An Incomplete Review of Fast Multipole Methods–from Static to Wideband–as Applied to Problems in Computational Electromagnetics

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***Abstract*—** Fast multipole methods (FMM) and their immediate predecessors, tree codes, were developed in response to the need for solving *N*-body problems that occur in applications as varied as biophysics, computational chemistry, astrophysics and electromagnetics. In all these areas, it is necessary to compute long range potentials of the form $1/R$ between a dense distribution of point charges, where R is the distance between any two charges. Often, repeated evaluation of these potentials is necessary. It is apparent that the cost of direct evaluation, which scales as $O(N^2)$ for N degrees of freedom, forms a fundamental bottleneck. FMM and tree methods ameliorate the cost associated with these computation; CPU times of these method scale as $O(N)$. It stands to reason that FMM has had a seminal impact on a multitude of fields, so much so, that it was recognized as one of the top ten algorithms of the past century. A method to rapidly compute potentials of the form e^{-jkR}/R soon followed. As the reader is aware, these potentials are the crux of integral equation based analysis tools in electromagnetics and the advent of these methods have transformed the face of computational electromagnetics. Consequently, the state of art of integral equation solvers has grown by leaps and bounds over the past decade. This paper attempts to present a detailed review of the state of art of FMM based methods that are used in computational electromagnetics, from the static to the high frequency regime.

***Index Terms*—** Fast Multipole Method (FMM), FMM review, Multiscale, Wideband FMM, Multipole methods, Cartesian expansions, ACE.

I. INTRODUCTION

The numerical solution of Maxwell's equations has typically proceeded along two different paths. The first, and perhaps the more popular, is the direct discretization of Maxwell's equations [1, 2]. Finite difference and finite element methods belong to this class of solvers. Their popularity stems from two salient features; (i) they are typically simpler to program and (ii) their memory and CPU cost scales as $O(N)$, where N denotes the number of degrees of freedom. The second methodology for solving Maxwell's equations are based on developing integral equations (IE) derived by evoking the Green's identity/equivalence theorems. While the latter was introduced in electromagnetics more than four decades ago [3], they were not a popular option for electromagnetic analysis. The bottlenecks to their adoption was due to both the memory and CPU complexity, both of which scale as $O(N^2)$. This is despite some of the inherent advantages of integral equations for analyzing open region problems, viz., better condition numbers, possibility of using surface integral equations and incorporation of the radiation boundary condition in the Green's function.

The introduction of the fast multipole methods (and tree codes) significantly altered the landscape. Both these methods were developed in response to accelerating pairwise potential evaluations in N-body problems in fields ranging from biophysics to computational chemistry to astrophysics, etc. Here, it is necessary to compute long-range Coulombic potentials repeatedly between N randomly distributed particles. The tree methods [4, 5] and the fast multipole method (FMM) [6–9] reduced the computational complexity of computing these pairwise potentials from $O(N^2)$ to $O(N)$. FMM and tree codes are based on a hierarchical decomposition of the computational domain, and using multipole/local expansions to compute the influence between sub-domains that are sufficiently separated. The FMM,

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as introduced in [7], exploits the representation of the potential in terms of spherical harmonics. As we shall see, this is a consequence of using addition theorems to represent the potential as a series wherein each term is a product of two functions. These functions depend either on the coordinates of the source or the observer only. The separation between source and observer is crucial to creating a fast scheme. At about the same time, an algorithm that achieves the same reduction on complexity, *albeit* using Cartesian tensors was introduced [10]. This derivation relies on using Taylor expansion of the potential function to provide the necessary addition theorems [11]. Cartesian expansions have been used extensively in tree codes. More recently, FMM codes based on Cartesian expansions have used recurrence relations to avoid derivatives [12]. Typically, FMMs derived using the Cartesian expansion are more expensive as spherical harmonics are optimal in representing Coulombic potentials. However, it was recently shown that it is possible to develop a FMM using Maxwell-Cartesian harmonics that are as optimal as using spherical harmonics with one singular advantage; it avoids the need for special functions [13]. Both FMM and tree codes have revolutionized analysis in various application domains ranging from molecular dynamics [14], elastostatics [15, 16], elastic wave equations [17], flow problems [18], capacitance [19] and impedance [20] extraction in micro-electronic circuits, evaluation of splines [21] and spherical harmonics [22, 23]. The FMM framework has also been extended to the solution of potentials resulting from parabolic equations [24–26].

However, direct extension of FMM to the solution of potentials arising from hyperbolic equations is not as straightforward. The first solution to this problem was presented in two dimensions [27, 28], and soon extended to three dimensions [29, 30]. The crux to developing these algorithms was the derivation of a diagonalized form of the translation operator [30–32]. Since then, there has been a virtual explosion in research in application of these methods to various problems in electromagnetics; see [33–35] and references therein. The state of art is such that problems of the order of tens million spatial degrees of freedom have been solved [36– 41]. However, the development of FMM based method continues on many fronts [42–50]. This paper reviews progress in FMM technology since its inception and details current trends in FMM research.

With this introduction, the rest of the paper is organized as follows; in Section II, we will outline the overall problem, introduce notation that is common to the article. Next, static FMM is presented in Section III. Methods based on both Spherical harmonics and Accelerated Cartesian expansions (ACE) will be presented. In Section IV, we will detail the development of FMMs and their variants for the Helmholtz equation. Finally, in Section IV-B we will elaborate on the methods used for wideband analysis. In all cases, we will first present theorems to implement a single level algorithm followed by the steps necessary to implement a multilevel version. The CPU time of all steps will be elucidated as well as "some" algorithmic improvements to these methods since they were first introduced. Finally, Section VI will summarize the state of art of FMM. In what follows, a time dependence of $e^{j\omega t}$ is tacitly assumed. As an aside, while we have tried to be as complete as possible in our citations, it is almost certain that we have unknowingly missed some.

II. HIERARCHICAL COMPUTATION SCHEME

The purpose of this section is to outline the structure of fast multipole methods and introduce notation that will be used in the rest of the paper.

A. Preliminaries

Consider a source distribution $q(r)$ such that *supp* $\{q(r)\} = \Omega \subset \mathbb{R}^3$. Likewise, it is assumed that the observers are also distributed in Ω. With no loss of generality, it is assumed that $q(r) = \sum_{i=1}^{N} q_i \delta(r - r_i)$, where N is the number of degrees of freedom. The field due to this source constellation at any point $r \in \mathbb{R}^3$ is given by

$$\phi(r) = g(|r|) * q(r) = \sum_{i=1}^{k} g(|r - r_i|) q_i \,, \qquad (1)$$

where $g(|r|)$ is the appropriate Green's function, and * denotes a spatial convolution. It is apparent from this expression that the field evaluation scales as $O(N^2)$ for N observation points. Ideas introduced by [4] to ameliorate this cost for static problems relies on exploiting the fact that the field

at a point due to a cluster of sources is rank deficient, where the rank depends on the distance between the point and the center of the cluster. In other words, for a given accuracy, potential at an observation point sufficiently separated from a cluster of sources can be computed with few multipole expansions. Similarly, for given accuracy, few local expansions can be used to compute potential at a cluster of observation point due to a well-separated source. These ideas were cast in a more formal framework as tree-codes [5] and FMM [6]. At this point, we note that there is rampant confusion in terminology; both FMM and tree codes are used interchangeably. While the two methods are closely related, there are subtle but significant differences between the two [51]. Tree codes compute interactions between source pairs using one of three methods: (i) directly, (ii) evaluating fields at each observation point using multipole expansion due to a cluster of sources, or (iii) using local expansion at observation clusters to find fields. The decision on the operation used depends on which one is computationally efficient. On the other hand, the algorithmic structure of FMM enables the computation of potentials in an optimal manner [51]. Two additional operations that permit this are aggregation and disaggregation functions. These permit the computation of information at coarser (or finer) levels using information at finer (or coarser) levels. Thus, FMM relies on a hierarchical decomposition of the computational domain. This is achieved using the following strategy [8]; the computational domain Ω is embedded in a fictitious cube that is then divided into eight sub-cubes, and so on. This process continues recursively until the desired level of refinement is reached; an N_l-level scheme implies N_l -1 recursive divisions of the domain, see Fig. 1. At any level, the (sub) domain that is being partitioned is called the parent of all the eight children that it is being partitioned into. At the lowest level, all source/observers are mapped onto the smallest boxes. This hierarchical partitioning of the domain is referred to as a regular octtree data structure. Regular oct-tree representations are optimal only for geometries with uniform distribution [52]; non-uniform distributions can be represented using compressed oct-trees [39, 51]. In compressed oct-trees, sub-division of a domain is stopped when number of source/observer in that domain becomes less than a pre-fixed limit. While many algorithm exist for constructing a tree, the one that we have found to be efficient is the use of key data-structure to represent the nodes of a tree. In this approach the root box enclosing the entire geometry is represented with integer value 1; each of the eight (four) children of a parent is identified with a three (two) bit code which is appended to the parent box key to obtain their global unique key. Figure 2 shows an example compressed oct-tree where each box is represented using key-codes. This representation has several advantages: the nodes of tree at each level automatically follow Morton ordering and it plays an important role when partitioning the boxes among processors in parallel algorithm, all antecedents of a box and essential information like size of box, center position, level etc. can be readily recovered from its key-code using bit manipulations [38, 53, 54]. Mapping the computational domain onto a tree facilitates partition/classification of interactions as being either in the near or farfield. This is done using the following rule: at any level in the tree, all boxes/sub-domains are classified as being either in the near or far field of each other using the following dictum: two sub-domains are classified as being in the farfield of each other if the distance between the centers is at least twice the side length of the domain, and their parents are in the near field of each other; see Fig. 3 for an illustration of these classification. Once, the interaction list have been built for all levels, the computation proceeds as follows; at the lowest level, interaction between the elements of boxes that are in the near-field of each other is computed directly, i.e., using (1). All other interactions are computed using a three stage algorithm: (i) compute multipoles of sources that reside in each box; (ii) convert these to local expansion at all boxes that are in its far field; (iii) from the local expansion, compute the field at each observer. This simple three stage scheme is called a one-level scheme, and necessitates the development of theorems for (i) computation of multipoles at leaf (or smallest boxes), (ii) translate multipole expansion to local expansion and (iii) finally, aggregate the local expansions in a box to compute the field at all the observers. It is apparent that one can derive a more efficient computational scheme by embedding this scheme within itself as shown in Fig. 4. That is, if two sets of sub-domains that interact with each other are

sufficiently far away, then these clusters may be combined to form large clusters that then interact with each other at a higher level and so on; this is referred to here as a multilevel scheme. This implies that it is necessary to develop additional theorems that enable (i) shifting the origins of multipole so that effects of small clusters can be grouped together to form larger clusters and (ii) move the origin of local expansion so that expansions at the origin of the parent may be disaggregated to those of its children. In concert, these theorems enable one to traverse up and down the tree, and are presented next. This said the various steps involved in the hierarchical computing are shown in Algorithm 1.

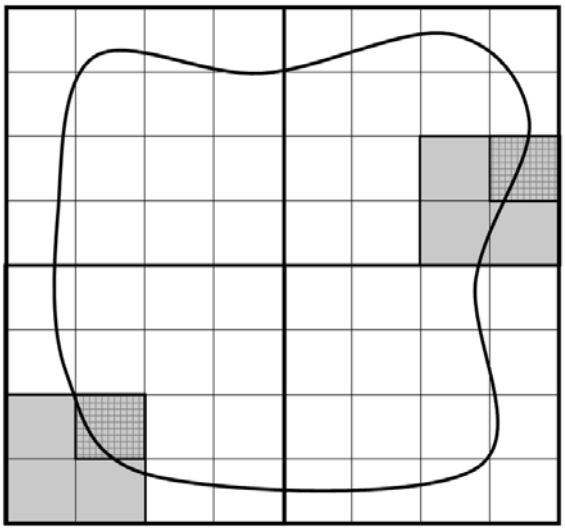

Fig. 1. Hierarchical decomposition of a 2D computational geometry.

Note that in single level algorithm the upward and downward tree traversal (steps 5 and 7) are absent. Next, we will detail these operations for different FMMs. Starting with well known static FMM to those for Helmholtz and finally to those for wideband FMM. Details are presented for the first two despite the fact that they are well known. The rationale for doing so is two fold (i) it is important to understand when FMM for Helmholtz fails and (ii) techniques developed for static FMM and some of the new FMM approaches find their way into the development of wideband FMM.

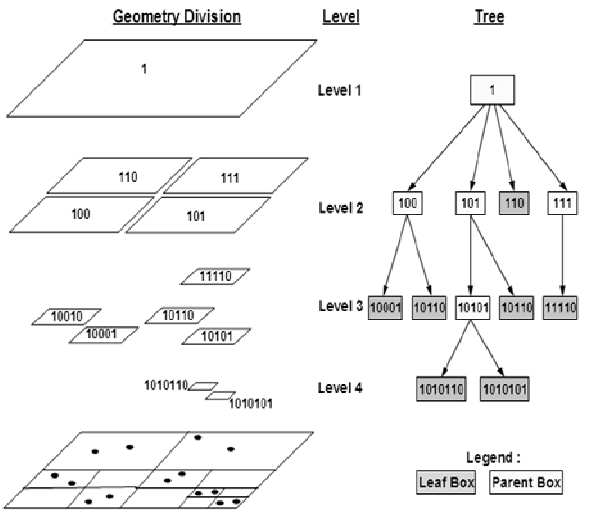

Fig. 2. Representation of 2D computational geometry using quad-trees. Boxes at different levels and corresponding nodes in tree are represented using binary keys.

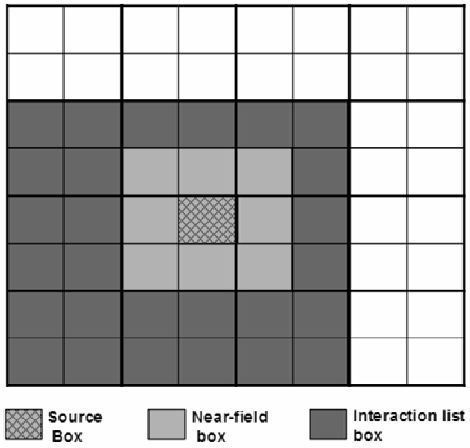

Fig. 3. Illustration of interaction list; dark boxes are contained in the interaction list of source box.

III. STATIC FAST MULTIPOLE METHOD

This section provides the appropriate theorems for fast evaluation of potential defined in terms of $g\left(|r|\right)=1/|r|$. Such potentials are commonly used in study of plasma dynamics, magnetostatic problems, eddy currents etc. While on first glance, one might be inclined to exclude methods developed for rapid evaluation of the Coulomb potential but these play an important role in developing fast methods for wideband problems.

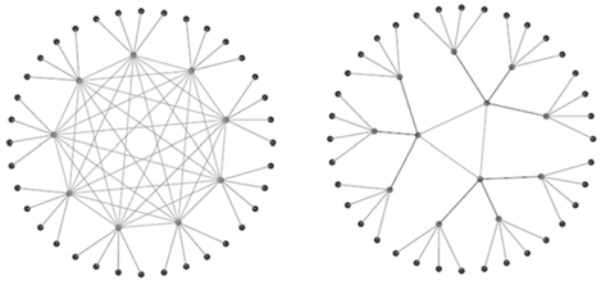

Fig. 4. Illustration of computational load in singleand multi-level FMMs. Dark nodes correspond to actual sources while light shaded nodes represent centers of multipole and local expansions.

Algorithm 1 Hierarchical computing

1: Construct the tree representation for the given geometry (distribution of discrete points).
2: Build interaction list using the above definition, for all boxes in the tree and the near-field list for leafless boxes.
3: **NF**: Use direct method for computation of nearfield potential at observation points in each leafless box from sources contained in its near-field boxes.
4: **S2M**: compute multipole expansions for each leafless boxes from sources contained within it.
5: **M2M** (upward traversal): for all parent boxes compute the multipole expansion by combining the multipole expansions at their children boxes.
6: **M2L** (translation): for all boxes in the tree convert the multipole expansions to local expansions about centers of boxes in their interaction list.
7: **L2L** (downward traversal): update the local expansion information at a child box using the local expansion of their parent box.
8: **L2O**: use the local expansions about each leafless box to compute the farfield potential at its observation points

A. Single Level Scheme

Consider two domains $\Omega_s \subset \mathbb{R}^3$. and $\Omega_o \subset \mathbb{R}^3$ that comprises of randomly located source and observer points, respectively. With no loss of generality, it is assumed that the number of sources and observers are *k*, these domains can be embedded in spheres of radius *a*. The centers of Ω_s and Ω_o are denoted by r_s and r_o, respectively. It is assumed that $\Omega_s \subset \bar{\Omega}_s$, and $\Omega_o \subset \bar{\Omega}_o$, and $\bar{\Omega}_s \bigcap \bar{\Omega}_o = \varnothing$, and the domains of Ω_s and Ω_o are sufficiently separated. In what follows, the domains $\bar{\Omega}_s$ and $\bar{\Omega}_o$ will be called parents of Ω_s and Ω_o, respectively. The parent domains can be embedded in a sphere of radius 2*a*, and their center are denoted by r_s^p and r_o^p, respectively. Next, we will present a single level FMM constructed using two methods; (i) spherical harmonics and (ii) Cartesian tensors.

1) Spherical harmonics: The theorems for a single and multilevel FMMs using spherical coordinates were introduced in a series of papers [7, 8], and have found extensive application in various disciplines; a sampling of these can be found in [6, 8, 19, 20, 55–57]. The genesis of the method is the well known generating function for Legendre polynomials [58],

$$\frac{1}{R} = \frac{1}{r\sqrt{1-2\frac{r'}{r}\cos\gamma+\left(\frac{r'}{r}\right)^2}} = \sum_{n=0}^{\infty}\frac{r'^n}{r^{n+1}}P_n\left(\cos\gamma\right) \tag{2}$$

with

$$\cos\gamma = \cos\theta\cos\theta' + \sin\theta\sin\theta'\cos\left(\phi-\phi'\right), \tag{3}$$

where $P_n(u)$ represents Legendre polynomial of degree *n*, $r' = \left(r',\theta',\phi'\right)$ and $r = \left(r,\theta,\phi\right)$. Legendre polynomials in (2) can be represented in terms of spherical harmonics $Y_{nm}(\theta, \phi)$ using the addition theorem [59],

$$P_n\left(\cos\gamma\right) = \sum_{m=-n}^{n} Y_{nm}^{*}\left(\theta,\phi\right)Y_{nm}\left(\theta',\phi'\right), \tag{4}$$

where the superscript * represents complex conjugate. Using (4) in (2) results in complete separation of source and observation quantities,

$$\frac{1}{R}=\sum_{n=0}^{\infty}\sum_{m=-n}^{n} r'^{n} Y_{nm}^{*}(\theta',\phi')\frac{Y_{nm}(\theta,\phi)}{r^{n+1}}. \qquad (5)$$

These expressions enable the derivation of the following theorems necessary for steps 4, 6 and 8 in Algorithm 1.

Theorem 3.1: *Multipole Expansion (S2M), spherical*: Let k charges of strengths $\{q_i, i=1,...,k\}$ be located at $r_i \in \Omega_s$ with $|r_i - r_s| < a$. Then for any $r \in \Omega_o$, the potential ϕ is given by,

$$\phi(r)=\sum_{n=0}^{\infty}\sum_{m=-n}^{n} M_n^m \frac{Y_{nm}(\theta,\phi)}{|r-r_i|^{n+1}}, \qquad (6)$$

where

$$M_n^m=\sum_{i=1}^{k} q_i |r_i-r_s|^n Y_{nm}^{*}(\theta',\phi'), \qquad (7)$$

where the parameters $\{\theta_i, \phi_i\}$ and $\{\theta, \phi\}$ are spherical coordinates of r_i and r w.r.t the origin at r_s. In Theorem 3.1, M_n^m is the multipole expansion at r_s constructed from the source quantities $q_i(r_i)$. Proofs for the error bounds in the above expressions can be obtained from [8, 9]. Next, these multipoles are translated from r_s to r_o.

Theorem 3.2: *Multipole to Local Translation operator* (*M2L*), *spherical*: Given a multipole expansion O_n^m about r_s, it can be mapped to local expansion L_n^m at r_o using

$$L_i^k=\sum_{n=0}^{\infty}\sum_{m=-n}^{n} \times \frac{O_n^m (-j)^{|k-m|-|m|-|k|} A_n^m Y_{i+n}^{m-k}(\theta,\phi)}{(-1)^n A_{i+n}^{m-k} |r_s-r_o|^{i+n+1}}, \qquad (8)$$

where $\{\theta, \phi\}$ are the spherical coordinates of the r_s w.r.t r_o, and $A_n^m=\dfrac{(-1)^n}{\sqrt{(n-m)!(n+m)!}}$.

Finally, the local expansions at any leaf node may be mapped onto the observers using the theorem presented next.

Theorem 3.3: *Local Expansions to Observer* (*L20*), *spherical*: The potential at a point $r \in \Omega_o$ due to local expansion L_n^m about origin is given by,

$$\phi(r)=\sum_{n=0}^{\infty}\sum_{m=-n}^{n} L_n^m |r-r_o|^n Y_{nm}(\theta,\phi). \qquad (9)$$

As before, the parameters $\{\theta, \phi\}$ are the spherical coordinates of r with respect to the origin at r_o. The above theorems, in a one level setting, permit the rapid computation of potentials at all points in Ω_o due to sources in Ω_s. It is evident that this scheme can be embedded within itself to create a multilevel scheme. But prior to doing so, it is instructive to re-examine the fundamentals of FMM from a Cartesian perspective.

2) Cartesian Tensors: While FMMs that were based on spherical harmonics and Cartesian tensors were introduced approximately at the same time [10], the latter did not receive much recognition as it was perceived to be more expensive and cumbersome. But, these expansions were used extensively in developing tree-codes [60], as well as FMM type algorithms for magnetostatics [12] and potentials of the form $R^{-\nu}$ [61, 62]. Our rationale for including this approach here is that there is an intimate relationship between spherical harmonics and the Cartesian tensors, and these connections are well known and have been explored extensively (as early as Maxwell!); see [63–65] and references therein. The following statements hold true: (i) components of a traceless tensor of rank n serve as constant coefficients in a spherical harmonic of degree n, and (ii) there is a class of traceless tensors of rank n whose components are n-degree spherical harmonics functions of x, y, z. Indeed, recurrence relationship that were conjectured for translating multipole expansions [12] can be rigorously derived using traceless tensors. Therefore, it stands to reason that the two seemingly disparate methods should have identical cost structure. In what follows, we shall briefly present theorems that permit an FMM algorithm using Cartesian tensors. Proofs for theorems presented here, and the myriad advantages of this method are detailed in [13].

In what follows, we will denote an nth rank tensor using the notation $\mathrm{A}^{(n)}$, and as is well known, it comprises of 3^n components and may be expressed in component form as $A_{\alpha_1,\ldots,\alpha_n}$. If the tensor is totally symmetric, i.e., its value is not altered with permutation of indices, then it

contains only $(n + 1)(n + 2)/2$ independent components. A n-fold contraction between an $(n + m)th$ rank tensor and $(n)th$ results in an $(m)th$ rank tensor and is denoted using $C^{(m)} = A^{(n)} \bullet n \bullet r^{(n)}$. Any homogeneous polynomial in r can be written in terms of tensors as $f(r) = A^{(n+m)} \bullet n \bullet B^{(n)}$, and if $A^{(n)}$ is totally traceless, then $f_n(r)$ is a solid spherical harmonic of degree n. Here, r^n is a polyadic. These concepts are illustrated using Taylor's expansion of the Green's function as

$$\begin{aligned}\frac{1}{R} &= \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} r'^n \bullet n \bullet \nabla^n r^{-1} \\ &= \sum_{n=0}^{\infty} \frac{1}{n!} \frac{r'^n}{r^{n+1}} \hat{r}'^n \bullet n \bullet D_n \hat{r}^n , \\ &= \sum_{n=0}^{\infty} \frac{r'^n}{r^{n+1}} P_n\left(\hat{r}' - \hat{r}\right)\end{aligned} \tag{10}$$

where D_n is called the detracer which extracts the traceless component of any tensor $A^{(n)}$. The equivalence between (2) and (10) is readily apparent as

$$P_n\left(\hat{r}' - \hat{r}\right) = \frac{1}{n!} \hat{r}'^n \bullet n \bullet D_n \hat{r}^n . \tag{11}$$

It should be noted that a traceless tensor contains only $(2n + 1)$ components. This mathematical apparatus is critical to theorems necessary to traverse the tree, and these are enumerated next.

Theorem 3.4: *Multipole Expansion* (*S2M*), *Cartesian:* The total potential at any point $r \in \Omega_o$ due to k sources q_i, $i = 1,\ldots, k$ located at points $r_i \in \Omega_s$ is given by

$$\begin{aligned}\phi(r) &= \sum_{n=0}^{\infty} M_t^{(n)} \bullet n \bullet \nabla^n \frac{1}{|r - r_s|} \\ &= \frac{1}{(2n-1)!!} D_n M^{(n)} \\ &= \sum_{n=0}^{\infty} (-1)^n \frac{q_i}{n!} (r - r_s)^n ,\end{aligned} \tag{12}$$

where (2n - 1)!! denotes a double factorial. This theorem is derived using a Taylor expansion of the potential function. It follows that a similar expansion can be used again to map these multipoles that exist at r_s to local expansions at r_o.

Theorem 3.5: *Multipole to Local* (*M2L*), *Cartesian:* Assume that the domains $\bar{\Omega}_s$ and $\bar{\Omega}_o$ are sufficiently separated, and the distance between their centers, $r_{os} = |r_{os}| = |r_o - r_s|$, is greater than $diam\{\Omega_s\}$ and $diam\{\Omega_o\}$. If a traceless multipole expansion $M_t^{(n)}$ for all n is located at r_s, then another expansion $L_t^{(n)}$ that produces the same field $\forall r \in \Omega_o$ is given as

$$L_t^{(n)} = \sum_{m=n}^{\infty} \frac{1}{n!} \nabla^n \frac{1}{r_{os}} \bullet (m - n) \bullet M_t^{(m-n)} , \tag{13}$$

where $\nabla^n r^{-1} = (-1)^n r^{-2n-1} D_n r^n$. This expression permits simple computation of derivatives and is a generalization of the formulae provided in [12]. Finally, as in the spherical case, the final step is mapping this local expansion onto the observers. This can be accomplished by exploiting Taylor expansion, and results in the following theorem.

Theorem 3.6: *Local to Observer* (*L2O*), *Cartesian*: Given a local expansion $L_t^{(n)}$ that exist in the domain Ω_o centered around r_o, it can be shifted to any point $r \in \Omega_o$ using

$$\phi(r) = \sum_{m=0}^{\infty} (r - r_0^c)_t^m \bullet m \bullet L_t^{(m)} . \tag{14}$$

These theorems indicate that the classical FMM can be expressed as a cascaded series of Taylor's expansion. And when properly formulated/ structured, it has identical computational complexity as the original FMM [13]; a consequence of the fact that traceless rank n tensors in the above expressions contain only $(2n + 1)$ independent components. It is also apparent that FMM-type algorithm can be developed without resorting to traceless tensors. The advantage of such representation is elaborated in [13] and will be detailed in the next section together with methods necessary for creating a multilevel algorithm. Finally, cost of the single level algorithm is computed in the following manner. Let s be the average number of source/observation points in each leafless box and P be the maximum order of harmonics used is above expansions. Then the cost of creating P^2 multipole coefficients from sources (in S2M) and computing potential from local expansions (in L2O) scales $O(P^2N)$; cost of translating multipole to local expansions (in M2L) scales as $O(N^2/s^2P^4)$

and $O(N_s)$ is the cost for direct evaluation for nearfield. It can be shown that for optimal s, the overall cost scales as $O(N^{4/3}P^{4/3})$.

B. Multilevel FMM algorithm

It is apparent that the $O(N^{4/3})$ cost of single level algorithm can be further reduced by embedding this scheme within itself, as is evident from Fig. 4. To implement such a scheme it is necessary to develop methods that enable one to construct multipole expansions at a parent level from those at their children. These are effected using the following theorems.

Theorem 3.7: *Multipole to Multipole (M2M), spherical*: A multipole expansion O_n^m about r_s can be mapped onto one that exists around r_s^p using

$$M_i^k = \sum_{n=0}^{i}\sum_{m=-n}^{n}(-j)^{|k|-|m|-|k-m|}O_{i-n}^{k-m} \times \frac{A_n^m A_{i-n}^{k-m}\left(r_s^{cp}\right)^n Y_{nm}^*(\theta,\phi)}{A_i^k}, \quad (15)$$

where $r_s^{cp} = \left|r_s^{cp}\right| = r_s - r_s^p$, and $\{\theta, \phi\}$ are the polar coordinates of r_s w.r.t. r_s^p.

Theorem 3.8: *Local to Local* (*L2L*), *spherical*: Given a local expansion O_n^m about r_o^p, it can be mapped to one around r_o using

$$L_i^k = \sum_{n=i}^{p}\sum_{m=-n}^{n}(-j)^{|m|-|k|-|m-k|}O_n^m \times \frac{A_{n-i}^{m-k}A_i^k\left(r_o^{cp}\right)^{n-i}Y_{n-i}^{m-k}(\theta,\phi)}{(-1)^{n+i}A_n^m}, \quad (16)$$

where $r_o^{cp} = \left|r_o^{cp}\right| = r_o - r_o^p$, and $\{\theta, \phi\}$ are the polar coordinates of r_o w.r.t. r_o^p. The equivalent theorems for Cartesian expansion likewise follow.

Theorem 3.9: *Multipole to Multipole (M2M), Cartesian:* A traceless multipole tensor $O_t^{(m)}$ at r_s is related to $M_t^{(m)}$ that is centered at r_s^p via

$$M_t^{(m)} = \sum_{n=0}^{m}\frac{(-1)^n}{n!}\frac{D_n\left(r_s^{pc}\right)^n}{(2n-1)!!}O_t^{(m-n)}, \quad (17)$$

where $r_s^{cp} = r_s^p - r_s$.

Theorem 3.10: *Local to Local* (*L2L*), *Cartesian*: Given a local expansion $O_t^{(n)}$ that exist in the domain $\bar{\Omega}_o$ centered around r_o^p, it can be shifted to the domain Ω_o centered around r_o using

$$L_t^{(m)} = \sum_{n=0}^{\infty}\binom{m+n}{m}O_t^{(m+n)} \bullet m \bullet (r_o^{cp})_t^n, \quad (18)$$

where $r_o^{cp} = r_o - r_c^p$.

These theorems, in concert, permit traversing up and down the oct-tree, see figure 5. While these theorems are the bare-bones presentation of the steps required, there have been several attempts to make these more efficient [7, 8, 14, 66]. As both methods are based on Taylor expansions the upperbounds in using these approaches can be readily derived. Such a derivation is presented in [8, 13]. Alternatively, another interesting algorithm was introduced in [13] that permits exact evaluation of the multipole expansion at the parent given the multipole expansion at the children–this has been shown both analytically and numerically for different potential functions. However, in order to get this exact expression, one has to abandon the use of traceless tensors. It follows that the cost of using exact multipole to multipole translations is higher. But in our experience, we have found that we need a smaller number of multipoles for the same precision, and this can significantly affect the total cost, especially for large data sets [13]. Abandoning the use of traceless operators has three salient benefits; (i) the algorithms can be used for any potential function whose Taylor's series converges rapidly, (ii) it does not depend on special functions and (iii) only the translation operator depends on the potential function which implies that multiple potentials may be easily combined [67].

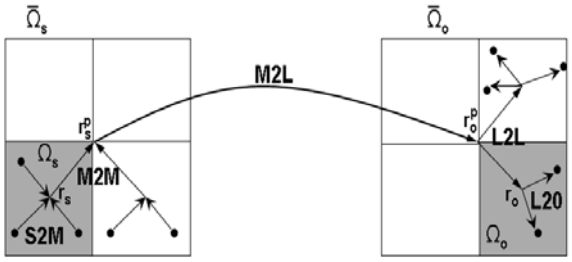

Fig. 5. Various operators involved in a multilevel FMM.

In all the above expressions, it was assumed that the number of multipoles used was infinite. The analytical estimates regarding truncation of this sum for both the spherical and Cartesian form can be found in [8, 13]. The cost analysis for multilevel approach is as follows: the total number of boxes in the tree is $O(N/s)$ and the cost for S2M and L2O operations remains the same; the cost of applying M2L translation operation across levels scales as $O(P^4N/s)$. In addition the cost of applying M2M and L2L operations for all boxes scales as $O(N/sP^4)$. Thus, the overall computational cost associated with both schemes scales as $O(P^4N)$. This cost is largely dominated by the time for multipole to local translation (M2L) and considerable research effort has been expended on reducing this cost. A closer examination of the M2L operation reveals that (i) the number of translations per box is 189 and (ii) the cost per translation scales as $O(P^4)$. The latter is due to the fact that this operation is not diagonal. Greengard et. al. [9] remedied this deficiency by introducing a novel algorithm that diagonalizes the translation operator. Additional modifications to the overall algorithm introduced there [42, 68] further reduces the number of translations, making the "revamped" FMM extremely efficient. Ideas behind this diagonalization can be exploited by either both varieties of FMM; spherical and Cartesian. It also plays a key role in FMMs for lowfrequency, and consequently, will be presented in some detail next. An FFT based implementation of above un-diagonalized form results in a overall cost that scale as $O(NP^2 \log P)$ [66], but will not be dwelt here.

C. Diagonalized Translation Operators

A diagonal translation operator may be derived using a spectral representation of the Green's function [9], viz.,

$$\frac{1}{R} = \frac{1}{2\pi}\int_0^\infty d\lambda\, e^{-\lambda z}\int_0^{2\pi} d\alpha\, e^{-j\lambda(x\cos\alpha + y\sin\alpha)} \quad (19)$$

for $z > 0$. It is apparent that the inner integral is in fact a zeroth order Bessel function. The computation of potentials using the above expressions hinge on the existence of an integration rule that is efficient to a given precision and scale invariant if this formula is to be used at different levels in the FMM tree. Given the existence of such a rule [69], the potential at any point can be written as [9]

$$\phi(r) = \sum_{k=1}^{s(\varepsilon)}\sum_{i=1}^{M(k)} W(k,i)\, e^{-\lambda_k z - j\lambda_k(x\cos\alpha_i + y\sin\alpha_i)} + O(\varepsilon), \quad (20)$$

where the coefficients $W(k, i)$ are a combination of the charges q_i and integration weights ω_k, $s(\varepsilon)$ and $M(k)$ denotes the number of integration points for ε accuracy. Evidently, in above discrete representation, the number of integration points $M(k)$ for evaluating α integral depends on k to account for the varying bandwidth, λ_k, of its integrand. The advantages of above scheme are immediately apparent in that it readily permits translation of the origin; translation of the origin is quite simply a shift in the exponentials. The similarity between (20) and those in Theorems (3.1), and (3.4) are readily apparent. The mapping from spherical harmonic multipole coefficients M_n^m onto exponential expansions $W(k, j)$ is given as [9],

$$W(k,i) = \frac{\omega_k}{M(k)}\sum_{m=-\infty}^{\infty}\sum_{n=|m|}^{\infty}(j)^{|m|} e^{-jm\alpha_i} \times \frac{M_n^m \lambda_k^n}{\sqrt{(n-m)!(n+m)!}}, \quad (21)$$

and given $W(k, i)$ coefficients the spherical harmonic local expansion L_n^m can be computed with,

$$L_n^m = \frac{(j)^{|m|}}{\sqrt{(n-m)!(n+m)!}} \sum_{k=1}^{s(\varepsilon)}(-\lambda_k)^n \sum_{i=1}^{M(k)} W(k,i) e^{-jm\alpha_i}. \quad (22)$$

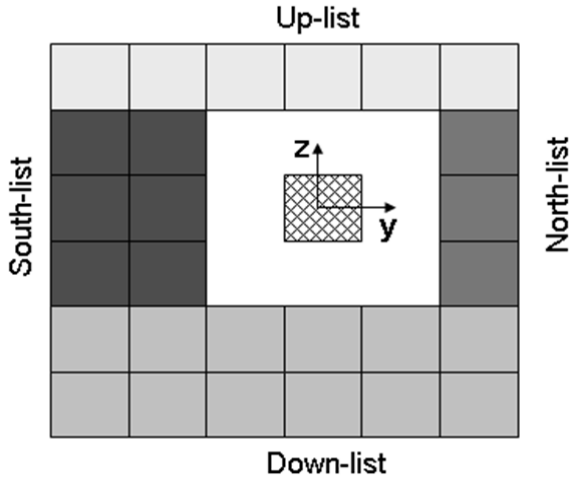

Fig. 6. Re-grouped boxes in original interaction list, in figure 3, for application of diagonal translation operator (20).

The multipole to local translation operation, with diagonalized translation forms, can be computed as a three stage process: multipole coefficients are mapped to $W(k, i)$, translate $W(k, i)$, and then map the translated coefficients back to local expansions, and then proceed as usual. It is evident that cost of all operators involving exponential expansions scale as $O(P^2)$. Various symmetry considerations in implementation reduces the number of total translation count from 189 to 40. Additionally, one can exploit symmetry in the expressions involved to further reduce the overall cost, if not the asymptotic complexity [56]. Thus, properly modifying and augmenting either spherical or Cartesian multipole based algorithms with plane wave translation operators can considerably ameliorate the cost. However, a couple of issues must be noted; (i) the plane wave expression is valid for $z > 0$, this implies that the interaction list must be modified [9]; (ii) additional operators must be introduced to rotate the multipole operators along the required axis; (iii) the operator developed should be scale invariant for the scheme to be efficient. In implementation the spherical harmonic multipole coefficient is converted into six plane wave expansions corresponding to each face of the cube and the interaction list definition is changed accordingly. For example, exponential expansions corresponding to $+z$ cube face is valid only for boxes present above x-y plane, as illustrated in Fig. 6. Boxes in original interaction list are divided into six new sets termed as up-list, down-list, north-list, south-list, east-list and west-list corresponding to $+z$, $-z$, $+y$, $-y$, $+x$ and $-x$ cube faces respectively [9]. Overall, the diagonalized version of the translation operator reduces both the total number of translation operation and per translation cost leading to a much faster algorithm. This approach is very similar to spectral approaches developed for alternative derivation of Helmholtz FMM [42, 70] and is the crux of many methods developed for wideband FMM.

IV. FMM FOR HELMHOLTZ EQUATIONS

Thus far, we have seen that cascaded Taylor expansions can be used to develop static FMM. While these ideas are readily extended to the solution of parabolic equations as well [24], they are not readily extendable to Helmholtz equation kernels, especially at high frequencies. Furthermore, as was evident from last section, the scheme developed should be diagonal. Consider a problem setting that is identical to what was described in Section II. We shall seek development of methods to accelerate the evaluation of the potential integral in (1) with $g\left(|r|\right) = \exp\left[-j\kappa|r|\right]/|r|$. One expression that readily suggests itself is the Gegenbauer addition theorem [31, 59, 71],

$$\frac{e^{-j\kappa|X+d|}}{|X+d|} = -j\kappa\sum_{l=0}^{\infty}(-1)^l(2l+1)jl(\kappa d)h_l^{(2)}(\kappa X)P_l\left(\hat{d}\cdot\hat{X}\right), \tag{23}$$

where X and d are position vectors such that r = X+d and $|X| = |d|$, j_l and $h_l^{(2)}$ are l^{th} order spherical Bessel and Hankel function of second kind, $X = |\mathrm{X}|$ and $d = |\mathrm{d}|$. Augmenting this theorem with another addition theorem for Legendre polynomials in (4) completes the separation between the source and observer coordinates.

$$\frac{e^{-j\kappa|X+d|}}{|X+d|} = -j\kappa\sum_{l=0}^{L}(-1)^l(2l+1)jl(\kappa d)h_l^{(2)}(\kappa X), \quad (24)$$
$$\times\sum_{m=-l}^{l} Y_{lm}(\theta_X,\phi_X)Y_{lm}^*(\theta_d,\phi_d)$$

where L is the number of terms used in the summation, $\{\theta_X, \phi_X\}$ and $\{\theta_d, \phi_d\}$ are the polar coordinates of $\hat{X}$ and $\hat{d}$ respectively. It is evident that one may use a sequence of addition theorems to create hierarchical computational methodology. However, the principal bottleneck to such a scheme is the fact that the operators involved are not diagonal. However, diagonal operators are easily developed by recognizing that

$$4\pi(-j)^l\, jl(\kappa d)P_l(\hat{d}\cdot\hat{X}) = \int d^2\hat{\kappa}e^{-j\kappa.d}P_l(\hat{\kappa}\cdot\hat{X}), \quad (25)$$

where $d^2\hat{\kappa} = \sin\theta d\theta d\phi$ and $\kappa = \kappa\hat{\kappa}$.. The relation (25) can be derived from well known orthogonality relation among spherical harmonics and expansion for plane waves given as.

$$\int d^2\hat{\kappa}Y_{lm}(\hat{\kappa})Y_{l'm'}^*(\hat{\kappa}) = \frac{4\pi}{2l+1}\delta_{mm'}\delta_{ll'}, \quad (26)$$

$$e^{-jz\cos\gamma} = \sum_{l=0}^{\infty} jl(z)P_l(\cos\gamma). \quad (27)$$

Substituting (25) in (23), interchanging the summation and the integral, and truncating the summation over l yields the final diagonalized form,

$$\frac{e^{-j\kappa|X+d|}}{|X+d|} = \frac{-j\kappa}{4\pi}\int d^2\hat{\kappa}e^{-j\kappa.d} \times\sum_{l=0}^{L}(-1)^l(2l+1)h_l^{(2)}(\kappa X)P_l(\hat{\kappa}\cdot\hat{X}). \quad (28)$$

Several derivation that result in above diagonalized form exist and are based on different set of starting formulas [30–32, 72, 73]. First scheme for diagonalizing (23) was presented in [30] with the use of forward and inverse far field transform defined as,

$$\tilde{f}(\theta,\phi) = \sum_{l=0}^{\infty}\sum_{m=-l}^{l}(-j)^l Y_{lm}(\theta,\phi)f_{lm}, \quad (29)$$

$$f_{lm} = \int d^2\hat{\kappa}(-j)Y_{lm}(\hat{\kappa})\tilde{f}(\theta,\phi). \quad (30)$$

Above definition is a simple spherical harmonic transform from $\hat{\kappa}$ to $\{l, m\}$ basis with direct analogy to Fourier transform. A simpler version of derivation in [30] is presented in [35, 73]. In [31], the expansion in (23) is represented as matrix vector multiplication which reveals a convolution relation in indices $\{l, m\}$. Such convolutions in $\{l, m\}$ can be computed as one-to-one multiplication in $\hat{\kappa}$ domain using the far field transform [31]. A detailed discussion on deriving the above diagonalized forms from the convolution representation of original multipole expansion for both Laplace and Helmholtz equation is presented in [31]. An alternate derivation based on similarity transform and their relation to group theory is presented in [32] to yield the same expansion in (28).

1) Single Level FMM: As before, assume that Ω_s and Ω_o denote the source and observation domain, and it is necessary to find the fields $\forall r \in \Omega_o$. It is further assumed that the domains are cubes, in keeping with the data structure of oct-tree and that each domain can be embedded in a sphere of radius a. Furthermore, the clusters are assumed to be well separated. The separation distance is closely related to error bounds [30, 71], and will be dealt with in later part of the paper. Given these conditions, traversal up and down the tree is effected using the following set of theorems:

Theorem 4.1: *Farfield signature*: The far field signature due a set of source q_i for i = 1, … , k located at $r_i \in \Omega_s$ is given by

$$M(r_s,\kappa) = \sum_{i=1}^{k} M(q_i,\kappa,r_s-r_i) = \sum_{i=1}^{k} q_i\exp\left[-j\kappa.(r_s-r_i)\right] \quad (31)$$

Theorem 4.2: *Translation operator*: If a farfield signature exists at a point r_s such that it is valid for all points outside the domain Ω_s, then the translation operator that maps this farfield to the local expansion that is centered around r_o and valid in the domain Ω_o is given by

$$T\left(\kappa, r_{os}\right) = \sum_{l=0}^{\infty} (-1)^{l} \left(2l+1\right) h_{l}^{(2)}\left(\kappa \left| r_{os} \right|\right) P_{l}\left(\hat{\kappa}.\hat{r}_{os}\right), \quad (32)$$

where $r_{os} = r_o - r_s$. Finally, the potential at any point $r \in \Omega_o$ can be constructed using

$$\phi(r) = \frac{-jk}{4\pi} \int d^2\hat{\kappa} M\left(1, -\kappa, r_o - r\right) \times T\left(\kappa, r_{os}\right) M\left(r_s, \kappa\right), \quad (33)$$

While these equations are readily derived from (28). More insight into the derivation of these equations can be obtained by realizing that the farfield (and local expansions) can be represented in terms of spherical harmonics. In turn, this interpretation leads to expressions that reveal convergence rates of these and error bounds as a function radius a and the separation distance. More importantly, this insight leads to the type of quadrature rules that must be used to implement these schemes numerically. In other words, the continuous integral is evaluated using

$$\phi(r) = \frac{-jk}{4\pi} \sum_{p=1}^{L} \sum_{q=-p}^{p} \omega_{pq} M\left(1, -\kappa, r_o - r\right) \times T\left(\kappa_{pq}, r_{os}\right) M\left(r_s, \kappa_{pq}\right), \quad (34)$$

where L is the order of the Gauss Legendre rule, ω_{pq} are the integration weights, p and q are the integration points in θ and ϕ axis,

$$\phi_q = \frac{2\pi q}{2L+1}$$

$$\theta_p \text{ is the } (p+1)^{th} \text{ zero of } P_{L+1}(\cos\theta)$$

$$\omega_{pq} = \frac{4\pi\left(1-\cos^2\theta_p\right)}{\left(2L+1\right)\left[\left(L+1\right)P_L\left(\cos\theta_p\right)\right]^2}$$

$$\hat{\kappa}_{pq} = \hat{x}\sin\theta_p\cos\phi_q + \hat{y}\sin\theta_p\sin\phi_q + \hat{z}\cos\theta_p \,. \quad (35)$$

As is apparent form the above equations, uniform sampling is used to evaluate the integral along ϕ. Other applicable rules may be found in [74]. We have yet to elaborate the underlying factors that decide the order of Gauss-Legendre rule that is used along θ. A number of formulae exist for choosing the number of Gauss-Legendre quadrature point [30, 71, 75]. However, examination of (28) yields interesting insight. If only the exponential terms are considered in this integral, it is apparent that these expressions can be represented using L = $O(\kappa d)$ = $O(2\kappa a)$ harmonics. This, in turn, implies that the summation is also truncated using L terms. Though the reasoning here is based on economical means to discretize the integral a deeper reason, arriving at same conclusion, exists for choice of L based on original multipole expansion [71]. Choice of L should be large enough for the series (24) to converge, but not too large to cause numerical instability due to the asymptotic behavior of spherical Bessel and Hankel functions. Given that only a finite number of terms are being used, one can explicitly derive error bounds that, in turn, depend on the translation distance also [30]. Deriving rigorous error bounds has been a focus of considerable work [43, 75–78], and the behavior of error is well understood [79, 80] as are the means to overcome these. A simple choice for truncation limit L applicable to most practical problems is,

$$L = \kappa d + C\log\left(\kappa d + \pi\right), \quad (36)$$

where C is a number that depends on the desired accuracy ε; typically for ε = {10^{-3}, 10^{-6}, 10^{-14}} the choice for C = {3, 5, 10}, respectively [71, 81]. This estimate is semi-empirical and assumes that the two boxes are well separated if they are one box apart. Other estimates [75, 79, 82] based on approximation of Bessel and Hankel function exists both in two- and three-dimensions and can account for multiple box separation between interacting boxes [78, 80]. Cost of this scheme can be computed in the same manner as in the static with $P = L$ and the diagonalized form of translation operator implies $O(P^2)$ cost per operation. However choice of L depends on size of box kd, which in turn dictates the number of unknowns per box s (assuming uniform discretization). It can be show that the optimal cost of the above scheme scales as $O(N^{3/2})$ for surface problems.

2) Multilevel FMM: While the above exposition details the necessary mathematics for implementing a single level scheme, nesting these in a hierarchical setting is the next logical extension. The first robust attempts to do so are [83–85]. Extension to multilevel is different from that encountered for the Laplace FMM; there, the number of multipoles at all level of the tree was

constant. But as is evident from (36) and (34) as the size of the source/receiver boxes increases, the bandwidth increase increases by a factor of two, which implies that the number of directions increase by a factor of four. This then creates a need for developing robust methods for going up and down the tree for the stages of aggregation and disaggregation. These operators can be thought of as filters. But before we proceed into intricate details of the methods to implement these, the theorems that help achieve these are as follows:

Theorem 4.3: *Translation of farfield signatures*: If the farfield signature $M(r_s, \kappa)$ around the point $r_s \in \Omega_s$ is known, then the farfield signature $M\left(r_s^p, \kappa\right)$ around the point $r_s^p \in \Omega_s^p$ is given by

$$M\left(r_s^p, \kappa\right) = M\left(r_s, \kappa\right) e^{-j\kappa.\left(r_s^p - r_s\right)}. \tag{37}$$

An identical theorem for can be derived for translating local expansion at the parent level to that of its child. Numerical implementation of these theorems is not as simple as it seems. To maintain uniform accuracy across levels, employing (36), the L for parent is approximately twice that of its child. This implies that the number of direction for parent box is approximately four times that of its child; thus the multipole expansions for the child and parent box are defined on different grids. This process of computing a higher bandwidth representation from lower bandwidth farfield signature is referred to as interpolation and anterpolation is its inverse analogue applied during downward tree traversal. Implementing the above theorems calls for efficient methods to interpolate (or anterpolate). Several methods that exist have been elaborated upon in [33] and summarized as well in [75]. An efficient and exact algorithm can be devised using the forward and inverse farfield transform for both interpolation and anterpolation [23, 35, 85, 86]. This algorithm relies on the fact that at any level the farfield signature can be represented in terms of spherical harmonics., viz.

$$M\left(.,\kappa\right) = \sum_{n=0}^{\infty} \sum_{m=-n}^{n} a_{nm} Y_{nm}\left(\theta, \phi\right). \tag{38}$$

As is well known, the farfield signature of a source constellation is bandlimited to $O(\kappa a)$ harmonics. This implies that the above expression can be truncated. Furthermore, since an L[th] order rule is chosen to evaluate the spectral integral in (28), it follows that the upper limit in the summation over n can be chosen to be L. This said, direct computation of a_{nm} is expensive. Alternate methods both exact and approximate have been discussed in [23, 87]. Consider the computation of a_{nm} from child farfield signature $M(r_s, \kappa_{kp})$ represented using $(2L^2 + 1)$ coefficients, i.e. $p = 1, \ldots, L$ and $q = 1, \ldots, (2L + 1)$,

$$\begin{aligned} a_{nm} &= \int d^2\hat{\kappa} M\left(r_s, \kappa\right) Y_{nm}^*\left(\theta, \phi\right) \\ &= \sum_{p=1}^{L} \omega_p P_{nm}\left(\cos\theta_p\right) \left(\sum_{q=1}^{(2L+1)} M\left(r_s, \kappa_{pq}\right) e^{jn\phi_q} \right) \\ &= \sum_{p=1}^{L} \omega_p P_{nm}\left(\cos\theta_p\right) \alpha_m\left(\theta_p\right) \end{aligned} \tag{39}$$

where ω_p are numerical quadrature weights. Since, the integration along ϕ is performed using uniform sampling, fast Fourier transform (FFT) can be used for summation inside the brackets. These coefficients are then used to compute samples along new polar coordinates $\left(\tilde{\theta}_p, \tilde{\phi}_q\right)$ with $p = 1, ..., \tilde{L}$ and $q = 1, ..., \left(2\tilde{L} + 1\right)$ as,

$$M\left(r_s, \tilde{\kappa}_{pq}\right) = \sum_{m=-\tilde{L}}^{\tilde{L}} e^{-jm\tilde{\phi}_q} \sum_{n=1}^{\tilde{L}} a_{nm} P_{nm}\left(\cos\tilde{\theta}_p\right). \tag{40}$$

Again, FFT can be used to evaluate the outer summation. In interpolation, $\tilde{L} > L$ to accommodate for the increase in bandwidth and $\tilde{\kappa}_{pq}$ represents the discrete directions of the farfield signature corresponding to the parent. The required multipole coefficients about parent origin r_s^p can be obtained using a simple shifting operation,

$$M\left(r_s^p, \tilde{\kappa}_{pq}\right) = M\left(r_s, \tilde{\kappa}_{pq}\right) e^{-j\tilde{\kappa}_{pq}.\left(r_s^p - r_s\right)}. \tag{41}$$

An inverse procedure is performed when translating local expansions from parent to child where anterpolation is used in place of interpolation. First, the parent local expansion about r_o^p is shifted about child origin r_o; then in anterpolation, the forward and inverse farfield transform are performed to reduce the bandwidth in an exact manner as described above but with $\tilde{L} < L$, where L represents the number of

harmonics in parent domain. Above procedure for interpolation/anterpolation can be further accelerated with the use of fast Legendre transform [23] where the coefficients a_{nm} are not computed explicitly. Though this approach scales favorably the break-even point is large and not suitable for most practical applications [35]. This can be overcome to some extent using the 1D FMM for fast Legendre transforms [87]. Cost of Interpolation/anterpolation using this approach scales as $O(Q\log Q)$, where Q denotes the number of directions in farfield signature. This said it can be shown that overall cost of the multilevel algorithm scales as $O(N \log^2 N)$ [35]. Other methods used for interpolation and anterpolation have been presented in detail in [33, 75, 84]. These include the use of polynomials and approximate prolate spheroidal wave functions. The singular advantage of these methods is their cost scales linearly with the number of samples, thus the overall cost scales as $O(N \log N)$. However, while interpolation is sufficiently accurate, one has to be more careful when anterpolating functions as it is necessary to remove higher order harmonics. While we have not digressed into implementation of these schemes for vector electromagnetic problems, we must caution that it is not a trivial extension. It is important to realize that the farfield component represented in terms of polar components in not bandlimited [88], whereas they are bandlimited when represented in terms of Cartesian components. This means that one either uses a fast scheme based on vector spherical harmonics [88] or converts these to Cartesian before interpolation/anterpolation. Another intriguing method for interpolation and anterpolation was introduced by Sarvas [48], wherein he introduced modifications that enabled the use of FFTs. In other words, bandlimited farfield signatures can be represented in terms of Fourier basis as

$$M(\kappa) \approx \sum_{p=-P}^{P} \sum_{q=-Q}^{Q-1} a(p,q) e^{i(p\theta+q\phi)}, \quad (42)$$

where,

$$\begin{aligned} a(p,q) &= DFT\{M(\kappa)\} \\ &= \sum_{p=-P}^{P} \sum_{q=-Q}^{Q-1} e^{-j(m\theta+n\phi)} M(\kappa_{pq}) \end{aligned}$$

$$\begin{aligned} \theta &= \frac{p2\pi}{2P+1}, \\ \phi &= \frac{\pi q}{Q}, \end{aligned} \quad (43)$$

where DFT(.) represents forward discrete Fourier transform, $2M$ and $2N$ are number of samples or basis function in θ and ϕ axis respectively. Then the integral over the surface of sphere can be written as,

$$\begin{aligned} &\int_{-\pi}^{\pi} d\phi \int_{0}^{\pi} d\theta \sin\theta U(\theta,\phi) \\ &= \int_{-\pi}^{\pi} d\phi \int_{-\pi}^{\pi} d\theta |\sin\theta| U(\theta,\phi) \end{aligned} . \quad (44)$$

Note that the above modification changes the limit on θ integral to [-π, π], thus it can also be evaluated in fast manner using FFT. In single level implementation, the integrand in (24) are first represented in terms of Fourier basis using (42) and then (44) is used for fast evaluation of integrals. In multilevel implementation the interpolation and anterpolation, for varying bandwidth of multipole and local expansion, can be achieved by zero padding and truncating the Fourier coefficients respectively. In anterpolation the Fourier coefficients of parent local expansions are symmetrically truncated before inverse Fourier transform, to obtain the local expansion about child domain with the desired bandwidth. Thus all operations, including the evaluation of integral, can be evaluated using FFT. Reader is referred to [48] for related theorems, proofs and numerical results. Finally, the numerical implementation of multilevel FMM has been scrutinized in terms of different errors and to ensure stability. This includes discussion on the relation between truncation and integration error in (34) [77], and interpolation/anterpolation error using Lagrange interpolation [79] and spherical transform [75]. In addition, errors due to round off and evaluation of special-function have been considered along with stability criterion [80]. Numerical experiments show that truncation error in (34) is lower bounded [43, 79]; thus for applications that routinely demand very high accuracies it is preferable to increase the distance between wellseparated boxes. Evidently this amounts to an increase in number of boxes in near-field interaction.

A. Other FMMs

The above exposition presented FMMs that are apt for analyzing very general problems. However, for certain problems it is possible to develop FMM schemes that take advantage of topological features of scatterer to reduce the asymptotic complexity. The first of such algorithm was the fast steepest descent path algorithm [89] that exploited spectral representation of the Green's function. The next incarnation of this was the steepest descent FMM. It was developed following realization that when analyzing scattering from objects whose height is considerably lesser that its lateral dimension, it is not particularly useful to expand the fields using the complete spectrum. In other words, SDFMM can be interpreted to be a windowed FMM, and results in a method whose complexity scales as $O(N)$. In SDFMM, it is achieved naturally using the Sommerfeld integral representation of the Green's function and evaluating this integral using a combination of two-dimension FMM and steepest descent. More specifically,

$$\frac{e^{-jkR}}{R} = \frac{-j}{2}\int_{-\infty}^{\infty} dk_z e^{-jk_z(z-z')} H_0^{(2)}\left(\kappa_\rho \left|\rho-\rho'\right|\right) \quad , \quad (45)$$

$$= \frac{-j}{2}\sum_{n=1}^{N_{sd}} \omega_n \kappa_\rho^{(n)} H_0^{(2)}\left(\kappa_\rho \left|\rho-\rho'\right|\right) e^{-jk_z(z-z')}$$

where N_{sd} is quadrature rule along the integration path, ω_n is the integration weight, $\kappa_\rho^{(n)} = \kappa \sin\alpha_n$ and $k_z = \kappa\cos\alpha_n$, and α is defined along steepest descent path. It is immediately apparent that the summation over Hankel functions can be accelerated using a generalization of the two-dimensional FMM, and as before, this algorithm can be cast within a multilevel framework. Another algorithm along these lines was the fast inhomogeneous plane wave algorithm (FIPWA) [47, 68]. This algorithm follows directly from Weyl's identity

$$\frac{e^{-j\kappa R}}{R} = \frac{-j}{2}\int_0^{2\pi} d\phi \int_{SIP} d\theta \sin\theta e^{-j\kappa.R} . \quad (46)$$

The path of integration yields contributions from both homogeneous and inhomogeneous plane waves. As written, the above integral is slowly converging, but the contour can be deformed along the steepest descent path. This integral is evaluated numerically. However, values of the radiation pattern for complex θ is obtained using interpolation/ extrapolation. Manipulation of the requisite equations results in a diagonal translation operator. This method has been extended for analysis of scattering from objects above a layered medium [45, 47]. Additionally, they have been modified for developing stable algorithms for broadband applications [90]. However, we shall describe these algorithms and others [42] for rapidly computing potentials for wideband applications in the next section.

Finally, other variants of FMM exist that exploit the fact that between well separated boxes, one may construct windowed translation operators to lower the cost. One such method is the ray propagation FMM (RPFMM) [72, 91]. Other windowed translation operators have been used in two-dimensions for the analysis of scattering from bianisotropic objects [92]. However, it follows from complexity analysis that these methods will be fruitful only when the objects are sufficiently far away from each other. This implies that the algorithm is most useful when used in a one-level setting and may not be effective with a multilevel implementation.

B. Wideband FMM

In above discussion, a significant highlight is the restrictive choice of L used to truncate the expansions. This choice, based on the asymptotic behavior of Bessel and Hankel function, reveals the behavior of above expansions when applied to low frequency problems where κ is very small. It is well known that Hankel function is singular at origin and as $\kappa \to 0$ the expansion in (28), though valid, becomes numerically unstable. This breakdown is referred to as low-frequency breakdown [42, 43]. Consequently for fixed κ the size of source domain, which also defines the translation distance, cannot be made arbitrarily small. This issue becomes significant when the geometry is densely discretized, much more than the conventional $\lambda/10$ criterion, mostly to represent intricate structural details.

1) Scaled expansions: At low frequencies the numerical instability can be averted by using a normalized form of the original expansion (28)

[44, 46]. This approach is motivated by the asymptotic behavior of spherical Bessel and Hankel function for small argument. Let t be a normalization constant such that $t = O(kd)$ then the multipole expansions in (24) can be written as,

$$th_0^{(2)}\left(-j\kappa\left|X+d\right|\right) = -j\kappa\sum_{l=0}^{L}\sum_{m=-l}^{l}(-1)^l(2l+1)\times \left[\frac{1}{t^l}jl(\kappa d)Y_{lm}^*(\theta_d,\phi_d)\right]t^{l+1}h_l^{(2)}(\kappa X)Y_{lm}(\theta_X,\phi_X) \quad (47)$$

In above expression, terms inside the square brackets are the new normalized multipole coefficients. As $\kappa\to 0$, using small argument approximation for spherical functions and with $t = \kappa$, it is a straightforward exercise to show that the normalized expansions reduces to the expansions (2) used in static case. While the normalized form ensures numerical stability, the low-frequency nature of the problem implies that one can choose the number of multipoles to be same at every level. This in turn implies that the multilevel version of this approach scales as $O(N)$ [46]. A constant normalization factor is sufficient when the geometry is uniformly discretized. However to accommodate wide variation in domain sizes and maintain the stability of expansion different normalization factor should be chosen in different parts [33]. This approach has been successfully used in integral equation solution for scattering from sub-wavelength structures [46, 93].

2) Spectral representation based plane wave expansions: An alternate approach, inspired by the diagonalized form for static FMM, was introduced in [42] and later implemented in [49, 50, 90]. It is based on the following well-known spectral representation of solution to Helmholtz equation [94],

$$\frac{e^{-j\kappa R}}{R} = \frac{1}{2\pi}\int_0^\infty d\lambda e^{-\sqrt{\lambda^2-\kappa^2}z}\frac{\lambda}{\sqrt{\lambda^2-\kappa^2}} \times \int_0^{2\pi} d\alpha\, e^{-j\lambda(x\cos\alpha+y\sin\alpha)}, \quad (48)$$

this relation is valid for $z > 0$. Further it is straightforward to identify the purely propagating part of spectrum as $0 \le \lambda \le \kappa$ and the evanescent part as $\kappa \le \lambda \le \infty$; with simple change of variables, above expression can be written as [42],

$$\frac{e^{-j\kappa R}}{R} = \left(\frac{e^{-j\kappa R}}{R}\right)_{evan.} + \left(\frac{e^{-j\kappa R}}{R}\right)_{prop.}, \quad (49)$$

where,

$$\left(\frac{e^{-j\kappa R}}{R}\right)_{evan.} = \frac{1}{2\pi}\int_0^\infty d\sigma\, e^{-\sigma z}\times \int_0^{2\pi} d\alpha\, e^{-j\sqrt{\sigma^2+\kappa^2}(x\cos\alpha+y\sin\alpha)},$$

$$\left(\frac{e^{-j\kappa R}}{R}\right)_{prop.} = \frac{j\kappa}{2\pi}\int_0^{\pi/2} d\alpha\, e^{-j\kappa\cos\theta}\times \int_0^{2\pi} d\theta\, e^{-j\kappa\sin\theta(x\cos\alpha+y\sin\alpha)}. \quad (50)$$

Notice that with $\kappa\to 0$ the propagating part vanishes and the evanescent part reduces to the diagnolized form (19) used in static FMM. Now it remains to discretize the above integrals for numerical evaluation and generalized Gaussian quadratures can be employed for this. However, unlike in static case, the integrand cannot be rendered scale independent and this means quadrature points and weights should be pre-computed for all possible translation distances at all levels. It is worthwhile to recount that the multipole and local expansions are computed and stored as they appear in original spherical harmonics expansion (28); they are converted to exponential expansions back and forth during multipole to local translation only and these relations can be found in [50]. This approach avoids the floating point overflow as all the computed quantities and operations are regular and numerically stable. Other approaches based on above spectral representation have been presented [49, 90, 95, 96] and they differ significantly in their numerical implementation and structure. In all these methods the multipole and local expansion are represented directly in terms of exponential expansion coefficients; hence they require new interpolation/anterpolation operators for multilevel implementation. In [90], an extension of FIPWA as introduced for multi-layered structures, the integrand is sampled along the steepest descent path (SDP) and extrapolation techniques to estimate the evanescent portion of

the spectrum from samples of the propagating portion. However, one has to treat "shallow" evanescent waves differently from "deep" evanescent waves. In [49], the evanescent integrand is sampled along the traditional Sommerfeld integral path (SIP) and singular value decomposition (SVD) of the integrand is used to obtain expressions for multipole coefficient and multilevel translation operators. An interpolation matrix approach is presented in [96] to relate exponential expansions at different levels. Using sample points in child and parent domain an overdetermined system of equation is formed and solved for the interpolation matrix entries in a least square sense. The advantage of latter approaches is that they avoid the spherical harmonic to exponential expansion and reverse mapping operations.

3) Cartesian harmonics: At sufficiently low frequencies Cartesian harmonics provides the following convergent series expansions,

$$\frac{e^{-jk|r-r_s|}}{|r-r_s|} = \sum_{p=0}^{\infty} \left(r_s\right)^{(p)} \bullet\, p \bullet \nabla^p \frac{e^{-jk|r|}}{|r|}, \quad (51)$$

where,

$$\nabla^n \frac{e^{-jkR}}{R}\left(n_1, n_2, n_3\right) =$$

$$\sum_{m_1=0}^{\left\lfloor \frac{n_1}{2} \right\rfloor} \sum_{m_2=0}^{\left\lfloor \frac{n_2}{2} \right\rfloor} \sum_{m_3=0}^{\left\lfloor \frac{n_3}{2} \right\rfloor} (-1)^{n+m} R^{2m-2n-1} g\left(n-m, R\right) \times \quad (52)$$

$$\begin{bmatrix} n_1 \\ m_1 \end{bmatrix} \begin{bmatrix} n_2 \\ m_2 \end{bmatrix} \begin{bmatrix} n_3 \\ m_3 \end{bmatrix} x^{n_1-2m_1} x^{n_1-2m_2} x^{n_3-2m3}$$

and

$$g\left(n, R\right) = \sqrt{2/\pi}\left(j\kappa r\right)^{(n+0.5)} K_{n+0.5}(jkR)$$

$$\begin{bmatrix} n \\ m \end{bmatrix} = \frac{n!}{2^m m!(n-2m)!} \quad (53)$$

In above expressions $K_n(.)$ represents the modified Hankel function of order n, $r = x\hat{x} + y\hat{y} + z\hat{z}$, $R = |r|$ and $\lfloor . \rfloor$ is the floor operation. Interestingly, it can be shown that above expansions reduces to the spherical harmonic expansions in (28) with low argument approximation for spherical Bessel and Hankel functions [97, 98]. This assures the accuracy and stability of above expansions for low frequency problems. Again the number of harmonics can remain constant for all levels and this results in an $O(N)$ algorithm. As mentioned earlier the multipole and local expansions are independent of the form of potential; thus definitions of all operators, except M2L, remains same as that for static.

4) Hybrid methods: In multiscale geometries the low-frequency breakdown occurs only in parts when the domain size is much smaller than the incident wavelength. This implies that both low and high-frequency FMM should be used simultaneously. Henceforth, the different versions of FMM are referred to as LF-FMM and HF-FMM. When using different forms of expansions a switchover between LF-FMM and HF-FMM quantities is necessary to handle both large and small domain sizes simultaneously. Such hybrid methods have been developed by combining HF-FMM with plane wave expansions [50] and with Cartesian harmonics [99]. A smooth transition between the two versions of FMM is a key necessity of these hybrid methods and operators are prescribed to map quantities from LF-FMM onto HF-FMM and vice versa. In plane wave based LF-FMM, the multipole and local coefficients are represented in terms of the original expansion; thus the conversion to farfield signature required in HF-FMM and the reverse mapping operation can be simply performed using the forward and inverse farfield transform using (30). Mapping the Cartesian harmonics, represented by tensors, onto farfield signatures is slightly involved. It follows from the observation that farfield signatures are essentially plane waves in different directions and following Taylor's series expansion is possible,

$$\begin{aligned} e^{-i\kappa.\left(r_s^p - r_s\right)} &= \sum_{p=0}^{\infty} r_s^{(p)} \bullet\, p \bullet \nabla^{(p)} e^{-i\kappa.r_s^p} \\ &= \sum_{p=0}^{\infty} r_s^{(p)} \bullet\, p \bullet T^{(p)}\left(\kappa, r\right) \end{aligned} \quad (54)$$

where,

$$\begin{aligned} T^{(p)}\left(\kappa, r\right)\left(p_1, p_2, p_3\right) &= \partial_{p_1} \partial_{p_2} \partial_{p_3} e^{-i\kappa.r_s^p} \\ &= \left(-j\right)^p \kappa_x^{p_1} \kappa_y^{p_2} \kappa_z^{p_3} e^{-i\kappa.r} \end{aligned} \quad (55)$$

In above expressions $p = p_1 + p_2 + p_3$, $\kappa = \kappa_x \hat{x} + \kappa_y \hat{y} + \kappa_z \hat{z}$, r_s and r_s^p represents the

center of child and parent domain respectively. Here $T^{(p)}$ is the mapping operator relating the Cartesian multipole coefficient about child box, r_s^p, to the farfield signature about parent box. Since the local expansions take the same form as multipole expansions in both Cartesian and spherical forms, the reverse mapping is same as $T^{(p)}$ and in addition should evaluate the spherical integral $\int d^2\kappa$. In implementation the infinite sum in (54) is truncated and it is fairly straight-forward to show that error due to this decreases rapidly as the translation distance ($\left|r_s^p - r_s\right|$) decreases or as the number of harmonics increases. This is shown numerically in Table 1, where the L2 norm error between multipole coefficients evaluated using farfield transform and by mapping from Cartesian multipoles are shown for different orders of Cartesian harmonics (P) and translation distance $a = \left|r_s^p - r_s\right| / \lambda$. As expected the error uniformly reduces to machine precision with increasing *P* and *a*.

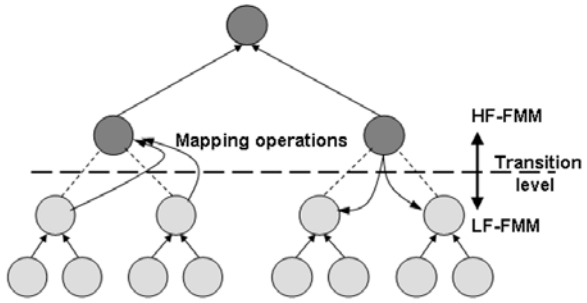

Fig. 7. Tree representation for hybrid methods.

Table 1: Error convergence of Cartesian to spherical harmonics mapping operator (55).

a	P=3	P=6	P=9	P=12
0.5	2.13	5.58E-3	9.62E-6	5.90E-9
0.25	2.58E-2	8.04E-6	1.51E-9	1.27E-13
0.125	3.49E-4	1.30E-8	1.55E-13	2.24E-15
0.0625	1.04E-5	5.34E-11	1.41E-15	1.41E-15

The overall execution of hybrid algorithm, in both cases, proceeds as follows: an adaptive (compressed) oct-tree is constructed to represent the geometry and a transition level is chosen such that LF-FMM and HF-FMM are stable for all boxes below and above this level respectively. Further, for simplicity, it is assumed that all leaf boxes are below this transition level, see Fig. 7 for an illustration. First the LF-FMM multipole coefficients are computed at all leaf boxes and upward tree traversal is executed until the transition level. At this point, for Cartesian harmonics based LF-FMM, the farfield signature of parent box above the transition level are computed from the Cartesian expansions in child boxes at the transition level using mapping operators (54); in plane wave based LF-FMM the farfield signature of box at the transition level is computed from the plane wave expansion coefficients of the same box. With this the upward tree traversal is performed for all boxes above the transition level. Next, the multipole coefficients in each box are translated into local coefficients of boxes in its interaction list which, by definition, are at the same level. In downward tree traversal, the local expansion coefficients of all boxes in HF-FMM region are updated with that of their parents. At transition level, for Cartesian harmonics based LF-FMM the child box Cartesian local expansions are computed by mapping the farfield signature about the parent box; for plane wave based LF-FMM the child box farfield signature is first computed and then converted as spherical harmonic local expansion using inverse farfield transform. Then the downward tree traversal continues for all boxes below the transition level. Finally the local expansion coefficients at leafless boxes are used to compute the farfield potential at their respective observation points. As in all algorithms the complete potential is computed by accounting for the near-field contribution by direct evaluation. The choice of transition level is influenced by different constraints in both the methods and in general it is determined through numerical experiments. In Cartesian-spherical harmonics hybrid method, the O(N) scaling of Cartesian expansion algorithm may favor a higher transition level, however, this implies more number of Cartesian harmonics to maintain the accuracy of both LF-FMM and the mapping operations. Similar considerations for plane wave expansion based LFFMM and HF-FMM hybrid algorithm are detailed in [50] along with numerical experiments. Tables 2 and 3 shows the convergence and efficiency of the hybrid method obtained by combining the Cartesian and spherical harmonics. Table 2 presents the error in potential

evaluation for different orders of Cartesian and spherical harmonics. Here χ_{sph} is the oversampling factor in spherical harmonics FMM i.e. the truncation L = χ_{sph} κd and P_{Cart} denotes the maximum order of Cartesian harmonics used. All errors reported here are the L2 norm error in farfield values computed using hybrid algorithm and direct evaluation; also, in the numerical experiments reported here, 16,000 source and observer points were randomly distributed following a uniform distribution within a domain of size $4\times2\times2\lambda$. The separation between interacting boxes was kept as three boxes to ensure uniform error convergence till machine precision (10^{-12}) in both LF- and HF-FMM. Note that this is the worst case error as including the near-field contribution will only decrease the error further. Table 3 shows the time taken by the hybrid algorithm for evaluation of farfield potentials as the total number of unknowns is varied. *N* source/observer points were randomly distributed, following a uniform distribution, within a domain size $4\times2\times2\lambda$ and the boxes were subdivided hierarchically until the number of unknowns per box was 64 on average. The orders of expansion in Cartesian and spherical harmonics were chosen so as to obtain an accuracy of $O(10^{-4})$. For this accuracy the distance between interacting boxes is reduced to one box separation. The time scaling (complexity) of this hybrid algorithm depends on the number of levels in LF- and HF-FMM i.e. the complexity would scales as $O(N)$ if the number of LF-FMM levels are much more than that in HF-FMM part. In essence the complexity of this algorithm has an O(N log N) upper-bound.

Table 2: Error convergence of Cartesian-spherical harmonics hybrid algorithm for 16,000 random points in $4\times2\times2\lambda$ domain.

χ_{sph}	P_{Cart}	ε
1.5	3	2.14E-2
1.7	4	3.91E-3
2.0	6	3.91E-4
2.2	8	6.74E-6
2.5	12	4.84E-9
3.2	16	6.08E-12

Table 3: Time vs. *N* corresponding to an error of $O(1.0E^{-4})$ in Cartesian-spherical harmonics hybrid algorithm

N	T_{fast}	T_{direct}
64000	9.38	468.81
128000	16.25	-
256000	34.40	-
512000	65.15	-
1024000	133.25	-
2000000	268.98	-

V. APPLICATIONS

This section provides an overview on applications of above discussed algorithms in different contexts. As mentioned in introduction, FMM and other fast methods, e.g. FFT and tree code based, were developed primarily to accelerate the evaluation of potential or field in N body problems. Integral equation solution, a common choice in simulation of many electromagnetic applications, sought through iterative solvers requires repeated evaluation of potential or field at source points itself. Thus fast algorithms play a significant role in solving real world problems within realistic time duration. The literature referenced here is only selective and not exhaustive as the use of these algorithms has become more common during recent years. Also, only topics related to electromagnetics are listed here; for applications in other research field refer to introduction.

First, electromagnetic application of static or Laplace FMM was evaluation of electrostatic potential in 2D [6, 100]. The extension to 3D has seen lot of applications, particularly, in plasma dynamics [8, 101]. FMM based FastCap and FastHenry are widely popular tools for extraction of equivalent capacitance and impedance among multiconnects in micro-electronic components [19, 20]. Static FMM is also used in integral equation solution of magnetostatic problems predominantly for analysis and design of electric machines [102]. Simulations with non-linear materials have benefited much as they demand multiple solution before attaining stability [56, 103, 104]. It has also been applied to quasi-static case especially in simulation of eddy-current phenomena [105, 106] and micromagnetics is another area of practical interest [107, 108].

The recently published book on fast methods in electromagnetics is a virtual treasure house of FMM methods and their applications to various problems in high frequency electromagnetics [33]. As is to be expected, Helmholtz FMM has been applied to accelerate iterative solution of surface and volume integral equations. The means to modify Helmholtz equation such that they are applicable to vector electromagnetics problems was first presented in [84]. More detailed description can be found [33, 81, 109]. Since their introduction, they have been applied extensively to scattering and radiation problems of different flavors; for instance, scattering from perfect electrically conducting surfaces [28, 71, 75, 84, 109–114], scattering from dielectric/ composite bodies [115–120], volume integral equations [76, 121–123], anisotropic objects [124, 125], scattering from rough surfaces [126–128], application to microstrips [129], EMC/EMI analysis [130–132], antennas [133–135]. Efficient implementation of FMM in solvers with higher order geometry and basis function representations have led to the development of fast and accurate solvers [116, 136, 137]. [138, 139].

Multipole accelerated algorithms have also been employed in various hybrid methods where solution is obtained with use of moment method combined with one or more of following techniques: to impose global radiation boundary conditions in finite element solvers [140–142], ray tracing and diffraction methods [143], multi-grid methods [144] and physical optics [145, 146]. These techniques are primarily used in applications with multi-scale scatterers like antenna interactions [147] and field predications for urban mobile communications [148]. Implementation of FMM was also modified to accommodate perfectly matched layer (PML) assisted integral equation methods used in simulation of monolithic microwave integrated circuit (MMIC) and photonic crystals [149–151]. Fast inhomogeneous plane wave (FIPWA) method and other forms of FMM have been used to accelerate solution of scattering simulations involving layered media structures with applications in design of microstrip antennas [138, 139, 152–157] and geophysical investigations for sub-surface scatterers [70, 158–166]. A combined FMM-FFT algorithm [167, 168] and SDFMM have been used in electromagnetic analysis of general quasi-planar structures with applications to rough surface scattering, grating structure design in quantum devices and radiation from microstrip patch antenna [127, 169–171]. Parallel versions of FMM [36, 38, 41, 54, 81, 172–176], especially on cluster computers with distributed memory, have been employed to solve problems with few millions of unknowns [37, 40], aided with developments in different preconditioning techniques [177–181]. Finally, we note that while FMMs reviewed here primarily accelerate the solution off frequency domain integral equations, equivalent models have been developed for time domain integral equations also [88, 182, 183].

VI. SUMMARY

Introduction of FMM changed the landscape of numerical simulation in many fields and the developments in past two decades have made it an ubiquitous tool for fullwave analysis. This paper reviews different FMMs and their applications to problems in electromagnetics. The development of FMM is traced from static to dynamic, and covers various methodologies that form the current state of art FMM. These include the spectral representation to obtain diagonalized operators in both static and dynamic FMM, farfield signatures for diagonalized forms in dynamic, Cartesian harmonics based expansions for static and low-frequency dynamic case and other application specific techniques to improve both accuracy and efficiency. This review also includes an overview of recent developments in combining different FMMs to obtain hybrid algorithms that are applicable to wideband analysis. But, all said and done, while we have tried to be as comprehensive as possible in this review, the papers cited herein provide only snapshot of the papers that exploit FMM for accelerating integral equation solvers.

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