The Error Cross-Section Method for Quantifying the Error in Electromagnetic Scattering Problems

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Abstract **─** The Error Cross-Section (ECS) is introduced to quantify the error associated with the numerical solution of electromagnetic scattering problems. The ECS accounts for different approximations and inaccuracies in the object discretization and numerical computations. The ECS definition is based on the power conservation principle and is visualized by comparing it to the radar cross-section of a thin wire for two-dimensional (2-D) problems or a small sphere for three-dimensional (3-D) problems. The proposed ECS method is independent of the adopted numerical technique and therefore can be used to give confidence in the obtained solution using several methods, such as the Method of Moments (MoM) and the Finite-Difference Frequency-Domain (FDFD) method. Application of the ECS to the optimization of certain parameters for some numerical formulations, such as the Combined-Field Integral Equation (CFIE) is also presented.

Index Terms – CFIE, numerical error, radar cross-section.

I. INTRODUCTION

Numerical treatment of Maxwell's equations has steadily advanced for decades and a variety of computational methods have been devised to solve electromagnetic problems, especially problems involving scattering from arbitrarily shaped objects [1]-[4]. Such problems require using geometrical discretization methods to model the objects in a manner amenable to computers, followed by approximations of the equations associated with the used formulation and finally adopting a numerical routine to evaluate such approximate forms.

As yet, there have been quite a few works on the quantification of the error associated with the above procedure in computational electromagnetics [5]-[8]. Commercial software packages do not provide a confidence level to the user in the accuracy of the produced results. Verifying the boundary conditions (frequently used to guarantee that the solution satisfies them) typically uses the same operator equation, which has been approximated and thus suffers from complications related to singularities, mesh inaccuracy etc. and is specific to the adopted method.

This work proposes an error quantification approach, which is not only independent of the adopted numerical method but also visualizes the error by the so-called Error Cross-Section (ECS). In section II, the numerical error in solving a general electromagnetic scattering problem is discussed. In the same section, the definition of the ECS is presented and its relation to the Radar Cross-Section (RCS) of a thin conducting wire is investigated. Also, the correlation between the proposed error measure and the actual error is studied. In section III, the ECS is computed for various scattering problems using different numerical methods, such as the MoM and the FDFD, with an application to the optimization of the mixing factors in the MoM combined-field formulation. Conclusions and discussions are given in section IV.

II. PROBLEM FORMULATION

A. Residual error in scattering problems

Figure 1 shows an arbitrary object illuminated by a uniform plane wave. The total fields in the region enclosing the scatterer are given by:

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$$\mathbf{E}^t = \mathbf{E}^i + \mathbf{E}^s, \tag{1}$$
$$\mathbf{H}^t = \mathbf{H}^i + \mathbf{H}^s. \tag{2}$$

Typically, the scattered fields are determined using a numerical technique for arbitrarily shaped objects. The inaccuracies associated with the adopted technique result in an error in the total fields, i.e.:

$$\delta\mathbf{E}^t = \mathbf{E}^t|_{num} - \mathbf{E}^t|_{exact}, \tag{3}$$
$$\delta\mathbf{H}^t = \mathbf{H}^t|_{num} - \mathbf{H}^t|_{exact}. \tag{4}$$

Estimation of the residual error $\delta\mathbf{E}^t$ and $\delta\mathbf{H}^t$ requires obtaining the exact solution $\mathbf{E}^t|_{exact}$ and $\mathbf{H}^t|_{exact}$, which is usually unknown for non canonical problems. Conventionally, numerical methods adopt certain convergence criteria to the solution by increasing the number of unknowns till the difference between the current and the previous solutions becomes acceptable. This, however, neither gives an indication about the error in the current solution with respect to the exact one, nor provides a physical insight into the quantified error. In this work, the goal is to define a new quantity, conveniently referred to as the Error Cross-Section (ECS), which is correlated with the actual residual error in a way that is independent of the adopted numerical technique. Unlike most error estimates [5]-[8], only few have a physical meaning like the Sobolev norm [9]. The ECS has this advantage and can be used to visualize the quantified error by comparing its definition to the RCS. Furthermore, the highest solution accuracy that can be achieved on a specific machine can be deduced by finding the lower limit of the ECS.

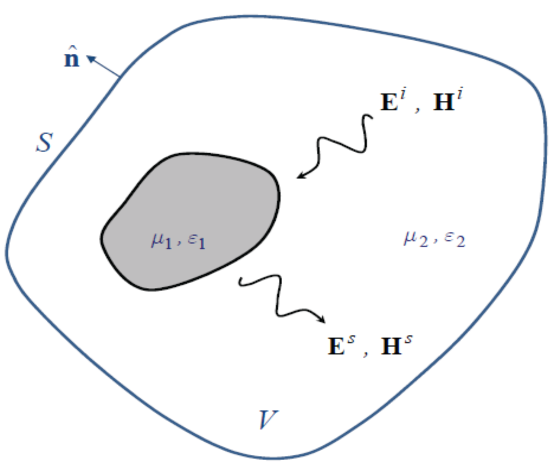


Fig. 1. Geometry of a general scattering problem.

B. Definition of the ECS

In electromagnetic scattering problems, the power conservation principle [10] requires that the total exiting power W^t must vanish if the media bounded by S were lossless, i.e.:

$$W^t = \frac{1}{2}\oiint_S \operatorname{Re}\left\{\mathbf{E}^t|_{exact} \times \mathbf{H}^{t^*}\big|_{exact}\right\} \cdot \hat{\boldsymbol{n}}dS = 0, \tag{5}$$

where $\hat{\boldsymbol{n}}$ is the outward normal to the surface S. Ideally, (5) should be satisfied; however, due to the errors in the scattered fields computation, the integral in (5) yields a residual value, viz.

$$W_{res}^t = \frac{1}{2}\oiint_S \operatorname{Re}\left\{\mathbf{E}^t|_{num} \times \mathbf{H}^{t^*}\big|_{num}\right\} \cdot \hat{\boldsymbol{n}}dS \neq 0. \tag{6}$$

Thus, the ECS is defined as follows:

$$\text{ECS} = \frac{W_{res}^t}{P_i|_{num}}, \tag{7}$$

where $P_i|_{num} = \frac{1}{2}\operatorname{Re}\left\{\mathbf{E}^i\big|_{num} \times \mathbf{H}^{i^*}\big|_{num}\right\}$ is the incident power density.

For 2-D problems, the Error Width (EW) is used instead of the more general term ECS and the integral in (6) is performed on a contour C rather than on a surface S. This is similar to using the scattering width in 2-D problems, instead of the radar cross-section used in 3-D problems.

A fundamental lower limit to the EW is attributed to the error in the numerical evaluation of (6) in the absence of the scatterer, i.e.:

$$\text{EW}^{min} = \frac{W_{res}^i}{P_i|_{num}}, \tag{8}$$

where $W_{res}^i = \frac{1}{2}\oint_C \operatorname{Re}\left\{\mathbf{E}^i\big|_{num} \times \mathbf{H}^{i^*}\big|_{num}\right\} \cdot d\mathbf{l} \neq 0$.

Although there could be different combinations of $\mathbf{E}^t|_{num}$ and $\mathbf{H}^t|_{num}$, which satisfy the power conservation principle, only one of them is correct in light of the uniqueness theorem [10]. Therefore, before finding the ECS, the boundary conditions should be verified to guarantee that the solution under consideration is actually the correct one. It is important to underline that the goal behind using any error estimate is not to decide whether the solution is correct or not, but to find out how accurate a correct solution is and to establish a confidence level in it. In light of this, error estimates can be employed for many purposes, i.e.: to minimize the computational effort using a specific numerical method by determining the optimum number of unknowns and to find the highest obtainable accuracy. Also, they can be used to compare the accuracy of different methods for a given problem and to provide a physical meaning for the error.

C. Physical meaning of the ECS

To visualize the quantified error and to have a feeling about how much this error is for a specific problem, the definition of the ECS is compared to the RCS or the scattering width (SW) in case of 2-D problems, which is defined as [10]:

$$\sigma_{2D} = \lim_{\rho\to\infty} 2\pi\rho \frac{|\mathbf{E}^s|^2}{|\mathbf{E}^i|^2}. \quad (9)$$

This can give an estimate of the error associated with the solution as a residual field $\delta\mathbf{E}^t$ and $\delta\mathbf{H}^t$, due to scattering from a fictitious thin wire, as compared to the original problem of scattering from the actual object. Figure 2 shows the SW of a thin conducting wire of radius a_w. The ordinate of Fig. 2 will be used to access the EW of the solution to determine the radius of the corresponding thin wire.

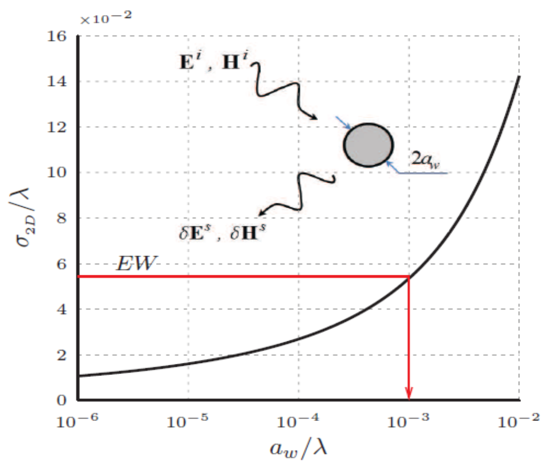


Fig. 2. Scattering width of a conducting wire with radius a_w normalized to the wavelength λ. The arrow shows how the figure is used to visualize the error width.

D. Correlation between the EW and the actual residual error

The correlation between the EW and the actual residual error $|\delta\mathbf{E}^t|$ for the problem of TM^z plane wave scattering from a 2-D circular PEC cylinder having a radius $a = \lambda/2$, is studied (see Fig. 3). In this example, the EFIE formulation of the moment method is adopted and the effect of varying the number of basis functions per wavelength (N_λ) is investigated. Invoking (6), the total exiting power is computed on a circular contour of radius $b = a + \lambda/4$. The exact solution $\mathbf{E}^t|_{exact}$ for this problem can be found analytically using [10]:

$$\mathbf{E}^t|_{exact} = -E_0 \sum_{n=-\infty}^{\infty} j^n \frac{J_n(k_0 a)}{H_n^{(2)}(k_0 a)} H_n^{(2)}(k_0 b) e^{jn\phi}, \quad (10)$$

where J_n and $H_n^{(2)}$ are the Bessel function of the first kind and the Hankel function of the second kind, respectively and k_0 is the free-space wavenumber. The residual error $|\delta\mathbf{E}^t|$ is computed by averaging its values on the circular contour C. Results as shown in Fig. 4, indicate that the EW and $|\delta\mathbf{E}^t|$ have the same asymptotic convergence rate.

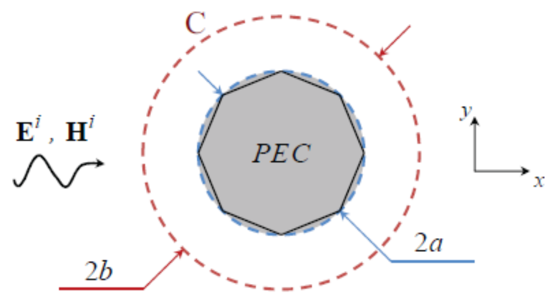


Fig. 3. TM^z plane wave scattering from a 2-D circular (approximated as an octagon) PEC cylinder.

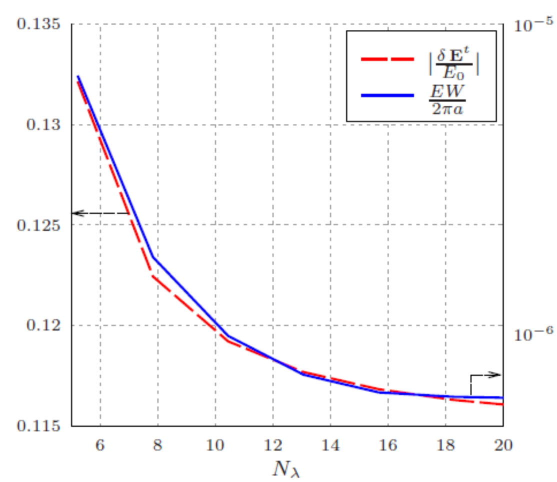


Fig. 4. Correlation between the EW and the actual residual error $|\delta\mathbf{E}^t|$ for the MoM solution of the problem shown in Fig. 3.

III. RESULTS

A. Method of moments

The proposed method is applied to selected numerical methods such as the MoM and the FDFD techniques. First, the example shown in Fig. 5 is considered to compare the EW for the problem of plane wave scattering from a 2-D PEC cylinder with equilateral triangular cross-section using different formulations. The scatterer is enclosed by a cylinder of radius $a = \lambda/2$. Figure 6 (a) shows the computed EW for the case of electric (EFIE) and magnetic field integral equation (MFIE) for TM^z illumination. It can be inferred from Fig. 6, that for this problem the EFIE has superior performance compared to the MFIE and that triangular basis functions give lower EW as compared to the pulses. Figure 6 (b) also shows that the EW is almost independent of b, for $b < b_{critical}$. This critical value depends on the accuracy of the numerical routine used to evaluate the integrals. Increasing this, accuracy results in a higher $b_{critical}$ and vice versa. To explain that, the scattered field intensity is noticed to be proportional to $1/b$ in 2-D problems and therefore the error in calculating $\mathbf{E}^s$ and $\mathbf{H}^s$ is also proportional to $1/b$. At the same time, when b increases, the integration contour C is enlarged with $2\pi(\Delta b)$ and the associated error increases with the same proportionality, provided that the accuracy is high enough. Therefore, the reduction in the error when calculating the scattered fields compensates the increase in the error due to the numerical evaluation of the integrals. This is true up to a critical value, $b_{critical}$ after which the error in evaluating the integrals, i.e.: EW^{min} is not linear anymore with b, as shown in Fig. 6 (c).

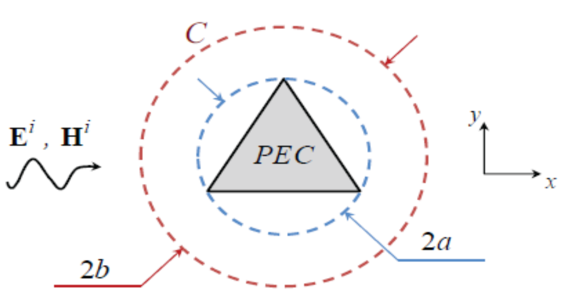
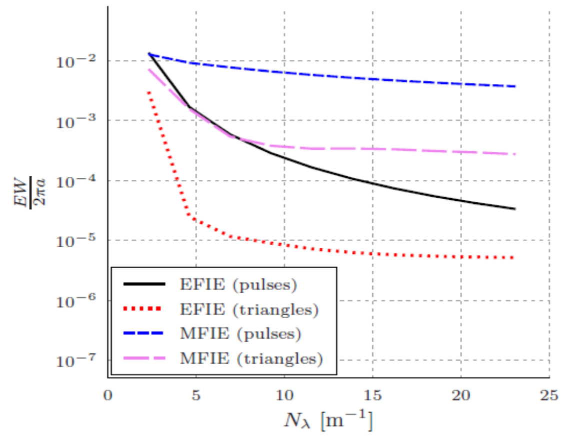
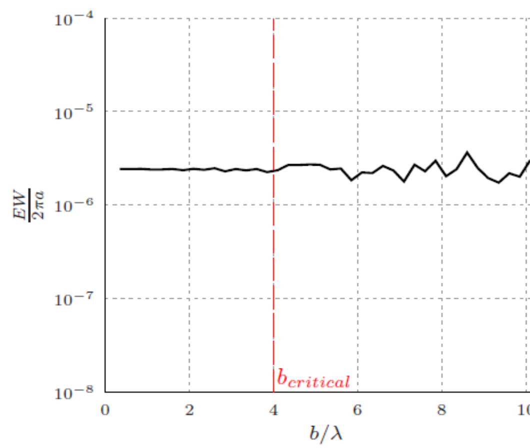


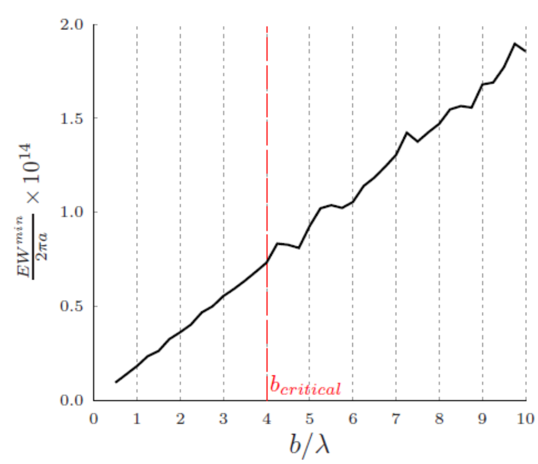
Fig. 5. TM^z plane wave scattering from a 2-D equilateral triangular PEC cylinder.



(a)



(b)



(c)

Fig. 6. (a) EW versus N_λ, (b) EW versus b and (c) EW^{min} for the MoM solution of the problem shown in Fig. 5.

An interesting observation regarding the MoM solution is that when the electrical size of the problem greatly increases, the number of unknowns increases likewise. There is a critical value for the number of unknowns at which the MoM matrix becomes ill-conditioned and the error in finding its inverse affects the overall accuracy. This critical value can be determined using the EW. To manifest this phenomenon, the example of Fig. 3 is considered again but the radius of the PEC cylinder is now made electrically huge, i.e.: $a = 5\lambda \gg \lambda$. Using the MoM-EFIE formulation, $N_\lambda|_{critical}$ and the corresponding $\text{EW}_{critical}$ are shown in Fig. 7. Based on the EW/SW analogy and referring to Fig. 2, the thin wire radius corresponding to the error at $N_\lambda|_{critical}$ is 1000.

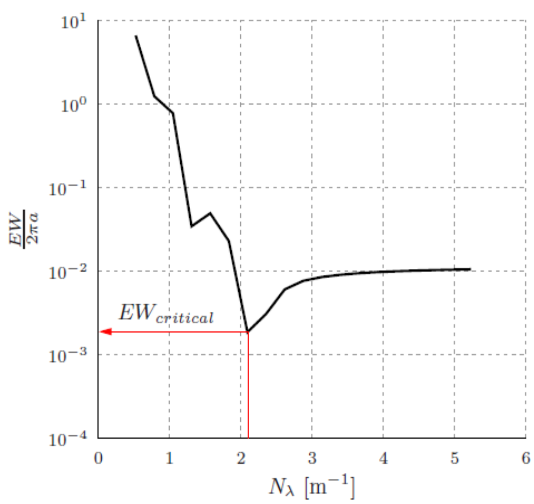


Fig. 7. The error width of the MoM solution using EFIE formulation for the problem shown in Fig. 3.

B. Finite-difference frequency-domain

Considering another numerical method, the error in the FDFD solution of plane wave scattering from a 2-D rectangular dielectric cylinder with a side length l and a dielectric constant ε is investigated. Due to the rectangular grid employed by the FDFD in defining the geometry and field points, it is more convenient for the integration contour to be rectangular with a side length L, as shown in Fig. 8. For $l = 2\lambda$, $L = 4\lambda$ and $\varepsilon = 4$, the EW is depicted in Fig. 9.

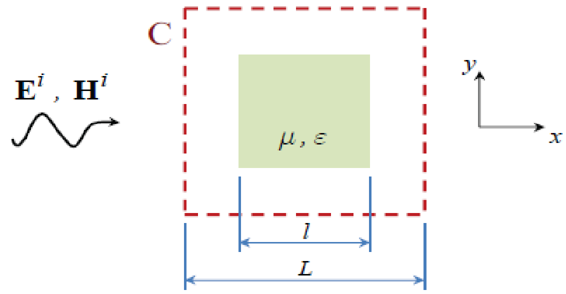


Fig. 8. Plane wave scattering from a 2-D rectangular PEC cylinder.

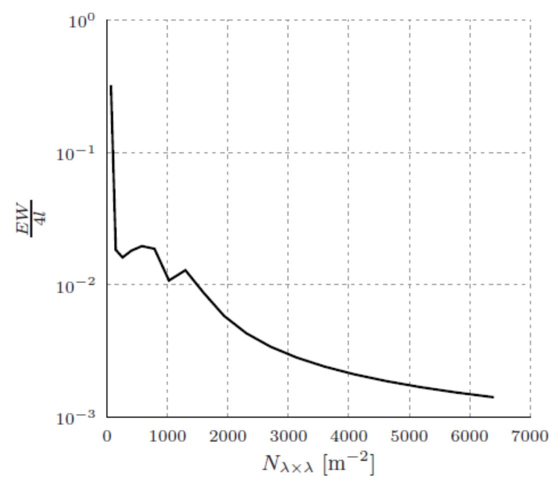


Fig. 9. The error width of the FDFD solution for the problem shown in Fig. 8 under TM^z illumination.

C. Optimization of the mixing factors in the MoM-CFIE

An interesting application to the proposed error estimate is to determine the best choice of the mixing factors used in the CFIE, commonly adopted in the MoM solution to remedy the internal resonance problem. The CFIE formulation is typically obtained as a weighted sum of the EFIE and MFIE with orthogonal weights [11], i.e.:

$$\text{CFIE} = \frac{\beta}{\eta_0}\text{EFIE} + \alpha\ \text{MFIE}, \qquad (11)$$

where CFIE, EFIE and MFIE are either the matrix of unknowns or the excitation vector and η_0 is the intrinsic impedance of free-space. In [12], the choice of the factors α and β was random. The EW concept can be employed to study the effect of this choice on the solution accuracy.

This is done by determining the combination that results in a minimum EW at each number of unknowns. In light of this, it was found that having a 90° phase difference between the mixing factors and keeping α constant for the given problem, the value of β is a very sensitive function of N_λ, i.e.:

$$\text{CFIE} = \frac{\beta(N_\lambda)}{\eta_0}\text{EFIE} + j\alpha\,\text{MFIE}. \tag{12}$$

The example in Fig. 5 is investigated again using the proposed formula in (12) and pulse basis functions. The variation of the EW with β for different values of N_λ, is shown in Fig. 10. Results manifest that for each N_λ there exists a certain value of β, which results in a minimum EW; hence, leads to a significant improvement in the solution accuracy compared to the EFIE or MFIE formulations.

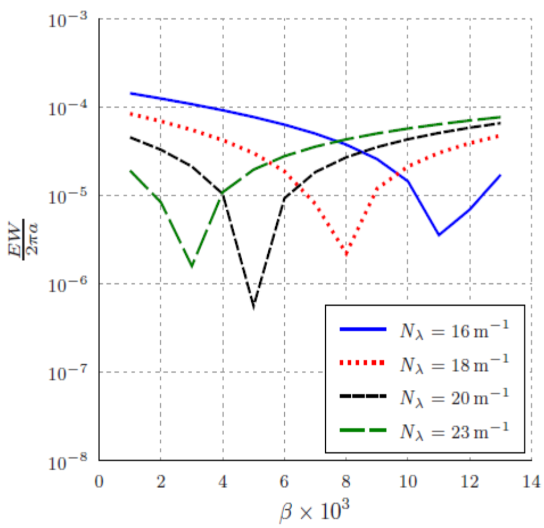


Fig. 10. MoM-CFIE solution using the formula in (12) for the scatterer shown in Fig. 5. The error width is plotted versus β coefficient for different values of N_λ.

IV. CONCLUSION

A general method based on power conservation and independent of the adopted technique is proposed to quantify the error. The definition of the ECS is introduced to compare the solution accuracy for different numerical techniques. For 2-D problems, the EW is used instead of the more general term ECS. The proposed method is applied to the MoM and FDFD solutions of plane wave scattering from 2-D objects. This approach can also be applied to 3-D objects in a straightforward manner. A comparison between the ECS and the RCS of a thin wire for 2-D problems or small sphere for 3-D problems is introduced to visualize the amount of error. An interesting application for the proposed ECS method is finding the critical value for the matrix size in MoM solution after which the accuracy starts to degrade. Another application is to estimate the accuracy of different formulations, as illustrated with the CFIE and to optimize the choice of the mixing factors. Moreover, incorporating the proposed method with the results obtained using commercial software packages as a post-processing step, is on-going with the goal of providing a unified benchmark for the error of these packages.

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