Study and Applications of an Unconditionally Stable Multi-Resolution Time-Domain Scheme

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***Abstract* —** An unconditionally stable locally one-dimensional multi-resolution time-domain (LOD-MRTD) algorithm is studied, which is free of the Courant-Friedrich-Levy (CFL) stability condition. The LOD-MRTD method is reformulated to get more efficient and simple formulations. The unconditional stability and dispersion equations of the LOD-MRTD in two dimension (2D) case are analyzed, and a 2D example is calculated to demonstrate these characteristics.

***Index Terms* -** Locally one-dimensional (LOD), multi-resolution time-domain (MRTD), and unconditional stability.

I. INTRODUCTION

The finite-difference time-domain (FDTD) method has been displayed in an effective way to provide accurate predictions to the field behaviors for varieties of electromagnetic interaction problems [1-2]. The FDTD method is very computationally intensive due to its two inherent physical constraints, one being the numerical dispersion, and another being the numerical stability. Because of the Courant – Friedrich – Levy (CFL) stability condition and the numerical dispersion errors, heavy meshing work and simulation cost time may be required to solve the electrically large structures. To reduce the value of the numerical dispersion, the spatial step of the FDTD method must be chosen fine, and normally it is less than one twentieth of wavelength. To make time-recursion stable, the time step must satisfy the CFL stability condition. Many efforts have been made in relaxing or removing the above two constraints to reduce the computational expenditures. To improve the numerical dispersion problems, some methods such as the MRTD method [3-5], the pseudo-spectral time domain (PSTD) method [6-7], and the higher order FDTD scheme [8], have been devoted to the way.

Among them, the space distribution functions of the MRTD scheme are expanded by the scaling and wavelet functions as basis functions [9-11]. The multi-resolution analysis (MRA) is applied to significantly reduce the dispersion errors. In order to further improve the computational efficiency of the MRTD methods and save the computational time, there are two approaches at least to be considered. One is the high-order method, in which the convergence of the time discretization is the same with the convergence of the spatial discretization, is called the RK-MRTD scheme [12]. Another approach is to use the ADI technology, which is applied into the MRTD scheme to form the ADI-MRTD method [13-14]. The ADI-MRTD method is free of the CFL stability condition and reduces significantly the computational time.

The locally one-dimensional (LOD) method is first introduced to the FDTD method by J. Shibayama [15]. The LOD-MRTD approach is proposed in [16], which shows the LOD-MRTD method more efficient compared with the ADI-MRTD method. In this paper, an improved LOD-MRTD method is studied. The scheme developed in [17] is applied in the conventional LOD-MRTD. The reformulated LOD-MRTD can get better efficiency than the conventional LOD-MRTD.

The remainder of the paper is organized as follows. In section II, the conventional and

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reformulated LOD-MRTD method is introduced. In section III, the numerical properties of the method in 2D case are analyzed, which includes the stability condition and the numerical dispersive characteristics. In section IV, a validation of an example and discussion of numerical results is presented. Finally, conclusions are summarized in section V.

II. LOD-MRTD FORMULATION

For simplicity, let us consider a two-dimensional (2D) TM_z wave propagation in a lossless isotropic medium with permittivity ε and permeability μ. Maxwell's equations are expressed as follows,

$$\mu\frac{\partial H_x}{\partial t} = -\frac{\partial E_z}{\partial y} \tag{1a}$$

$$\mu\frac{\partial H_y}{\partial t} = +\frac{\partial E_z}{\partial x}, \tag{1b}$$

$$\varepsilon\frac{\partial E_z}{\partial t} = \frac{\partial H_y}{\partial x} - \frac{\partial H_x}{\partial y}. \tag{1c}$$

According to [15], applying the LOD method to the time discretization of equation (1), Maxwell's equations can be solved using two steps, the first step is

$$H_x^{n+\frac{1}{2}} = H_x^n - \frac{\Delta t}{2\mu}\left(\frac{\partial E_z^{n+\frac{1}{2}}}{\partial y} + \frac{\partial E_z^n}{\partial y}\right), \tag{2a}$$

$$H_y^{n+\frac{1}{2}} = H_y^n, \tag{2b}$$

$$E_z^{n+\frac{1}{2}} = E_z^n - \frac{\Delta t}{2\varepsilon}\left(\frac{\partial H_x^{n+\frac{1}{2}}}{\partial y} + \frac{\partial H_x^n}{\partial y}\right), \tag{2c}$$

and the second step is

$$H_x^{n+1} = H_x^{n+\frac{1}{2}}, \tag{2d}$$

$$H_y^{n+1} = H_y^{n+\frac{1}{2}} + \frac{\Delta t}{2\mu}\left(\frac{\partial E_z^{n+1}}{\partial x} + \frac{\partial E_z^{n+\frac{1}{2}}}{\partial x}\right), \tag{2e}$$

$$E_z^{n+1} = E_z^{n+\frac{1}{2}} + \frac{\Delta t}{2\varepsilon}\left(\frac{\partial H_y^{n+1}}{\partial x} + \frac{\partial H_y^{n+\frac{1}{2}}}{\partial x}\right). \tag{2f}$$

In the MRTD scheme, the Daubechies compactly supported scaling functions are adopted as basis functions in the expansion of the fields. Using the orthogonal relations for the scaling functions [4], the derivative of the update equations (2) are represented as follows, for step n to step n+1/2,

$$\begin{aligned} &{}_{\phi x}H_{i,j+\frac{1}{2}}^{n+\frac{1}{2}} \\ &= {}_{\phi x}H_{i,j+\frac{1}{2}}^{n} - \frac{\Delta t}{2\mu\Delta y}\sum_{v=-N_v}^{N_v-1} a(v)\left({}_{\phi z}E_{i,j+v+1}^{n+\frac{1}{2}} + {}_{\phi z}E_{i,j+v+1}^{n}\right) \end{aligned} \tag{3a}$$

$${}_{\phi y}H_{i+\frac{1}{2},j}^{n+\frac{1}{2}} = {}_{\phi y}H_{i+\frac{1}{2},j}^{n}, \tag{3b}$$

$$\begin{aligned} &{}_{\phi z}E_{i,j}^{n+\frac{1}{2}} \\ &= {}_{\phi z}E_{i,j}^{n} - \frac{\Delta t}{2\varepsilon\Delta y}\sum_{v=-N_v}^{N_v-1} a(v)\left({}_{\phi x}H_{i,j+v+\frac{1}{2}}^{n+\frac{1}{2}} + {}_{\phi x}H_{i,j+v+\frac{1}{2}}^{n}\right). \end{aligned} \tag{3c}$$

For step n+1/2 to step n+1

$${}_{\phi x}H_{i,j+\frac{1}{2}}^{n+1} = {}_{\phi x}H_{i,j+\frac{1}{2}}^{n+\frac{1}{2}}, \tag{3d}$$

$$\begin{aligned} &{}_{\phi y}H_{i+\frac{1}{2},j}^{n+1} \\ &= {}_{\phi y}H_{i+\frac{1}{2},j}^{n+\frac{1}{2}} + \frac{\Delta t}{2\mu\Delta x}\sum_{v=-N_v}^{N_v-1} a(v)\left({}_{\phi z}E_{i+v+1,j}^{n+1} + {}_{\phi z}E_{i+v+1,j}^{n+\frac{1}{2}}\right), \end{aligned} \tag{3e}$$

$$\begin{aligned} &{}_{\phi z}E_{i,j}^{n+1} \\ &= {}_{\phi z}E_{i,j}^{n+\frac{1}{2}} + \frac{\Delta t}{2\varepsilon\Delta x}\sum_{v=-N_v}^{N_v-1} a(v)\left({}_{\phi y}H_{i+v+\frac{1}{2},j}^{n+1} + {}_{\phi y}H_{i+v+\frac{1}{2},j}^{n+\frac{1}{2}}\right), \end{aligned} \tag{3f}$$

where the coefficients $a(v)$ are resulted from the scaling functions and satisfy the relation of $a(v) = -a(-v-1)$, $N_v = 2N-1$ and N is the order of vanishing moments of the Daubechies scaling functions used in the LOD-MRTD.

The 2D LOD-MRTD formulations can be further simplified, after some tedious derivations, we finally obtain the 2D LOD-MRTD iterative equations, and there are only four equations to be calculated.

$$\begin{aligned} &{}_{\phi z}E_{i,j}^{n+\frac{1}{2}} - \frac{\Delta t^2}{4\varepsilon\mu\Delta y^2}\sum_{v_1,v_2=-N_v}^{N_v-1} a(v_1)a(v_2)\,{}_{\phi z}E_{i,j+v_1+v_2+1}^{n+\frac{1}{2}} \\ &= {}_{\phi z}E_{i,j}^{n} + \frac{\Delta t^2}{4\varepsilon\mu\Delta y^2}\sum_{v_1,v_2=-N_v}^{N_v-1} a(v_1)a(v_2)\,{}_{\phi z}E_{i,j+v_1+v_2+1}^{n} \\ &\quad - \frac{\Delta t}{\varepsilon\Delta y}\sum_{v_1=-N_v}^{N_v-1} a(v_1)\,{}_{\phi x}H_{i,j+v_1+\frac{1}{2}}^{n} \end{aligned}, \tag{4a}$$

$$
\begin{aligned}
&{}_{\phi x}H^{n+1}_{i,j+\frac{1}{2}} \\
&= {}_{\phi x}H^{n}_{i,j+\frac{1}{2}} - \frac{\Delta t}{2\mu\Delta y}\sum_{v_2=-N_v}^{N_v-1} a(v_2)\left({}_{\phi z}E^{n+\frac{1}{2}}_{i,j+v_2+1} + {}_{\phi z}E^{n}_{i,j+v_2+1}\right),
\end{aligned} \tag{4b}
$$

$$
\begin{aligned}
&{}_{\phi z}E^{n+1}_{i,j} - \frac{\Delta t^2}{4\varepsilon\mu\Delta x^2}\sum_{v_1,v_2=-N_v}^{N_v-1} a(v_1)a(v_2)\,{}_{\phi z}E^{n+1}_{i+v_1+v_2+1,j} \\
&= {}_{\phi z}E^{n+\frac{1}{2}}_{i,j} + \frac{\Delta t^2}{4\varepsilon\mu\Delta y^2}\sum_{v_1,v_2=-N_v}^{N_v-1} a(v_1)a(v_2)\,{}_{\phi z}E^{n+\frac{1}{2}}_{i+v_1+v_2+1,j} \\
&\quad + \frac{\Delta t}{\varepsilon\Delta x}\sum_{v_1=-N_v}^{N_v-1} a(v_1)\,{}_{\phi y}H^{n}_{i+v_1+\frac{1}{2},j}
\end{aligned}, \tag{4c}
$$

$$
\begin{aligned}
&{}_{\phi y}H^{n+1}_{i+\frac{1}{2},j} \\
&= {}_{\phi y}H^{n}_{i+\frac{1}{2},j} + \frac{\Delta t}{2\mu\Delta x}\sum_{v_2=-N_v}^{N_v-1} a(v_2)\left({}_{\phi z}E^{n+1}_{i+v_2+1,j} + {}_{\phi z}E^{n+\frac{1}{2}}_{i+v_2+1,j}\right).
\end{aligned} \tag{4d}
$$

It needs to solve a banded matrix from the updated equations, and the width of the banded coefficient matrix is determined by the order of vanishing moments of the Daubechies scaling functions. In the 3D LOD-MRTD all the six components of the E and H should be solved. The iterative equations of the LOD-MRTD method [16] in the three-dimensional formula are expressed as follows, such as E_x and H_z components for step n to step n+1/2,

$$
\begin{aligned}
&{}_{\phi x}E^{n+\frac{1}{2}}_{i+\frac{1}{2},j,k} - \frac{\Delta t^2}{4\varepsilon\mu\Delta y^2}\sum_{v_1,v_2=-N_v}^{N_v-1} a(v_1)a(v_2)\,{}_{\phi x}E^{n+\frac{1}{2}}_{i+\frac{1}{2},j+v_1+v_2+1,k} \\
&= {}_{\phi x}E^{n}_{i+\frac{1}{2},j,k} + \frac{\Delta t^2}{4\varepsilon\mu\Delta y^2}\sum_{v_1,v_2=-N_v}^{N_v-1} a(v_1)a(v_2)\,{}_{\phi x}E^{n}_{i+\frac{1}{2},j+v_1+v_2+1,k} \\
&+ \frac{\Delta t}{\varepsilon\Delta y}\sum_{v_1=-N_v}^{N_v-1} a(v_1)\,{}_{\phi y}H^{n}_{i+\frac{1}{2},j+v_1+\frac{1}{2},k},
\end{aligned} \tag{5a}
$$

$$
\begin{aligned}
&{}_{\phi z}H^{n+\frac{1}{2}}_{i+\frac{1}{2},j+\frac{1}{2},k} \\
&= {}_{\phi z}H^{n}_{i+\frac{1}{2},j+\frac{1}{2},k} \\
&\quad + \frac{\Delta t}{2\mu\Delta y}\sum_{v_2=-N_v}^{N_v-1} a(v_2)\left({}_{\phi x}E^{n+\frac{1}{2}}_{i+\frac{1}{2},j+v_2+1,k} + {}_{\phi x}E^{n}_{i+\frac{1}{2},j+v_2+1,k}\right)\ .
\end{aligned} \tag{5b}
$$

It is clear that all the electric field equations are implicit iteration equations related with coefficients $a(v)$, but the magnetic field equations are the explicit iterative.

For the 2D problem, because the number of the update electric and updated magnetic fields are different, the required memory used in the LOD-MRTD algorithm is more than one third that of the original MRTD method; but in the three-dimensional case, the electric and the magnetic fields requires all six components, the total cost memory is 1.5 times to the classical MRTD method.

According to [17], the LOD-FDTD method can be reformulated, which makes the right-hand sides much simpler and more concise. We use this method to reformulate the LOD-MRTD algorithm to improve its efficiency. First an auxiliary variable P is defined as,

$$
P^{n+\frac{1}{2}} = E^{n+\frac{1}{2}} + E^{n} \quad . \tag{6}
$$

Then substitute equation (6) to equations (4) and (5), the formulations of the conventional LOD-MRTD method can be rewritten as,

$$
\begin{aligned}
&{}_{\phi z}P^{n+\frac{1}{2}}_{i,j} - \frac{\Delta t^2}{4\varepsilon\mu\Delta y^2}\sum_{v_1,v_2=-N_v}^{N_v-1} a(v_1)a(v_2)\,{}_{\phi z}P^{n+\frac{1}{2}}_{i,j+v_1+v_2+1} \\
&= 2\,{}_{\phi z}E^{n}_{i,j} - \frac{\Delta t}{\varepsilon\Delta y}\sum_{v_1=-N_v}^{N_v-1} a(v_1)\,{}_{\phi x}H^{n}_{i,j+v_1+\frac{1}{2}}
\end{aligned} \tag{7a}
$$

$$
{}_{\phi x}H^{n+1}_{i,j+\frac{1}{2}} = {}_{\phi x}H^{n}_{i,j+\frac{1}{2}} - \frac{\Delta t}{2\mu\Delta y}\sum_{v_2=-N_v}^{N_v-1} a(v_2)\,{}_{\phi z}P^{n+\frac{1}{2}}_{i,j+v_2+1}, \tag{7b}
$$

$$
{}_{\phi z}E^{n+\frac{1}{2}}_{i,j} = {}_{\phi z}P^{n+\frac{1}{2}}_{i,j} - {}_{\phi z}E^{n}_{i,j}, \tag{7c}
$$

$$
\begin{aligned}
&{}_{\phi z}P^{n+1}_{i,j} - \frac{\Delta t^2}{4\varepsilon\mu\Delta x^2}\sum_{v_1,v_2=-N_v}^{N_v-1} a(v_1)a(v_2)\,{}_{\phi z}P^{n+1}_{i+v_1+v_2+1,j} \\
&= 2\,{}_{\phi z}E^{n+\frac{1}{2}}_{i,j} + \frac{\Delta t}{\varepsilon\Delta x}\sum_{v_1=-N_v}^{N_v-1} a(v_1)\,{}_{\phi y}H^{n}_{i+v_1+\frac{1}{2},j}
\end{aligned}, \tag{7d}
$$

$$
{}_{\phi y}H^{n+1}_{i+\frac{1}{2},j} = {}_{\phi y}H^{n}_{i+\frac{1}{2},j} + \frac{\Delta t}{2\mu\Delta x}\sum_{v_2=-N_v}^{N_v-1} a(v_2)\,{}_{\phi z}P^{n+1}_{i+v_2+1,j}, \tag{7e}
$$

$$
{}_{\phi z}E^{n+1}_{i,j} = {}_{\phi z}P^{n+1}_{i,j} - {}_{\phi z}E^{n+\frac{1}{2}}_{i,j}, \tag{7f}
$$

for 2D TM_z case, and

$$
\begin{aligned}
&{}_{\phi x}P^{n+\frac{1}{2}}_{i+\frac{1}{2},j,k} - \frac{\Delta t^2}{4\varepsilon\mu\Delta y^2}\sum_{v_1,v_2=-N_v}^{N_v-1} a(v_1)a(v_2)\,{}_{\phi x}P^{n+\frac{1}{2}}_{i+\frac{1}{2},j+v_1+v_2+1,k} \\
&= 2\,{}_{\phi x}E^{n}_{i+\frac{1}{2},j,k} + \frac{\Delta t}{\varepsilon\Delta y}\sum_{v_1=-N_v}^{N_v-1} a(v_1)\,{}_{\phi z}H^{n}_{i+\frac{1}{2},j+v_1+\frac{1}{2},k}
\end{aligned}, \tag{8a}
$$

$$\begin{aligned}&{}_{\phi z}H^{n+\frac{1}{2}}_{i+\frac{1}{2},j+\frac{1}{2},k}\\&= {}_{\phi z}H^{n}_{i+\frac{1}{2},j+\frac{1}{2},k} + \frac{\Delta t}{2\mu\Delta y}\sum_{v_2=-N_v}^{N_v-1} a(v_2)\, {}_{\phi x}P^{n+\frac{1}{2}}_{i+\frac{1}{2},j+v_2+1,k}\end{aligned} \quad ,(8b)$$

for the 3D case, respectively.

It can be found that the right-hand sides of equations (7) and (8) become much simpler. Table 1 is listed in the number of arithmetic operations for the two methods. Furthermore, it is easy to know that the reformulated equations do not change the accuracy of the LOD-MRTD method.

Table 1: Number of multiplication / division operations for the conventional LOD-MRTD and the reformulated LOD-MRTD algorithms.

3D MRTD method		Conventional LOD-MRTD	Reformulated LOD-MRTD
Implicit	multiplication/ division	$(2*(2N_v)^2+7)+(2N_v+3)$	$2N_v+4$
	For D2 scaling function (N=2, Nv=3)	88	10
Explicit	multiplication/ division	$2N_v+4$	$2N_v+4$
	For D2 scaling function (N=2, Nv=3)	10	10
Total	multiplication/ division	$(2*(2N_v)^2+8)+2*(2N_v+3)$	$4N_v+8$
	For D2 scaling function (N=2, Nv=3)	98	20

III. CHARACTERISTICS ANALYSIS

The unconditional stability and dispersion equations of the LOD-MRTD in 3D case have been discussed in [16]. Here, we talk about these characteristics in 2D case. For the unconditional stability of the LOD-MRTD method, we employ the Fourier method described in [18-19] to obtain. The field components in the spatial spectral domain for the TMz wave can be written as,

$$\begin{aligned}{}_zE^n_{I,J} &= E^n_z e^{j(k_x I\Delta x + k_y J\Delta y)}\\ {}_xH^n_{I,J+1/2} &= H^n_x e^{j\left[k_x I\Delta x + k_y (J+\frac{1}{2})\Delta y\right]}\\ {}_yH^n_{I+1/2,J} &= H^n_y e^{j\left[k_x (I+\frac{1}{2})\Delta x + k_y J\Delta y\right]}\end{aligned} \quad (9)$$

where k_x and k_y are wave numbers along the x- and y-directions, respectively, and denote the field vector in the spatial spectral domain as,

$$\mathrm{X}^n = \begin{bmatrix} E^n_z \\ H^n_x \\ H^n_y \end{bmatrix}. \quad (10)$$

The Fourier analysis can be performed by substituting equations (9) and (10) into equation (3), then the following equation is obtained,

$$\mathrm{X}^{n+\frac{1}{2}} = \Lambda_1 \mathrm{X}^n \quad \text{and} \quad \mathrm{X}^{n+1} = \Lambda_2 \mathrm{X}^{n+\frac{1}{2}} \quad (11)$$

where,

$$\Lambda_1 = \begin{bmatrix} \dfrac{1+\frac{W_y^2}{\varepsilon\mu}}{1-\frac{W_y^2}{\varepsilon\mu}} & \dfrac{-\frac{2W_y}{\varepsilon}}{1-\frac{W_y^2}{\varepsilon\mu}} & 0 \\ \dfrac{-\frac{2W_y}{\mu}}{1-\frac{W_y^2}{\varepsilon\mu}} & \dfrac{1+\frac{W_y^2}{\varepsilon\mu}}{1-\frac{W_y^2}{\varepsilon\mu}} & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\Lambda_2 = \begin{bmatrix} \dfrac{1+\frac{W_x^2}{\varepsilon\mu}}{1-\frac{W_x^2}{\varepsilon\mu}} & 0 & \dfrac{\frac{2W_x}{\varepsilon}}{1-\frac{W_x^2}{\varepsilon\mu}} \\ 0 & 1 & 0 \\ \dfrac{\frac{2W_x}{\mu}}{1-\frac{W_x^2}{\varepsilon\mu}} & 0 & \dfrac{1+\frac{W_x^2}{\varepsilon\mu}}{1-\frac{W_x^2}{\varepsilon\mu}} \end{bmatrix},$$

with $W_x = j\dfrac{\Delta t}{\Delta x}\sum_{v=0}^{N_v-1} a(v)\sin(k_x(v+\frac{1}{2})\Delta x)$ and

$$W_y = j\frac{\Delta t}{\Delta y}\sum_{v=0}^{N_v-1} a(v)\sin(k_y(v+\frac{1}{2})\Delta y).$$

Therefore,

$$X^{n+1} = \Lambda X^n = \Lambda_2 \Lambda_1 X^n \tag{12}$$

$$\Lambda = \begin{bmatrix} \frac{1+\frac{W_x^2}{\varepsilon\mu}}{1-\frac{W_x^2}{\varepsilon\mu}} \cdot \frac{1+\frac{W_y^2}{\varepsilon\mu}}{1-\frac{W_y^2}{\varepsilon\mu}} & -\frac{1+\frac{W_x^2}{\varepsilon\mu}}{1-\frac{W_x^2}{\varepsilon\mu}} \cdot \frac{\frac{2W_y}{\varepsilon}}{1-\frac{W_y^2}{\varepsilon\mu}} & \frac{\frac{2W_x}{\varepsilon}}{1-\frac{W_x^2}{\varepsilon\mu}} \\ -\frac{\frac{2W_y}{\mu}}{1-\frac{W_y^2}{\varepsilon\mu}} & \frac{1+\frac{W_y^2}{\varepsilon\mu}}{1-\frac{W_y^2}{\varepsilon\mu}} & 0 \\ \frac{\frac{2W_x}{\mu}}{1-\frac{W_x^2}{\varepsilon\mu}} & -\frac{\frac{2W_x}{\mu}}{1-\frac{W_x^2}{\varepsilon\mu}} \cdot \frac{\frac{2W_y}{\varepsilon}}{1-\frac{W_y^2}{\varepsilon\mu}} & \frac{1+\frac{W_x^2}{\varepsilon\mu}}{1-\frac{W_x^2}{\varepsilon\mu}} \end{bmatrix}. \tag{13}$$

Here, Λ is called the amplification matrix. In order to value the eigen values of the matrix Λ, the following eigen equation of Λ can be used.

$$\begin{aligned} &\left| I\lambda - \Lambda \right| \\ &= \left(\lambda - \frac{1-r_x^2}{1+r_x^2}\frac{1-r_y^2}{1+r_y^2} \right)\left(\lambda - \frac{1-r_x^2}{1+r_x^2} \right) \\ &\times \left(\lambda - \frac{1-r_y^2}{1+r_y^2} \right) - \left(\frac{2r_x}{1+r_x^2}\frac{2r_y}{1+r_y^2} \right)^2 \\ &+ \left(\lambda - \frac{1-r_x^2}{1+r_x^2} \right)\left(\frac{1-r_x^2}{1+r_x^2} \right)\left(\frac{2r_y}{1+r_y^2} \right)^2 \\ &+ \left(\lambda - \frac{1-r_y^2}{1+r_y^2} \right)\left(\frac{1-r_y^2}{1+r_y^2} \right)\left(\frac{2r_x}{1+r_x^2} \right)^2 = 0 \ . \end{aligned} \tag{14}$$

The eigen values of Λ can be found, with the help of Matlab, as

$$\lambda_1 = 1 \tag{15a}$$

$$\lambda_{2,3} = -\frac{r_x^2 + r_y^2 + r_x^2 r_y^2 - 1 \pm 2j\sqrt{r_x^2 + r_y^2 + r_x^2 r_y^2}}{r_x^2 + r_y^2 + r_x^2 r_y^2 + 1}, \tag{15b}$$

where $r_x = \frac{c\Delta t}{\Delta x}\sum_{v=0}^{N_v-1} a(v)\sin(k_x(v+\frac{1}{2})\Delta x)$, $c = \frac{1}{\sqrt{\varepsilon\mu}}$,

$r_y = \frac{c\Delta t}{\Delta y}\sum_{v=0}^{N_v-1} a(v)\sin(k_y(v+\frac{1}{2})\Delta y)$.

It can be found that all the three eigen values have magnitude of unity, so the LOD-MRTD is unconditionally stable. Further, we can derive the dispersion equation of the LOD-MRTD method [20],

$$\tan^2(\frac{\omega\Delta t}{2}) = r_x^2 + r_y^2 + r_x^2 r_y^2 \ . \tag{16}$$

Equation (16) is the same as that of the ADI-MRTD method, which means that the LOD-MRTD and the ADI-MRTD methods have the same order dispersion errors.

IV. NUMERICAL RESULTS

To examine the performance of the LOD-MRTD, we consider a 2D parallel plate resonator. The total size is 1m × 1m and $\Delta x = \Delta y = \Delta s = 0.05$m for the TM_z polarization model. The CFL number α for the MRTD scheme is chosen as 0.3, and the parameters $\varepsilon = \varepsilon_0$ and $\mu = \mu_0$. It is noted that when one calculates the resonant frequency, we need to set the adequate time step, so that the frequency resolution of FFT meets the requirements to ensure the validity of the results data of the resonant frequency.

Figure 1 shows the relation of the computational errors increasing with the decrease of N_c and the CFLN. Figure 2 is the comparison of error of the proposed method with those of the ADI-MRTD method when CFLN takes different values. The calculation accuracy of the LOD-MRTD method is consistent with the ADI-MRTD method, and also agrees with the theoretical analysis described in the above section. From Fig. 2, we find that the simulation results of the LOD-MRTD method are reasonable and acceptable when the stability coefficient is taken some times as the classical MRTD method. Table 2 lists the parameters used in the LOD-MRTD and the MRTD methods, here CFLN = $\alpha_{LOD\text{-}MRTD} / \alpha_{MRTD}$, $N_c = \lambda/\Delta s$.

V. CONCLUSIONS

In this paper, an improved LOD-MRTD method is studied. The improved LOD-MRTD is more efficient than the conventional LOD -MRTD without loss of accuracy. A 2D case is calculated to demonstrate the characteristics of the LOD-MRTD.

Table 2: Analytic solutions and the results of the MRTD and the LOD-MRTD methods.

Mode (m, n)	1,1	1,3	3,3	1,5	3,5	5,5	3,7	5,7	1,9
N_c	28.28	12.65	9.43	7.84	6.86	5.66	5.25	4.65	4.42
Analytic (GHz)	0.2121	0.4743	0.6364	0.7649	0.8746	1.0607	1.1422	1.2904	1.3583
MRTD/ $\alpha_{MRTD} = 0.3$	0.2124	0.4749	0.6378	0.7678	0.8789	1.0675	1.1542	1.3049	1.3837
LOD-MRTD/ CFLN=1	0.2118	0.4736	0.6354	0.7629	0.8722	1.0565	1.1377	1.2842	1.3519
LOD-MRTD/ CFLN=2	0.2118	0.4712	0.6311	0.7526	0.8606	1.0388	1.1102	1.2506	1.2970
LOD-MRTD/ CFLN=4	0.2115	0.4630	0.6165	0.7166	0.8200	0.9427	0.9760	1.0193	1.1362
LOD-MRTD/ CFLN=5	0.2109	0.4565	0.6060	0.6934	0.7935	0.8921	0.9360	0.9658	1.0523
LOD-MRTD/ CFLN=10	0.2075	0.4150	-	-	-	-	-	-	-
LOD-MRTD/ CFLN=15	0.2018	0.3682	-	-	-	-	-	-	-
LOD-MRTD/ CFLN=20	0.1948	0.3252	-	-	-	-	-	-	-

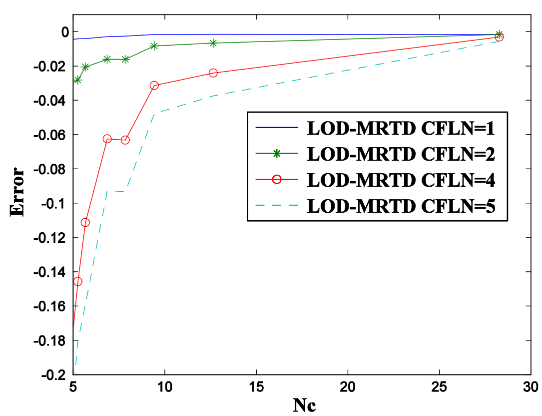

Fig. 1. The error varied of the LOD-MRTD method for different N_c and CFLN.

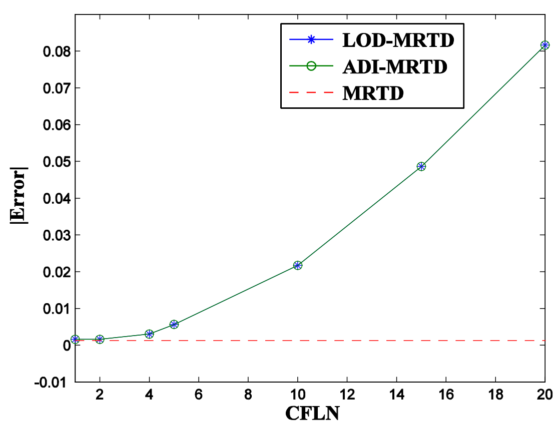

Fig. 2. The error comparisons of the LOD-MRTD and the ADI-MRTD methods.

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