SLF/ELF Electromagnetic Field of a Horizontal Dipole in the Presence of an Anisotropic Earth-Ionosphere Cavity

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Abstract **─** In this paper, the region of interest is a cavity between a spherical and electrically homogeneous earth and an anisotropic homogeneous ionosphere. Both the dipole (Vertical Electric Dipole (VED), Vertical Magnetic Dipole (VMD), or Horizontal Electric Dipole (HED)) and the observation point are assumed to be located on or near the spherical surface of the earth. The approximate formulas are obtained for the electromagnetic field of a vertical magnetic dipole and that of a vertical electric dipole in the presence of an anisotropic earth-ionosphere cavity, respectively. Based on the results obtained, the approximate formulas are derived readily for the electromagnetic field of a horizontal electric dipole in the presence of an anisotropic earth-ionosphere cavity by using a reciprocity theorem. Analysis and computations in SLF/ELF ranges are carried out specifically.

Index Terms **─** Anisotropic earth-ionosphere cavity, reciprocity theorem, SLF/ELF electromagnetic field.

I. INTRODUCTION

The subject on the VLF/ULF/SLF/ELF electromagnetic waves (VLF: 3 kHz - 30 kHz; ULF: 300 Hz - 3 kHz; SLF: 300 Hz - 30 Hz; ELF: 30 Hz - 3 Hz) in the earth-ionosphere waveguide or cavity were intensively investigated for over 60 years because of its myriad applications in submarine communication and navigation, geophysical prospecting and diagnostics, and earthquake electromagnetic detection [1-7].

Since the 1950s, driven by the defense requirement, the VLF radio wave propagation theory was investigated by many researchers, especially including several pioneers, such as Budden [3], Wait [4,8-10], and Galejs [14-17]. In early works by Wait and Galejs, detailed analysis was carried out on the VLF radio wave propagation in an earth-ionosphere waveguide. Naturally, the VLF radio wave propagation theory can be extended in the study on SLF/ELF radio wave propagation. Since the 1960s, the SLF/ELF wave propagation were studied widely [11-26]. In particular, it was noted that the outstanding contributions on the SLF/ELF wave propagation and application were made by Galejs, and the detailed findings were well summarized in the classic book [5]. With the extension of the pioneering works by Budden, Wait, and Galejs, some important developments on the SLF/ELF wave propagation were accomplished [18-26]. It was also pointed that some excellent works on the SLF/ELF wave propagation were also carried out in China, and those works were summarized in recent book [28].

In SLF/ELF ranges, when considering the effect by geomagnetic field, it is necessary that the ionosphere should be idealized as anisotropic plasma. Unfortunately, the complete analytical solution on SLF/ELF electromagnetic field of a horizontal electric dipole source in the presence of an anisotropic earth-ionosphere waveguide or cavity are still in dark by now. In what follows, we will attempt to outline the complete approximate solution for the SLF/ELF electromagnetic field of a horizontal electric dipole in the presence of an anisotropic earth-ionosphere cavity. The region of interest is a cavity between a spherical and electrically homogeneous earth and an anisotropic homogeneous ionosphere, and both the dipole source and the observation point are assumed on

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or near the earth's surface. First, the approximate formulas are obtained for the electromagnetic field of a vertical magnetic dipole in an anisotropic earth-ionosphere cavity. Based on the obtained results of vertical magnetic dipole and the available results of vertical electric dipole, the approximate formulas are derived readily for the electromagnetic field of a horizontal electric dipole in the presence of an anisotropic earth-ionosphere cavity by using a reciprocity theorem. Finally, computations and analysis in SLF/ELF ranges are carried out specifically.

II. EM FIELD IN AN ANISOTROPIC EARTH-IONOSPHERE CAVITY

The geometry under consideration is shown in Fig. 1. The cavity is occupied by the air characterized by the permeability μ_0, uniform permittivity ε_0, and conductivity $\sigma_0 = 0$. The spherical earth is characterized by μ_0, uniform permittivity ε_g, and conductivity σ_g. Assuming that the earth's magnetic field $\boldsymbol{B}_0$, which has the angle θ_0 with $z-$direction, is in the $x-z$ plane, the ionosphere can be characterized by a tensor permittivity $\overset{=}{\varepsilon}$ [3]. It is:

$$\overset{=}{\varepsilon} = \varepsilon_0([\mathbf{I}] + [\mathbf{M}]), \tag{1}$$

where ε_0 is the free-space permittivity, $[\mathbf{I}]$ is 3×3 unit matrix, the susceptibility $[\mathbf{M}]$ is expressed in the following form:

$$[\mathbf{M}] = -\frac{X}{U(U^2 - l_x^2 y^2)} \times \begin{bmatrix} U^2 - l_x^2 y^2 & il_z Uy & -l_x l_z y^2 \\ -il_z Uy & U^2 & il_x Uy \\ -l_x l_z y^2 & -il_x Uy & U^2 - l_z^2 y^2 \end{bmatrix}, \tag{2}$$

where $U = 1 + i(\varsigma/\omega)$ and ς is the effective electron collision frequency of the ionosphere, $y = (\omega_H)/(\omega)$, $X = (\omega_0^2)/(\omega^2)$, ω_H and ω_0 are the gyrofrequency of the electrons and the angular plasma frequency of the ionosphere, respectively; $l_x = \sin\theta_0$ and $l_z = \cos\theta_0$ are the directional cosines of the Earth's magnetic field in the x and z directions, respectively.

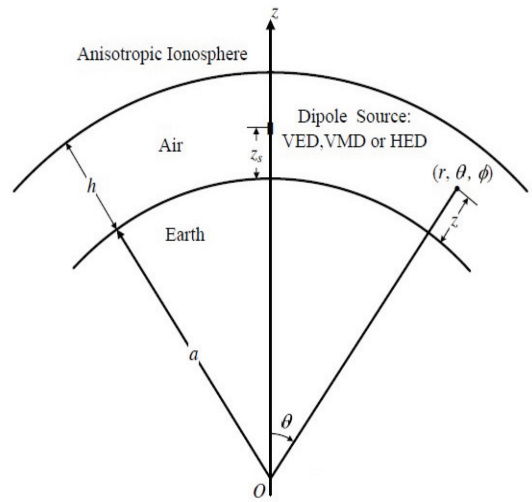

Fig. 1. Geometry of a dipole source in the presence of an anisotropic earth-ionosphere cavity.

In general, a SLF/ELF radiation source is usually employed a horizontal linear antenna. So that it is necessary to investigate SLF/ELF electromagnetic field of a horizontal electric dipole in the presence of an anisotropic earth-ionosphere cavity. In available reference [27], the electromagnetic field of a vertical electric dipole in the presence of an anisotropic earth-ionosphere cavity is addressed specifically. If the corresponding electromagnetic field of a vertical magnetic dipole, the analytical solution on the electromagnetic field of a horizontal electric dipole in the presence of an anisotropic earth-ionosphere cavity can be obtained readily. Next, we will attempt to derive the approximated formulas for the electromagnetic field of a vertical magnetic dipole in the presence of an anisotropic earth-ionosphere cavity.

A. Field of vertical magnetic dipole

Assume that a vertical magnetic dipole is represented by its moment $\hat{z}Ida\delta(x)\delta(y)\delta(z-b)$, where da is the area of the loop, $b = a + z_s$, and $z_s > 0$ denotes the height of the magnetic dipole above the earth's surface. Considering anisotropic properties of the ionosphere, the field components radiated by a vertical electric dipole in the presence of an anisotropic earth-ionosphere cavity can be expressed in the terms of the potential functions U and V, which is addressed in the book by Pan [27]. In this paper, we assume that the characteristics of SLF/ELF wave propagation are determined by the propagation path from the dipole source to the field point, namely, the propagation characteristics are not affected by other propagation paths. Then, we have $\partial/\partial\phi = 0$. Additionally, the surface impedances of the earth and the ionosphere are changed very slowly to the propagation distances. For mathematical convenience, the surface impedances of the earth and the ionosphere are regarded as constants. Thus, we write:

$$E_r = \left(\frac{\partial^2}{\partial r^2} + k^2\right)(Ur), \tag{3a}$$

$$E_\theta = \frac{1}{r}\frac{\partial^2}{\partial\theta\partial r}(Ur), \tag{3b}$$

$$E_\phi = -\frac{i\omega\mu}{r}\frac{\partial}{\partial\theta}(Vr), \tag{3c}$$

$$H_r = \left(\frac{\partial^2}{\partial r^2} + k^2\right)(Vr), \tag{3d}$$

$$H_\theta = \frac{1}{r}\frac{\partial^2}{\partial\theta\partial r}(Vr), \tag{3e}$$

$$H_\phi = \frac{i\omega\varepsilon}{r}\frac{\partial}{\partial\theta}(Ur). \tag{3f}$$

It is noted that the potential functions U and V satisfy the following wave equation:

$$\left(\nabla^2 + k^2\right)\begin{bmatrix} U \\ V \end{bmatrix} = 0. \tag{4}$$

With similar procedure for the electromagnetic field of a vertical electric dipole as addressed in the book by Pan [27], by using the boundary conditions at $r = a$ and $r = a + h$, the potential functions U and V are expressed in the following forms:

$$Ur = \sum_n \Lambda_n F_n^{(M)}(z) P_v\left(\cos(\pi - \theta)\right), \tag{5a}$$

$$\overline{V}r = \sum_n \Lambda_n G_n^{(M)}(z) P_v\left(\cos(\pi - \theta)\right), \tag{5b}$$

where $\overline{V} = V\eta$, η is wave impedance in free space, Λ_n is the excitation factor, and $P_v(\cos(\pi - \theta))$ is the potential function U is used to express the electromagnetic field components when the dipole source is a vertical electric dipole (TM mode), and the potential function V is used when the dipole source is a vertical magnetic dipole (TE mode).

For the electromagnetic field radiated by a vertical magnetic dipole, the "height-gain" function $G_n^{(M)}(z)$ of the potential function $\overline{V}$ should be normalized, namely, $G_n^{(M)}(z = 0) = 1$. Then, we have:

$$G_n^{(M)}(z) = \frac{1}{1 + R_{hn}}\left\{\exp\left[-ik\int_0^z (C_n^2 + \frac{2z}{a}S_n^2)^{1/2}dz\right] + R_{hn}\exp\left[ik\int_0^z (C_n^2 + \frac{2z}{a}S_n^2)^{1/2}dz\right]\right\}. \tag{6}$$

The normalized "height-gain" functions $F_n^{(M)}(z)$ of the potential function U is expressed in the form of:

$$F_n^{(M)}(z) = \frac{1/M_n}{1 + R_{hn}}\left\{\exp\left[-ik\int_0^z (C_n^2 + \frac{2z}{a}S_n^2)^{\frac{1}{2}}dz\right] + R_{gn}\exp\left[ik\int_0^z (C_n^2 + \frac{2z}{a}S_n^2)^{\frac{1}{2}}dz\right]\right\}. \tag{7}$$

In the above two formulas, C_n are the roots of the following mode equation:

$$\begin{aligned}&\left[\left(1 + R_h e^{2ikH}\right) - C'\Delta_{22}\left(1 - R_h e^{2ikH}\right)\right] \\ &\quad\left[C'\left(1 - R_g e^{2ikH}\right) + \Delta_{11}\left(1 + R_g e^{2ikH}\right)\right] \\ &\quad + C'\Delta_{21}\Delta_{12}\left(1 + R_g e^{2ikH}\right)\left(1 - R_h e^{2ikH}\right) = 0,\end{aligned} \tag{8}$$

where

$$R_g = \frac{C - \Delta_g}{C + \Delta_g}, \tag{9}$$

$$R_h = \frac{C - 1/\Delta_g}{C + 1/\Delta_g}, \tag{10}$$

$$H = \int_0^h \left(C^2 + \frac{2z}{a}S^2\right)^{\frac{1}{2}} dz, \tag{11}$$

$$C' = \left(C^2 + \frac{2h}{a}S^2\right)^{\frac{1}{2}}, \tag{12}$$

and Δ_g is the normalized surface impedance of the earth. At lower frequencies, the normalized surface impedance Δ_g at the earth's surface is approximated as:

$$\Delta_g = \frac{k}{k_g}\sqrt{1 - \left(\frac{k}{k_g}\right)^2}, \tag{13}$$

where Δ_g is the wave number of the earth. The normalized surface impedance matrix of the ionosphere $[\Delta]$ is written in the form:

$$[\Delta]=\begin{bmatrix}\Delta_{11} & \Delta_{12}\\ \Delta_{21} & \Delta_{22}\end{bmatrix}. \tag{14}$$

It is noted that the normalized surface impedance matrix of ionosphere $[\Delta]$ can be determined and computed by using the proposed method in the classic book by Budden [3]. The parameters C and S are defined by:

$$\frac{\nu(\nu+1)}{k^2a^2}=S^2;\quad C^2=1-S^2. \tag{15}$$

And M_n in (7) is the coupling factor to show the coupling relationship between TE wave and TM wave. It is written in the form:

$$M_n=\frac{\Delta_{12}C_n'(1-R_{hn}e^{2ikH_n})}{C_n'(1-R_{gn}e^{2ikH_n})+\Delta_{11}(1+R_{gn}e^{2ikH_n})}. \tag{16}$$

It is noted that the excitation factor for the electromagnetic field of a vertical magnetic dipole is different to that for the electromagnetic field of vertical electric dipole. It is necessary to examine the "orthogonality" between waves of different orders. The inner product of the two waves with different orders is defined by:

$$A_{mn}=\int_0^h\left[G_n^{(M)}(z)G_m^{(M)}(z)+\rho_1F_n^{(M)}(z)F_m^{(M)}(z)\right]dz, \tag{17}$$

where $\rho_1=\Delta_{21}/\Delta_{12}$. Since $G_n^{(M)}$, $G_m^{(M)}$, $F_n^{(M)}$ and $F_m^{(M)}$ satisfy the following differential equation:

$$Z''(r)+k^2\left[1-\frac{\nu(\nu+1)}{k^2r^2}\right]Z(r)=0, \tag{18}$$

we get:

$$\begin{aligned}\frac{d}{dz}&\left(F_n^{(M)}F_m'^{(M)}-F_m^{(M)}F_n'^{(M)}\right)\\ &=F_n^{(M)}F_m''^{(M)}-F_m^{(M)}F_n''^{(M)}\\ &=k^2\left[\frac{\mu(\mu+1)}{k^2r^2}-\frac{\nu(\nu+1)}{k^2r^2}\right]F_n^{(M)}F_m^{(M)}\\ &\approx k^2\left(S_m^2-S_n^2\right)F_n^{(M)}F_m^{(M)},\end{aligned} \tag{19}$$

where

$$S_m^2=\frac{\mu(\mu+1)}{k^2a^2};\quad S_n^2=\frac{\nu(\nu+1)}{k^2a^2}. \tag{20}$$

Considering the boundary conditions, we have:

$$\begin{aligned}\int_0^h F_n^{(M)}F_m^{(M)}dz&=\frac{1}{k^2\left(S_m^2-S_n^2\right)}\\ &\left[F_n^{(M)}(z)F_m'^{(M)}(z)-F_m^{(M)}(z)F_n'^{(M)}(z)\right]_0^h\\ &=-\frac{1}{k^2\left(S_m^2-S_n^2\right)}\frac{ik\Delta_{12}}{\Delta_{22}}\\ &\left[F_n^{(M)}(h)G_m^{(M)}(h)-F_m^{(M)}(h)G_n^{(M)}(h)\right].\end{aligned} \tag{21}$$

Similarly, we write:

$$\begin{aligned}\int_0^h G_n^{(M)}G_m^{(M)}dz&=-\frac{1}{k^2\left(S_m^2-S_n^2\right)}\frac{ik\Delta_{21}}{\Delta_{22}}\\ &\left[G_n^{(M)}(h)F_m^{(M)}(h)-G_m^{(M)}(h)F_n^{(M)}(h)\right].\end{aligned} \tag{22}$$

It is seen that, when $\Delta_{12}=\Delta_{21}=0$, both TE and TM waves are not coupled each other. When $n\neq m$, we have:

$$A_{mn}=\int_0^h\left(G_n^{(M)}G_m^{(M)}+\rho_1F_n^{(M)}F_m^{(M)}\right)=0. \tag{23}$$

Then, the factor A_{nn} is expressed in the following form:

$$\begin{aligned}A_{nn}&=\int_0^h\left(G_n^{(M)}G_n^{(M)}+\rho_1F_n^{(M)}F_n^{(M)}\right)dz\\ &=\frac{1}{2k^2S_n}\left\{\left[G_n^{(M)}(h)\frac{dG_n'(h)}{dS_n}-G_n'(h)\frac{dG_n^{(M)}(h)}{dS_n}\right]\right.\\ &\left.+\rho_1\left[F_n^{(M)}(h)\frac{dF_n'(h)}{dS_n}-F_n'(h)\frac{dF_n^{(M)}(h)}{dS_n}\right]\right\}.\end{aligned} \tag{24}$$

With substitutions (6) and (7) into (23), we have:

$$\begin{aligned}A_{nn}=&\frac{1}{2k^2S_n(1+R_h)^2}\\ &\left\{-4k^2C_n'\left(\frac{\rho_1}{M_n^2}R_g+R_h\right)\frac{\partial H}{\partial S_n}\right.\\ &+\left[\left(\frac{\rho_1}{M_n^2}R_g^2+R_h^2\right)\exp(2ikH)\right.\\ &\left.-\left(\frac{\rho_1}{M_n^2}+1\right)\exp(-2ikH)\right]ik\frac{\partial C_n'}{\partial S_n}\\ &\left.+2ikC_n'\left(\frac{\rho_1}{M_n^2}\frac{\partial R_g}{\partial S_n}+\frac{\partial R_h}{\partial S_n}\right)\right\}.\end{aligned} \tag{25}$$

Then, the field component E_ϕ can be expressed in the form:

$$E_\phi=-\frac{i\omega\mu}{r\eta}\sum\Lambda_nG_n^{(M)}(z)\frac{\partial}{\partial\theta}P_\nu(\cos(\pi-\theta)). \tag{26}$$

In the next step, the main task is to determine the excitation factor Λ_n . It is noted that the excitation source (vertical magnetic dipole) can be regarded as a small loop antenna which is placed at the height of z_s . When the observation point is infinitely close to the excitation source, the effects by the earth and the ionosphere can be neglected. Thus, we write:

$$\lim_{\theta\to 0} 2\pi r_s E_\varphi\left(r,\theta\right) = iu_0\omega H\left(r,0\right)da \rightarrow \frac{i\mu_0\omega Ida}{\theta}\delta\left(r-r_s\right). \quad (27)$$

Multiplying the function $G_n^{(M)}(z)$ on both sides of (27) and integrating from 0 to h for z , with (26), it follows that:

$$\lim_{\theta\to 0}\frac{i\omega\mu_0 Ida}{2\pi r_s\theta}G_n^{(M)}\left(z_s\right) = \lim_{\theta\to 0} -\frac{i\omega\mu_0}{r_s\eta}\Lambda_n A_{nn}\frac{\partial}{\partial\theta}P_\nu\left(\cos\left(\pi-\theta\right)\right). \quad (28)$$

Considering,

$$\lim_{\theta\to 0}\frac{\partial P_\nu(\cos(\pi-\theta))}{\partial\theta} = \frac{2\sin\nu\pi}{\pi\theta}, \quad (29)$$

we have:

$$\Lambda_n = -\frac{\eta Ida G_n^{(M)}(z_s)}{4A_{nn}\sin\nu\pi}. \quad (30)$$

In SLF/ELF ranges, the normalized surface impedance Δ_g is small, so that the excitation factor of a vertical magnetic dipole Λ_n is very small. In other words, for the electromagnetic field in SLF/ELF ranges radiated by a vertical magnetic dipole in the presence of an earth-ionosphere cavity, the excitation efficiency is very low.

From (3a)-(3f), (5a)-(5b), and (30), the six field components E_r , E_θ , E_ϕ , H_r , H_θ and H_ϕ of the electromagnetic field radiated by a vertical magnetic dipole at $\left(a+z_s,0,0\right)$ in the presence of an anisotropic earth-ionosphere cavity, are expressed in the following forms:

$$E_r\left(r,\theta,\varphi\right) = \frac{\eta Ida}{4r^2}\sum_n\frac{G_n^{(M)}\left(z_s\right)F_n^{(M)}}{A_{nn}}\frac{\partial^2 P_\nu\left(\cos\left(\pi-\theta\right)\right)}{\sin\nu\pi\partial^2\theta}, \quad (31)$$

$$E_\theta\left(r,\theta,\varphi\right) = \frac{-\eta Ida}{4r}\sum_n\frac{G_n\left(\nu z_s\right)}{A_{nn}}\frac{\partial F_n^{(M)}\left(z\right)}{\partial z}\frac{\partial P_\nu\left(\cos\left(\pi-\theta\right)\right)}{\sin\nu\pi\partial\theta}, \quad (32)$$

$$E_\varphi\left(r,\theta,\varphi\right) = \frac{i\omega\mu Ida}{4r}\sum_n\frac{G_n^{(M)}\left(z_s\right)G_n^{(M)}\left(z\right)}{A_{nn}}\frac{\partial P_\nu\left(\cos\left(\pi-\theta\right)\right)}{\sin\nu\pi\partial\theta}, \quad (33)$$

$$H_r\left(r,\theta,\varphi\right) = \frac{Ida}{4r^2}\sum_n\frac{G_n^{(M)}\left(z_s\right)G_n^{(M)}\left(z\right)}{A_{nn}}\frac{\partial^2 P_\nu\left(\cos\left(\pi-\theta\right)\right)}{\sin\nu\pi\partial^2\theta}, \quad (34)$$

$$H_\theta\left(r,\theta,\varphi\right) = \frac{-Ida}{4r}\sum_n\frac{G_n^{(M)}\left(z_s\right)}{A_{nn}}\frac{\partial G_n^{(M)}\left(z\right)}{\partial z}\frac{\partial P_\nu\left(\cos\left(\pi-\theta\right)\right)}{\sin\nu\pi\partial\theta}, \quad (35)$$

$$H_\varphi\left(r,\theta,\varphi\right) = \frac{-i\omega\varepsilon\eta Ida}{4r}\sum_n\frac{G_n^{(M)}\left(z_s\right)F_n^{(M)}\left(z\right)}{A_{nn}}\frac{\partial P_\nu\left(\cos\left(\pi-\theta\right)\right)}{\sin\nu\pi\partial\theta}, \quad (36)$$

where $r=z+a$.

B. Field of vertical electric dipole

If the excitation source is replaced by a vertical electric dipole, which is represented by the current density $\hat{z}Idl\delta(x)\delta(y)\delta(z-b)$, the formulas for the components E_r , E_θ , E_ϕ , H_r , H_θ and H_ϕ of the electromagnetic field radiated by a vertical electric dipole at (a+z,0,0) in the presence of an anisotropic earth-ionosphere cavity, were derived specifically in the monograph by Pan [27]. We write:

$$E_r\left(r,\theta,\varphi\right) = \frac{iIda}{4\omega\varepsilon r^2}\sum_m\frac{F_m^{(E)}\left(z_s\right)F_m^{(E)}\left(z\right)}{A_{mm}}\frac{\partial^2 P_\mu\left(\cos\left(\pi-\theta\right)\right)}{\sin\mu\pi\partial^2\theta}, \quad (37)$$

$$E_\theta\left(r,\theta,\varphi\right) = \frac{-iIdl}{4\omega\varepsilon r}\sum_m\frac{F_m^{(E)}\left(z_s\right)}{A_{nn}}\frac{\partial F_m^{(E)}\left(z\right)}{\partial z}\frac{\partial P_\mu\left(\cos\left(\pi-\theta\right)\right)}{\sin\mu\pi\partial\theta}, \quad (38)$$

$$E_\varphi(r,\theta,\varphi) = \frac{-\mu Idl}{4r\eta\varepsilon}\sum_m \frac{F_m^{(E)}(z_s)G_m^{(E)}(z)}{A_{mm}} \frac{\partial P_\mu(\cos(\pi-\theta))}{\sin\mu\pi\partial\theta}, \tag{39}$$

$$H_r(r,\theta,\varphi) = \frac{iIdl}{4\eta\omega\varepsilon r^2}\sum_n \frac{F_m^{(E)}(z_s)G_m^{(E)}(z)}{A_{mm}} \frac{\partial^2 P_\mu(\cos(\pi-\theta))}{\sin\mu\pi\partial^2\theta}, \tag{40}$$

$$H_\theta(r,\theta,\varphi) = \frac{-iIdl}{4\eta\omega\varepsilon r}\sum_m \frac{F_m^{(E)}(z_s)}{A_{mm}}\frac{\partial G_m^{(E)}(z)}{\partial z} \frac{\partial P_\mu(\cos(\pi-\theta))}{\sin\mu\pi\partial\theta}, \tag{41}$$

$$H_\varphi(r,\theta,\varphi) = \frac{Idl}{4r}\sum_n \frac{F_m^{(E)}(z_s)F_m^{(E)}(z)}{A_{mm}} \frac{\partial P_\mu(\cos(\pi-\theta))}{\sin\mu\pi\partial\theta}, \tag{42}$$

where

$$A_{mm} = \frac{1}{2k^2 S_n (1+R_g)^2} \Big\{-4ikC_n'\left(R_g+\rho^2 M_m^2 R_h\right)\frac{\partial H}{\partial S_m} + \Big[\left(1+\rho_2 M_m^2\right)\exp(-2ikH_m) - \left(R_g^2+\rho_2 M_n^2 R_h^2\right)\exp(2ikH)\Big]\frac{\partial C_m'}{\partial S_m} - 2C_n'\left(\frac{\partial R_g}{\partial S_m}+\rho_2 M_n^2\frac{\partial R_h}{\partial S_m}\right)\Big\}, \tag{43}$$

$$\rho = \Delta_{12}/\Delta_{21}. \tag{44}$$

It is noted that the "height-gain" function $F_m^{(E)}(z)$ of the potential function U is normalized as $F_m^{(E)}(z=0)=1$. Then, we write:

$$F_m^{(E)}(z) = \frac{1}{1+R_{gm}}\left\{\exp\left[-ik\int_0^z (C_m^2+\frac{2z}{a}S_m^2)^{\frac{1}{2}}dz\right] + R_{gm}\exp\left[ik\int_0^z (C_m^2+\frac{2z}{a}S_m^2)^{\frac{1}{2}}dz\right]\right\}. \tag{45}$$

Correspondingly, the normalized "height-gain" function $G_m^{(E)}(z)$ of the potential function $\overline{\mathrm{V}}$ is written as follows:

$$G_m^{(E)}(z) = \frac{M_m}{1+R_{gm}}\left\{\exp\left[-ik\int_0^z (C_m^2+\frac{2z}{a}S_m^2)^{\frac{1}{2}}dz\right] + R_{hm}\exp\left[ik\int_0^z (C_m^2+\frac{2z}{a}S_m^2)^{\frac{1}{2}}dz\right]\right\}. \tag{46}$$

C. Field of a horizontal electric dipole

The electromagnetic field of a horizontal electric dipole can be obtained from the electromagnetic fields of vertical electric and magnetic dipoles by the reciprocity theorem [4-5]. Following similar manner addressed in Sec. 4.2.4 in the monograph by Galejs [5], the vertical field components E_r^{he} and H_r^{he} at $(a+z,0,0)$, where the subscript he designates the horizontal electric dipole, can be derived readily. We write:

$$E_r^{he}(r,\theta,\varphi) = -\frac{iIds^{he}}{4\omega\varepsilon_0 r}\cos\varphi\sum_m \frac{F_m^{(E)}(z)}{A_{mm}} \frac{\partial F_m^{(E)}(z_s)}{\partial z}\frac{\partial P_\mu(\cos(\pi-\theta))}{\sin\mu\pi\partial\theta}, \tag{47}$$

$$H_r^{he}(r,\theta,\varphi) = -\frac{Ids^{he}}{4r}\sin\varphi\sum_n \frac{G_n^{(M)}(z_s)G_n^{(M)}(z)}{A_{nn}} \frac{\partial P_\nu(\cos(\pi-\theta))}{\sin\nu\pi\partial\theta}. \tag{48}$$

From Maxwell's equations, the other field components E_θ^{he}, E_ϕ^{he}, H_θ^{he} and H_ϕ^{he} can be expressed in the terms of E_r^{he} and H_r^{he}, we have

$$\left(k^2+\frac{\partial^2}{\partial r^2}\right)\left(r\sin\theta H_\theta^{he}\right) = -i\omega\varepsilon_0\frac{\partial E_r^{he}}{\partial\varphi}+\sin\theta\frac{\partial^2 H_r^{he}}{\partial r\partial\theta}, \tag{49}$$

$$\left(k^2+\frac{\partial^2}{\partial r^2}\right)\left(r\sin\theta E_\theta^{he}\right) = \frac{\partial^2 E_r^{he}}{\partial r\partial\varphi}-i\omega\mu_0\sin\theta\frac{\partial H_r^{he}}{\partial r\partial\theta}, \tag{50}$$

$$\left(k^2+\frac{\partial^2}{\partial r^2}\right)\left(r\sin\theta H_\varphi^{he}\right) = i\omega\varepsilon_0\sin\theta\frac{\partial E_r^{he}}{\partial\theta}+\frac{\partial^2 H_r^{he}}{\partial r\partial\varphi}, \tag{51}$$

$$\left(k^2+\frac{\partial^2}{\partial r^2}\right)\left(r\sin\theta E_\theta^{he}\right) = \sin\theta\frac{\partial^2 E_r^{he}}{\partial r\partial\theta}+i\omega\mu_0\frac{\partial H_r^{he}}{\partial\theta}. \tag{52}$$

It is seen that the height functions of all modes of the electromagnetic field components in the earth-ionosphere cavity should be satisfied to the equation (18).

With substitutions (47) and (48) into (49)-(52), and considering the relation $\nu(\nu+1)/r \approx k^2 a$, the analytical formulas of the remaining four components are obtained readily. We write:

$$H_\theta^{he}(r,\theta,\phi) = \frac{Ids^{he}}{4k^2a^2}\sin\phi\left[\frac{1}{\sin\theta}\sum_m \frac{F_m^{(E)}(z)}{A_{mn}S_m^2} \frac{\partial F_m^{(E)}(z_s)}{\partial z}\frac{\partial P_\mu(\cos(\pi-\theta))}{\sin\mu\pi\partial\theta} - \sum_n \frac{1}{A_{nn}S_n^2}\frac{\partial G_n^{(M)}(z)}{\partial z}G_n^{(M)}(z_s) \frac{\partial^2 P_\nu(\cos(\pi-\theta))}{\sin\upsilon\pi\partial^2\theta}\right], \quad (53)$$

$$H_\phi^{he}(r,\theta,\phi) = \frac{Ids^{he}}{4k^2a^2}\cos\phi\left[\sum_m \frac{F_m^{(E)}(z)}{A_{mn}S_m^2} \frac{\partial F_m^{(E)}(z_s)}{\partial z}\frac{\partial^2 P_\mu(\cos(\pi-\theta))}{\sin\mu\pi\partial^2\theta} - \frac{1}{\sin\theta}\sum_n \frac{1}{A_{nn}S_n^2}\frac{\partial G_n^{(M)}(z)}{\partial z}G_n^{(M)}(z_s) \frac{\partial P_\nu(\cos(\pi-\theta))}{\sin\upsilon\pi\partial\theta}\right], \quad (54)$$

$$E_\phi^{he}(r,\theta,\phi) = \frac{iIdS}{4k^2a^2}\sin\phi \left[\frac{1}{\omega\varepsilon_0\sin\theta}\sum_m \frac{1}{A_{mm}S_m^2}\frac{\partial F_m^{(E)}(z)}{\partial z} \omega\mu_0\sum_n \frac{1}{A_{nn}S_n^2}G_n^{(M)}(z)G_n^{(M)}(z_s) \frac{\partial^2 P_\nu(\cos\pi-\theta)}{\sin\nu\pi\partial^2\theta}\right], \quad (55)$$

$$E_\theta^{he}(r,\theta,\phi) = -\frac{iIds^{he}}{4k^2a^2}\cos\phi \left[\frac{1}{\omega\varepsilon_0}\sum_m \frac{1}{A_{mn}S_m^2}\frac{\partial F_m^{(E)}(z)}{\partial z} \frac{\partial F_m^{(E)}(z_s)}{\partial z}\frac{\partial^2 P_\mu(\cos(\pi-\theta))}{\sin\mu\pi\partial^2\theta} + \frac{\omega\mu_0}{\sin\theta}\sum_n \frac{G_n^{(M)}(z)G_n^{(M)}(z_s)}{A_{nn}S_n^2} \frac{\partial P_\nu(\cos(\pi-\theta))}{\sin\upsilon\pi\partial\theta}\right]. \quad (56)$$

III. FIELD OF A HORIZONTAL ELECTRIC DIPOLE

From above derivations and analysis, it is seen that the approximated formulas are derived for the electromagnetic field of a horizontal electric dipole in the presence of an anisotropic earth-ionosphere cavity. In VLF ranges, the electromagnetic fields of a dipole source (vertical electric dipole, vertical magnetic dipole, and horizontal electric dipole) can be computed readily by using (47), (48), and (53)-(56). Obviously, the computations can also be carried out for the electromagnetic field in SLF/ELF ranges of a dipole source (VED, VMD, or HED) in the presence of anisotropic earth-ionosphere cavity.

In practical applications, the horizontal antenna is usually placed on or near the earth's surface, and considering the height of the antenna is very small comparing to the wavelength of the electromagnetic wave in SLF/ELF ranges, the horizontal dipole source and the observation point can be regarded to be placed on the surface of the earth; namely, $z_s \sim 0$ and $z \sim 0$. Then, we have:

$$F_m^{(E)}(0) = 1, \qquad G_m^{(E)}(0) = M_m(1+R_{hm})/(1+R_{gm}),$$

$$G_n^{(M)}(0) = 1, \qquad F_n^{(M)}(0) = (1+R_{gn})/M_n(1+R_{hn}),$$

$$\frac{\partial F_m^{(E)}(z)}{k\partial z}\Big|_{z=0} = -i\Delta_g, \quad \frac{\partial G_n^{(M)}(z)}{k\partial z}\Big|_{z=0} = \frac{-i}{\Delta_g}.$$

At large distance between the dipole source and the observation point, namely, $k\rho >> 1$, and the observation is not close to an antipole, the Legendre function of the first kind is approximated by:

$$\frac{P_\nu(\cos(\pi-\theta))}{\sin\upsilon\pi} \approx -\sqrt{\frac{2}{\pi kaS_n\sin\theta}}\exp(ikaS_n\theta + \frac{i\pi}{4}). \quad (57)$$

Thus, we have:

$$\frac{\partial P_\nu(\cos(\pi-\theta))}{\sin\upsilon\pi\partial\theta} \approx -i\sqrt{\frac{2kaS_n}{\pi\sin\theta}} \exp\left[i\left(\nu+\frac{1}{2}\right)\theta + \frac{i\pi}{4}\right], \quad (58)$$

$$\frac{\partial^2 P_\nu(\cos(\pi-\theta))}{\sin\upsilon\pi\partial^2\theta} \approx \sqrt{\frac{2}{\pi\sin\theta}}(\mathrm{ka}S_\mathrm{n})^{3/2} \exp\left[i\left(\nu+\frac{1}{2}\right)\theta + \frac{i\pi}{4}\right]. \quad (59)$$

When the observation point is close to an antipole of the dipole source, the approximate formulas of the functions $\frac{\partial P_\nu(\cos(\pi-\theta))}{\partial\theta}$ and $\frac{\partial^2 P_\nu(\cos(\pi-\theta))}{\partial^2\theta}$ are written in the following forms:

$$\frac{\partial P_\nu(\cos(\pi-\theta))}{\partial\theta} \approx \frac{\delta}{2}\left[\left(\nu+\frac{1}{2}\right)^2-\frac{1}{4}\right]-\frac{\delta^3}{16}\left(\nu+\frac{1}{2}\right)^2 \left[\left(\nu+\frac{1}{2}\right)^2-\frac{7}{6}\right], \quad (60)$$

$$\frac{\partial^2 P_\nu(\cos(\pi-\theta))}{\partial\theta^2} \approx \frac{-1}{2}\left[\left(\nu+\frac{1}{2}\right)^2-\frac{1}{4}\right] +\frac{3\delta^2}{16}\left(\nu+\frac{1}{2}\right)^2\left[\left(\nu+\frac{1}{2}\right)^2-\frac{7}{6}\right], \quad (61)$$

where $\delta=\pi-\theta$.

At short distance between the dipole source and the observation point, namely, $k\rho<<1$, we have:

$$\frac{\partial P_\nu(\cos(\pi-\theta))}{\sin\nu\pi\partial\theta} \approx i\left(\nu+\frac{1}{2}\right)H_1^{(1)}\left[\left(\nu+\frac{1}{2}\right)\theta\right], \quad (62)$$

$$\frac{\partial^2 P_\nu(\cos(\pi-\theta))}{\sin\nu\pi\partial^2\theta} \approx \frac{i(\nu+\frac{1}{2})H_1^{(1)}[(\nu+\frac{1}{2})\theta]}{\theta}. \quad (63)$$

By using above approximate formulas, the computations can be simplified greatly. In the following computations, the electron density of the lower ionosphere is described by a two-parameter exponential profile [29,30],

$$N_e(z)=1.43\times10^7 e^{-0.15h}e^{(\beta-0.15)(z-h)}cm^{-3}, \quad (64)$$

where the two parameters h in kilometers and β in km^{-1} control the altitude of the profile and the sharpness of the ionospheric transition, respectively. Then, by using the proposed method in the book by Budden [3], the normalized surface impedance matrix of the anisotropic ionosphere $[\Delta]$ can be computed readily.

With a=6378 km, $\phi=\pi/4$, $\sigma_g=10^{-4}$ S/M, h=85 km, and β=0.3 km^{-1}, the magnitudes of E_θ and H_ϕ at f=100 Hz are computed and shown in Figs. 2 and 3, respectively. In above computations, numerical results are obtained for both isotropic and anisotropic case by using the proposed method, respectively. Comparing to the available results for isotropic case, it is seen that the numerical results for isotropic case by using the proposed method in this paper are in good agreement with the corresponding results by using spherical harmonic series solution addressed in Chapter 2 in the book [28]. From Figs. 2 and 3, it is seen that when the propagation distance ρ is close to 20000 km, the magnitudes of the field components are enlarged. This is caused by the multi-path effects.

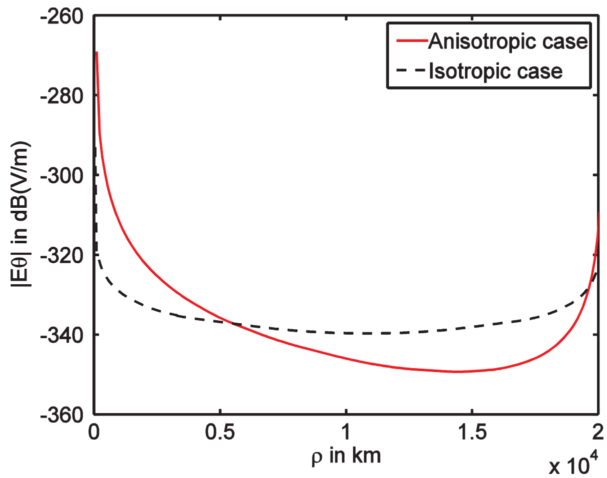

Fig. 2. The electric field $|E_\theta|$ in V/m at f=100 Hz with a=6378 km, $\phi=\pi/4$, $\sigma_g=10^{-5}$ S/m, h=85 km, and β=0.3 km^{-1}.

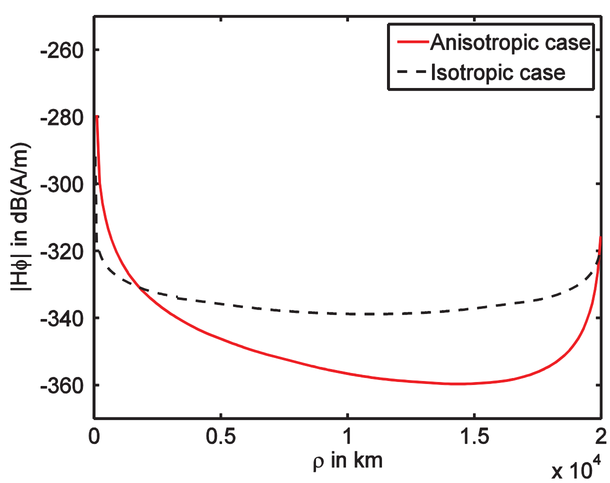

Fig. 3. The magnetic field $|H_\phi|$ in A/m at f=100 Hz with a=6378 km, $\phi=\pi/4$, $\sigma_g=10^{-5}$ S/m, h=85 km, and β=0.3 km^{-1}.

When the observation point is close to the antipole of the dipolesource, the multipath effects should be considered. The electric field components E_r, E_θ, and E_ϕ, and the magnetic field components H_θ and H_ϕ versus the distance ρ from the antipole at f=100 Hz are computed and shown in Figs. 4 and 5, respectively. The electromagnetic wave, which propagates along the large circular path should not be neglected. The total field included the electromagnetic wave propagating along both the short and large circular paths. Obviously, it is seen that when the observation point is close to the antipole, the interference between the electromagnetic waves along different propagation paths occurs.

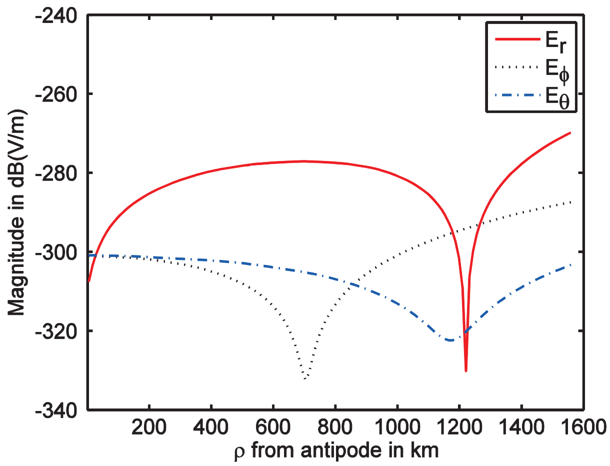

Fig. 4. Magnitudes of the components E_r, E_θ, E_ϕ at f=100 Hz with a=6378 km, $\phi = \pi/4$, $\sigma_g = 10^{-5}$ S/m, h=85 km, and β =0.3 km^{-1}.

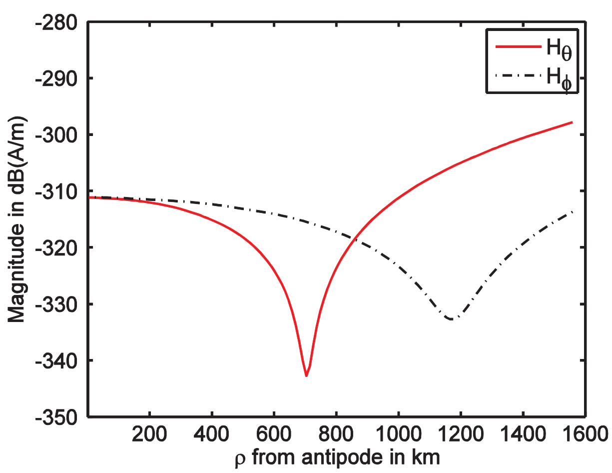

Fig. 5. Magnitudes of the components H_θ and H_ϕ at f=100 Hz with a=6378 km, $\phi = \pi/4$, $\sigma_g = 10^{-5}$ S/m, h=85 km, and β =0.3 km^{-1}.

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REFERENCES

[1] H. Bremmer, "Terrestrial radio waves," *New York: Elsevier*, 1949.

[2] C. J. Bouwkamp and H. B. G. Casimir, "On multipole expansions in the theory of electromagnetic radiation," *Physica*, vol. 20, pp. 539-554, 1954.

[3] K. G. Budden, "Radio waves in the ionosphere," *Cambridge: Cambridge University Press*, 1961.

[4] J. R. Wait, "Electromagnetic waves in stratified media," *New York: Pergamon Press*, 1970.

[5] J. Galejs, "Terrestrial propagation of long electromagnetic waves," *New York: Pergamon Press*, 1972.

[6] L. B. Felsen and N. Marcuvitz, "Radiation and scattering of waves, englewood cliffs," *New York: Prentice-Hall*, 1973.

[7] A. P. Nickolaenko and M. Hayakawa, "Resonances in the earth-ionosphere cavity," *Kluwer Acad. Pub*., 2002.

[8] J. R. Wait, "The mode theory of VLF ionospheric propagation for finite ground conductivity," *Proceedings of the IRE*, vol. 45, no. 6, pp. 760-767, 1957.

[9] J. R. Wait, "Terrestrial propagation of very-low-frequency radio waves, a theoretical investigation," *J. Res. Nat. Bureau Stand.*, vol. 64D, pp. 153-204, 1960.

[10] J. R. Wait and K. P. Spies, "Influence of finite ground conductivity on the propagation of VLF radio waves," *Radio Sc.*, vol. 3, pp. 787-791, 1965.

[11] W. O. Schumann, "On the radiation free self-oscillations of conducting sphere which is surrounded by an air layer and an ionospheric shell," (In German), *Zeitschrift und Naturfirschung*, 72: 149-154, 1952.

[12] W. O. Schumann, "On the damping of electromagnetic self-oscillations of the system earth-air-ionosphere," (In German), *Zeitschrift und Naturfirschung*, 72: 250-252, 1952.

[13] J. Galejs and R. Row, "Propagation of ELF waves below an inhomogeneous anisotropic ionosphere," *IEEE Trans. Antennas Propagat.*, vol. 12, no. 1, pp. 74-83, 1964.

[14] J. Galejs, "Terrestrial extremely-low-frequency propagation, natural electromagnetic phenomena below 30 kc/s," Bleil DF (ed.), New York, NY, *USA: Plenum Press*, 205-258, 1964.

[15] J. Galejs, "ELF and VLF fields of a horizontal electric dipole," *IEEE Trans. Antennas Propagat.*, vol. 16, no. 6, pp. 689-700, 1968.

[16] J. Galejs, "ELF and VLF propagation for models of a perturbed ionosphere," *Radio Sci.*, vol. 5, no. 7, pp. 1041-1044, 1970.

[17] J. Galejs, "Stable solutions of ionospheric fields in the propagation of ELF and VLF waves," *Radio Sci.*, vol. 7, no. 6, pp. 549-561, 1972.

[18] P. R. Bannister, E. A. Wolkoff, J. R. Katan, and F. J. Williams, "Far-field, extremely-low-frequency propagation measurements," *Radio Sci.*, vol. 8, no. 7, pp. 623-631, 1973.

[19] V. K. Tripathi, C. L. Chang, and K. Papadopoulos, "Excitation of the earth-ionosphere waveguide by an ELF source in the ionosphere," *Radio Sci.*, vol. 17, no. 5, pp. 1321-1326, 1982.

[20] P. R. Bannister, "ELF propagation update," *IEEE J. Oceanic Eng*., vol. 9, no. 3, pp. 179-188, 1984.

[21] K. J. Carroll and A. J. Ferraro, "Computer simulation of ELF injection in the earth-ionosphere waveguide," *Radio Sci*., vol. 25, no. 6, pp. 1363-1367, 1990.

[22] A. C. Fraser-Smith and P. R. Bannister, "Reception of ELF signals at antipodal distance," *Radio Sci*., vol. 33, no. 1, pp. 83-88, 1998.

[23] D. E. Barrick, "Exact ULF/ELF dipole field strengths in the earth-ionosphere cavity over the schumann resonance region: idealized boundaries," *Radio Sci*., vol. 34, no. 1, pp. 209-227, 1999.

[24] S. A. Cummer, "Modeling electromagnetic propagation in the earth-ionosphere waveguide," *IEEE Trans. Antennas Propagat*., vol. 48, no. 9, pp. 468-474, 2000.

[25] S. T. Rybachek and M. J. Ponomariev, "Efficiency of the earth-ionosphere waveguide excitation by ELF sources located in an anisotropic ionosphere," *Int. J. Geomagn. Aeron.*, vol. 7, GI1005, doi:10.1029/2005GI000123, 2007.

[26] V. V. Kirillov and A. E. Pronin, "Normal waves of the anisotropic earth-ionosphere waveguide in the VLF-ULF range," *Int. J. Geomagn. Aeron*., vol. 7, GI2006, doi:10.1029/2005GI000126G, 2007.

[27] W. Y. Pan, "LF/VLF/ELF wave propagation," (In Chinese), *UESTC Press*, Chengdu, China, 2004.

[28] W. Y. Pan and K. Li, "Propagation of SLF/ELF electromagnetic waves," *Springer-Verlag jointly with ZJU Press*, 2014.

[29] K. Rawer, D. Bilitza, and S. Ramakrishnan, "Goals and status of the international reference ionosphere," *Rev. Geophys*., vol. 16, pp. 177-181, 1978.

[30] S. A. Cummer, U. S. Inan, and T. F. Bell, "Ionospheric d region remote sensing using VLF radio atmospherics," *Radio Sci.*, vol. 33, pp. 1781-1792, November-December 1998.

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