Normal Directional NURBS Arithmetic of Conformal PML

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Abstract **─** Conformal Perfectly Matched Layer (PML) is a high-efficiency absorbing boundary condition for the finite element analysis of electromagnetic fields. Accurate calculation of normal direction of conformal PML is essential for the geometric modelling of conformal shell elements and constitutive parameters of conformal PML, especially for sophisticated and arbitrary shape scatterers. Consequently, a Non-Uniform Rational B-Splines (NURBS) arithmetic is proposed for describing the conformal surface accurately in this study. Based on the NURBS arithmetic, four weighted average formulas are presented for calculating the common normal direction of adjacent surface elements of conformal shell. Numerical experiments show the availability of NURBS arithmetic and precision of weighted average formulas in the geometrical modelling of conformal PML.

Index Terms **─** Common normal direction, conformal PML, finite element modeling, NURBS arithmetic.

I. INTRODUCTION

As an efficient artificial absorbing media, conformal Perfectly Matched Layer (PML) has been attracting more and more attention since the half-space Cartesian PML is extended into conformal absorbing boundary of cylindrical and spherical geometries in approximate PML formulations by Kuzuoglu and Mittra [1]. The conformal mesh truncation defined by exact PML formulations [2-4], which encloses the scatterer a small distance away, is very advantageous for saving spatial scattering elements, especially in the Finite Element Method (FEM). In curvilinear coordinates and general orthogonal curvilinear coordinates, numerical efficiency of absorbing boundary conditions of complex geometries are improved by conformal PML with grid generation technique [5]. The dynamic stability of the Cartesian, cylindrical, spherical and conformal PMLs is analyzed and presented in [6] and [7]. Some basic conclusions have played an important role on the design of conformal PMLs. However, how to mesh the conformal shell elements well is still a basic issue for realizing and developing the conformal PML [8-10]. Accurate calculation of normal direction of conformal PML is a basic requirement for meshing the conformal elements, especially for the high-fidelity mapped shell (hexahedron) elements. Fortunately, Non-Uniform Rational B-Splines (NURBS), which possesses the excellent characteristics on defining the complicated surfaces and generating the curvilinear elements [11-13], is a valid scheme for describing the conformal surface and calculating the normal direction of conformal PML.

The contents of this paper include the normal directional NURBS arithmetic of conformal PML, arithmetic implementation and numerical experiments, which demonstrate both the availability and precision of the arithmetic.

II. FEM IMPLEMENTATION OF CONFORMAL PML

In the FEM for electromagnetic scattering problems, the PML is an efficient mesh truncation boundary. As a more efficient absorbing boundary condition, the conformal PML is introduced for solving the computational scale problem of large-size scatterer. The basic implementation steps of applying the conformal PML into FEM are presented as follows.

Step 1. Mesh the outer surface of scatterer and

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generate the basic surface elements of scatterer. (When the shape of scatterer is arbitrary or unknown, the basic surface of scatterer should be reconstructed accurately by the surface elements. NURBS in Section III is advantageous for this situation.)

Step 2. Calculate the normal directions of nodes of basic surface elements and create the nodes of conformal surface along the normal directions, shown in Fig. 1. Because the precision and quality of conformal PML elements depend largely on these normal directions, this step is fundamental and key for computational accuracy and numerical efficiency of conformal PML.

Step 3. Based on the nodes of conformal surface, generate one layer of shell elements of conformal PML, similarly generate multilayer of shell elements of conformal PML.

Step 4. According to the geometric information of shell elements, compute the constitutive parameters $\overline{\overline{\mu}} = \mu_r \overline{\overline{\Lambda}}$ and $\overline{\overline{\varepsilon}} = \varepsilon_r \overline{\overline{\Lambda}}$ of conformal PML [2]. The matrix of $\overline{\overline{\Lambda}}$ in local coordinate system (u,v,w) is given by:

$$\overline{\overline{\Lambda}}_{u.v,w} = \begin{bmatrix} \frac{s_2 s_3}{s_1} & 0 & 0 \\ 0 & \frac{s_1 s_3}{s_2} & 0 \\ 0 & 0 & \frac{s_1 s_2}{s_3} \end{bmatrix}. \qquad (1)$$

Where $s_1 = \dfrac{r_{01} + \int_0^w s(\zeta) d\zeta}{r_1}$, $s_2 = \dfrac{r_{02} + \int_0^w s(\zeta) d\zeta}{r_2}$, $s_3 = s$, s is the complex stretching variable [9,10] in the w -direction. r_1 and r_2 are rincipal radiis [9,10] on the nodes of shell elements.

Step 5. Apply the constitutive parameters of conformal PML to vector wave equations of scattering field:

$$\nabla \times \left(\frac{1}{\mu_r} \overline{\overline{\Lambda}}^{-1} \cdot \nabla \times \boldsymbol{E}^s \right) - k_0^2 \varepsilon_r \overline{\overline{\Lambda}} \cdot \boldsymbol{E}^s = \boldsymbol{0} . \qquad (2)$$

Where $\boldsymbol{E}^s$ is scattering electric field. After interface boundary conditions of conformal PML are applied, the conformal absorbing boundary is completed in the FEM.

In the above implementation steps of conformal PML, the key step is to calculate accurately the normal directions of nodes of basic surface elements because the normal directions control the geometric shapes of conformal shell elements and constitutive parameters of conformal PML. Therefore, the calculation arithmetic of normal direction of conformal PML is presented in detail in the following sections.

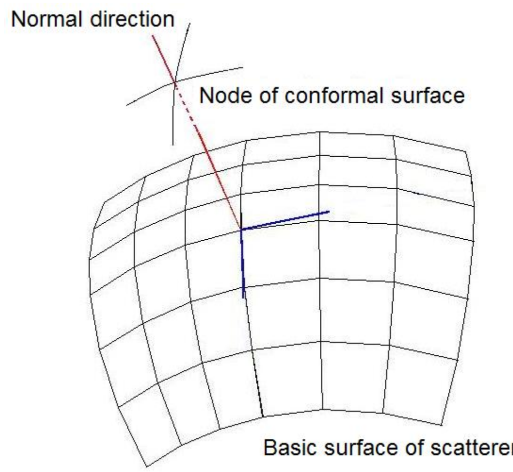

Fig. 1. Normal direction of conformal PML.

III. NURBS SURFACE

Since the Non-Uniform Rational B-Splines (NURBS) is introduced into the computational electromagnetic applications by Valle, Rivas and Citedra [14], it is always a quite sophisticated geometrical modelling method of arbitrary shape bodies and complex scatterers. As a quite powerful modelling tool, NURBS plays a fundamental role in integral and differential methods of computational electromagnetics. Presently, the rapid development of complex curve/surface construction and grid generation technique largely depends upon the progress of NURBS.

In view of the advantage on the geometric description, NURBS shows great promise as an ideal discrete approximation for the complex surface, especially on the large curvature surface. Generally, numerical accuracy and efficiency of NURBS are very high for describing and constructing the basic surface of arbitrary shape scatterers. Therefore, in our work the NURBS is employed to describe the basic surface of scatterer and convex surface of conformal PML.

In this section, we start by reviewing the basic definition of NURBS surface [15]; only the equations relevant to our implementation are presented.

If a NURBS surface is pth order on u direction and qth order on v direction, its piecewise rational vector function is expressed as:

$$\boldsymbol{S}(u,v)=\frac{\sum_{i=0}^{n}\sum_{j=0}^{m}N_{i,p}(u)N_{j,q}(v)w_{i,j}\boldsymbol{P}_{i,j}}{\sum_{i=0}^{n}\sum_{j=0}^{m}N_{i,p}(u)N_{j,q}(v)w_{i,j}},\quad 0\le u,v\le 1 \cdot \tag{3}$$

Where $\{\boldsymbol{P}_{i,j}\}$ define control points on u and v directions, $\{w_{i,j}\}$ are weighting factors, $\{N_{i,p}(u)\}$ and $\{N_{j,q}(v)\}$ are nonrational B-splines basis functions defined on vector $\boldsymbol{U}$ and $\boldsymbol{V}$ respectively [15],

$$\begin{cases} U=\left\{\underbrace{0,\cdots,0}_{p+1},u_{p+1},\cdots,u_{r-p-1},\underbrace{1,\cdots,1}_{p+1}\right\} \\ V=\left\{\underbrace{0,\cdots,0}_{q+1},u_{q+1},\cdots,u_{s-q-1},\underbrace{1,\cdots,1}_{q+1}\right\} \end{cases}. \tag{4}$$

Where $r=n+p+1$, $s=m+q+1$.

In (3), the numerator and denominator of piecewise rational vector function are respectively rewritten as:

$$\boldsymbol{A}(u,v)=\sum_{i=0}^{n}\sum_{j=0}^{m}N_{i,p}(u)N_{j,q}(v)w_{i,j}\boldsymbol{P}_{i,j}, \tag{5}$$

$$w(u,v)=\sum_{i=0}^{n}\sum_{j=0}^{m}N_{i,p}(u)N_{j,q}(v)w_{i,j}. \tag{6}$$

Hence, first order partial derivative [15] of piecewise rational vector function is given by:

$$\boldsymbol{S}_{\alpha}(u,v)=\frac{\boldsymbol{A}_{\alpha}(u,v)-w_{\alpha}(u,v)\boldsymbol{S}(u,v)}{w(u,v)}. \tag{7}$$

Where α indicates u or v partial derivative of $\boldsymbol{S}(u,v)$.

IV. NURBS ARITHMETIC OF COMMON NORMAL DIRECTION

Using the NURBS, we generate accurately the basic surface elements of scatterer in step 1 (in Section II). Otherwise, the normal direction definition of nodes of basic surface elements becomes a very knotty problem when the shell elements of conformal PML will be generated in step 2 (in Section II). As following Fig. 2, the normal directions of adjacent surface elements are different on the common node O. For instance, on the common node O, the normal direction $\boldsymbol{Z}_1$ of element no.1 is defined by two tangential directions $\boldsymbol{S}_{u1}$ and $\boldsymbol{S}_{v1}$ of element no.1. Similarly, the normal directions of other adjacent surface elements are also obtained.

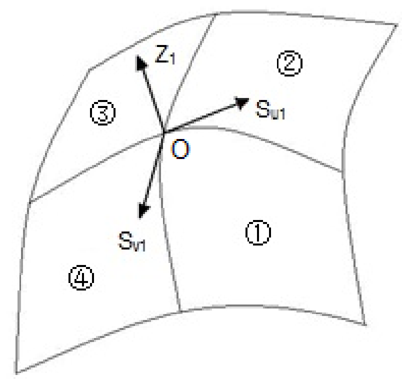

Fig. 2. Normal direction of adjacent surface elements.

In consideration of the conciseness of representation, the normal direction of the ith surface element on the node O is unitized as:

$$\boldsymbol{Z}_i(u,v)=\left.\frac{\boldsymbol{S}_{ui}(u,v)\times\boldsymbol{S}_{vi}(u,v)}{\left|\boldsymbol{S}_{ui}(u,v)\times\boldsymbol{S}_{vi}(u,v)\right|}\right|_{(u_0,v_0)}. \tag{8}$$

Where $\boldsymbol{Z}_i(u,v)$ is the unit normal direction of the ith surface element on the common node O. $\boldsymbol{S}_{ui}$ and $\boldsymbol{S}_{vi}$ are two tangential directions of the ith surface element on the common node O respectively.

For calculating accurately the common normal direction of adjacent surface elements, four weighted average formulas are proposed. In these formulas, different weighting factors are introduced to present the contribution of normal directions of adjacent elements to the common normal direction. Moreover, some geometric and discretization impacts are also considered. The detailed weighted average formulas are described as following.

(i) If the common normal direction is expressed as a simple average of normal directions of adjacent elements, the common normal direction is defined as:

$$\boldsymbol{Z}_0(u,v)=\frac{\sum_{i=1}^{n}\boldsymbol{Z}_i(u,v)}{n}. \tag{9}$$

Where n is total number of adjacent surface elements. Actually, the weighting factor is 1.0 in (9).

(ii) Considering that the shapes of adjacent surface elements are important factors for the common normal direction, area percentages of adjacent elements are introduced as the weighting factors. Thus, the common normal direction is defined as:

$$\boldsymbol{Z}_0(u,v)=\frac{\sum_{i=1}^{n}\Omega_i \boldsymbol{Z}_i(u,v)}{\sum_{i=1}^{n}\Omega_i}. \tag{10}$$

Where Ω_i is the area of the ith element. The area percentage $\Omega_i \Big/ \sum_{i=1}^{n}\Omega_i$ is the weighting factor of the ith element. This means that the common normal direction will be close to the normal directions of big elements.

(iii) Considering that the geometric curvatures of adjacent surface elements are also important factors for the common normal direction, Gauss curvatures of surface elements are introduced as the weighting factors. Thus, the common normal direction is defined as:

$$\boldsymbol{Z}_0(u,v)=\frac{\sum_{i=1}^{n}\rho_i \boldsymbol{Z}_i(u,v)}{\sum_{i=1}^{n}\rho_i}. \tag{11}$$

Where ρ_i is Gauss curvature of the ith surface element on the common node O. The curvature percentage $\rho_i \Big/ \sum_{i=1}^{n}\rho_i$ is the weighting factor of the ith element. This means that the common normal direction will be close to the normal directions of large curvature elements.

(iv) In order to ensure the geometric shape and computational precision of finite element, the size of conformal shell element must be so fine (1/10 wavelength) that the geometric information of element is enough for the discrete approximation on the large curvature domain. Therefore, area percentages and curvatures of surface elements should be considered globally. Based on the harmonic average formula, the common normal direction is defined as:

$$\boldsymbol{Z}_0(u,v)=\frac{\sum_{i=1}^{n}\frac{\rho_i}{\Omega_i}\boldsymbol{Z}_i(u,v)}{\sum_{i=1}^{n}\frac{\rho_i}{\Omega_i}}. \tag{12}$$

Where ρ_i is Gauss curvature of the ith surface element. Ω_i is the area of the ith surface element. The mixed percentage $\frac{\rho_i}{\Omega_i} \Big/ \sum_{i=1}^{n}\frac{\rho_i}{\Omega_i}$ is the weighting factor of the ith element. This means that the common normal direction will be close to the normal directions of large curvature (unsmooth) and small elements.

V. NUMERICAL EXPERIMENTS

In this section, in order to verify the modelling accuracy of NURBS arithmetic, we implement the arithmetic in two classical experiments.

(1) Sphere, its diameter is 2 cm, as following Fig. 3.

(2) Ellipsoid, its major axis is 4 cm, its minor axis is 2 cm, as following Fig. 4.

In the above experiments, all programs are developed in Matlab2009 compiled language.

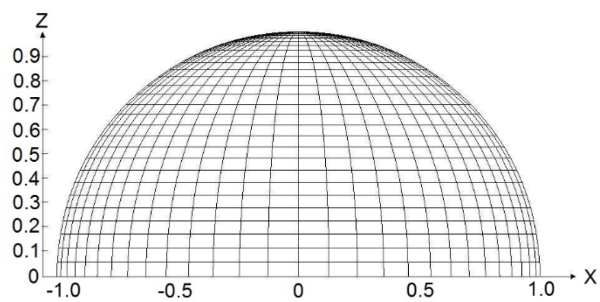

Fig. 3. Sphere.

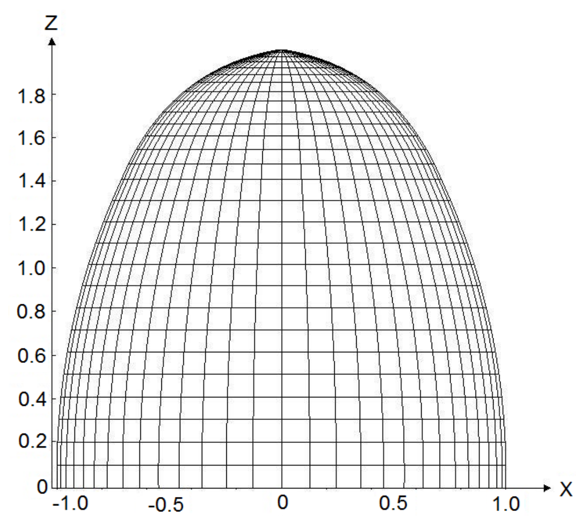

Fig. 4. Ellipsoid.

In Table 1, we compare the common normal directions calculated by four weighted average formulas with the analytic normal direction on the common node. The analytic normal direction and calculated common normal directions are expressed as the vector format. The spherical coordinate system (θ and φ) is employed for describing the conformal surface conveniently. The unit of spherical coordinates θ and φ is the degree. In consideration of the generality of four adjacent surface elements, division of surface is controlled by the angular intervals of spherical coordinates θ and φ in every case (in Table 1) like the longitude and latitude of the earth.

Table 1: Calculation results of common normal directions

Model	Mesh Sizes (^{0})	Analytic Normal Vector	Formula (i)	Formula (ii)	Formula (iii)	Formula (iv)
Sphere	θ: 20, 30, 40 φ: 1, 10, 20	0.150384 0.086824 0.984808	0.150388 0.086837 0.984806	0.15044 0.086878 0.984794	0.150388 0.086837 0.984806	0.150334 0.086795 0.984818
	θ: 0, 10, 20 φ: 20, 30, 40	0.492404 0.086824 0.866025	0.49239 0.086814 0.866034	0.49241 0.086817 0.866022	0.49239 0.086814 0.866034	0.492369 0.086811 0.866046
	θ: 0, 10, 40 φ: 20, 50, 60	0.754406 0.133022 0.642788	0.751913 0.13369 0.645565	0.751475 0.133856 0.64604	0.751988 0.133681 0.645479	0.752624 0.133381 0.6448
	θ: 20, 50, 60 φ: 1, 10, 40	0.111619 0.133022 0.984808	0.112833 0.133589 0.984593	0.11316 0.133792 0.984527	0.112797 0.133582 0.984598	0.111965 0.132882 0.984787
	θ: 30, 70, 90 φ: 5, 30, 60	0.17101 0.469846 0.866025	0.173981 0.468775 0.866014	0.175156 0.469796 0.865224	0.173793 0.46882 0.866028	0.172363 0.467737 0.866898
	θ: 5, 30, 60 φ: 30, 70, 90	0.813798 0.469846 0.34202	0.811349 0.469217 0.348638	0.81086 0.469319 0.349637	0.811438 0.469277 0.348351	0.811996 0.469193 0.347161
Ellipsoid	θ: 20, 30, 40 φ: 1, 10, 20	0.288022 0.16629 0.943075	0.292844 0.169097 0.941089	0.294023 0.169771 0.9406	0.292393 0.168845 0.941274	0.291268 0.168202 0.941738
	θ: 0, 10, 20 φ: 20, 30, 40	0.744445 0.131266 0.654654	0.748462 0.131963 0.649916	0.748397 0.131952 0.649993	0.748539 0.131985 0.649822	0.748601 0.131996 0.649749
	θ: 0, 10, 40 φ: 20, 50, 60	0.908121 0.160126 0.38688	0.917875 0.163196 0.361763	0.918985 0.163692 0.358708	0.919571 0.163598 0.357246	0.918558 0.162952 0.360135
	θ: 20, 50, 60 φ: 1, 10, 40	0.213778 0.25477 0.943075	0.238914 0.282892 0.92892	0.244353 0.288933 0.92564	0.233345 0.276594 0.932226	0.21844 0.259424 0.940735
	θ: 30, 70, 90 φ: 5, 30, 60	0.258543 0.710341 0.654654	0.272015 0.73291 0.62358	0.273087 0.732454 0.623646	0.270861 0.733465 0.623428	0.269443 0.73418 0.623202
	θ: 5, 30, 60 φ: 30, 70, 90	0.852031 0.49192 0.179044	0.854928 0.494413 0.157015	0.854999 0.494862 0.155202	0.855228 0.495324 0.152444	0.855205 0.494868 0.154045

The calculation errors of four weighted average formulas are shown in Table 2. The calculation errors are defined as the angles between the analytic normal direction and calculated common normal directions, as following equation (13). The unit of error is the degree.

$$\theta_{error} = 180 \arccos\left(\frac{\boldsymbol{Z}_a \cdot \boldsymbol{Z}_{0i}}{|\boldsymbol{Z}_a\|\boldsymbol{Z}_{0i}|}\right) \Big/ \pi . \qquad (13)$$

Where $\boldsymbol{Z}_a$ is the analytic normal direction. $\boldsymbol{Z}_{0i}$ is the common normal directions calculated by the ith formulas. $\arccos()$ is arc cosine function.

Although, the element size is very fine (1/10 wavelength) in the FEM of electromagnetic problems actually, the large size elements are proposed to show obviously the calculation errors between the analytic and numerical **s**olution. In Table 2, the results show that the calculation errors of four weighted average formulas are very small, and the calculation errors of formula (iv) are almost less than those of other formulas, especially on the large curvature domain. This means that the weighted average approach to the analytic normal direction will be very accurate in the FEM of actual electromagnetic problems.

Table 2: Calculation errors of common normal directions

Model	Mesh Sizes (^{0})	Formula (i) Error (^{0})	Formula (ii) Error (^{0})	Formula (iii) Error (^{0})	Formula (iv) Error (^{0})
Sphere	θ: 20, 30, 40 φ: 1, 10, 20	0.0008	0.00456	0.00078	0.00334
	θ: 0, 10, 20 φ: 20, 30, 40	0.0011	0.00054	0.0011	0.00247
	θ: 0, 10, 40 φ: 20, 50, 60	0.2172	0.2553	0.2107	0.1553
	θ: 20, 50, 60 φ: 1, 10, 40	0.0778	0.1000	0.0757	0.0214
	θ: 30, 70, 90 φ: 5, 30, 60	0.1809	0.2420	0.1699	0.1520
	θ: 5, 30, 60 φ: 30, 70, 90	0.4059	0.4687	0.3885	0.3144
Ellipsoid	θ: 20, 30, 40 φ: 1, 10, 20	0.3393	0.4220	0.3079	0.2291
	θ: 0, 10, 20 φ: 20, 30, 40	0.3582	0.3523	0.3652	0.3707
	θ: 0, 10, 40 φ: 20, 50, 60	1.5538	1.7421	1.8312	1.6529
	θ: 20, 50, 60 φ: 1, 10, 40	2.3084	2.8107	1.7908	0.4005
	θ: 30, 70, 90 φ: 5, 30, 60	2.3321	2.3360	2.3356	2.3460
	θ: 5, 30, 60 φ: 30, 70, 90	1.2811	1.3869	1.5474	1.4537

VI. CONCLUSION

For calculating accurately the common normal direction of conformal PML elements, we develop the NURBS arithmetic of conformal surface and four weighted average formulas of common normal direction. In view of its precision in the experiments, the NURBS arithmetic shows high availability as an ideal approach for the common normal direction of conformal PML.

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