - q[(33q^4+114q^2+37)-2m^2(23q^2+25)+13m^4]2^{-9}c^{-3} - [(63q^6+340q^4+239q^2+14) - \\ & 10m^2(10q^4+23q^2+3)+3m^4(13q^2+6)-2m^6]2^{-10}c^{-4} - q[(527q^6+4139q^4+5221q^2+1009)- \\ & m^2(939q^4+3750q^2+1591)+5m^4(93q^2+127)-53m^6]2^{-13}c^{-5} - [(9387q^8+101836q^6+ \\ & 205898q^4+86940q^2+3747)-12m^2(1547q^6+9575q^4+8657q^2+701)+6m^4(1855q^4+5078q^2+939)- \\ & 12m^6(167q^2+85)+51m^8]2^{-17}c^{-6} + 0(c^{-7})\end{aligned} \tag{41}$$

V. DISCUSSION

In order to obtain the accurate spheroidal eigenvalues, we should substitute the expression of spheroidal eigenvalues with hgigher order terms for Bouwkamp's or asymptotic expansion, as the initial values, into Eq. (2). Since the terms in the multipole expansion have been developed in such a high order that the initial eigenvalues will be immediately bound within the convergence circles. Therefore, the final eigenvalues λ_{mn} can be easily reached by iterated procedures at very fast convergence rates.

The spheroidal eigenvalues λ_{mn}, the spheroidal angular functions S_{mn}, and the spheroidal radial functions R_{mn}, together with the boundary conditions matching, make the computational electromagnetic scattering problems solvable.

REFERENCES

1. C.J. Bouwkamp, "On Spheroidal Wave Functions of Order Zero", J. Math. Phys., 26, 79 (1947).

2. J. Meixner, "Asymptotische Entwicklung der Eigenwerte und Eigenfunktionen der Differentialgleichungen der Sphäroidfunktionen und der Mathieuschen Funktionen", Z. angew. Math. Mech., 28, 304 (1948).

3. J. Meixner and F.W. Schäfke, "Mathieusche Funktionen und Sphäroidfunktionen", Springer-Verlag, Berlin (1954).

4. J.A. Stratton, P.M. Morse, L.J. Chu, J.D.C. Little, and F.J. Corbató, Spheroidal Wave Functions, The Massachusetts Institute of Technology Press (1956).

5. C. Flammer, Spheroidal Wave Functions, Stanford University Press (1957).

6. Wan-xian Wang, "Corrections and Developments on the Theory of Scattering by Spheroid — Comparison with Experiments", presented at 1986 CRDEC Scientific Conference on Obscuration and Aerosol Research, Aberdeen, MD (June 1986).

7. Wan-xian Wang, "Power-Series Expansion of the Eigenvalues for Spheroidal Differential Equation", presented at 1986 CRDEC Scientific Conference on Obscuration and Aerosol Research, Aberdeen, MD (June 1986).

8. R. Sips, "Représentation Asymptotique des Fonctions de Mathieu et des Fonctions d'onde Sphéroidales", Trans. Amer. Math. Soc., 66, 93 (1949).

9. E.L. Ince, "Researches into the Characteristic Numbers of the Mathieu Equation", Royal Soc. Edinburgh Proc., 46, 316 (1926).