Coulombian Model for 3D Analytical Calculation of the Torque Exerted on Cuboidal Permanent Magnets with Arbitrary Oriented Polarizations

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Abstract **─** This paper proposes improved analytical expressions of the torque on cuboidal permanent magnets. Expressions are valid for any relative magnet position and for any polarization direction. The analytical calculation is made by replacing polarizations by distributions of magnetic charges on the magnet poles (coulombian approach). The torque exerted on the second magnet is calculated by Lorentz force formulas for any arbitrary position. The three components of the torque are written with functions based on logarithm and arc-tangent. Results have been verified and validated by comparison with finite-element calculation. Further, the torque can be obtained with respect to any reference point. Although these equations seem rather complicated, they enable an extremely fast and accurate calculation of the torque exerted between two permanent magnets.

Index Terms **─** Analytical calculation, coulombian approach, force, permanent magnet, torque.

I. INTRODUCTION

Analytical expressions are very powerful, giving a very fast method to calculate magnetic interactions. It is why the analytical expressions of all the interactions, energy, forces, and torques between two cuboidal magnets are very important results. Many problems can be solved by the addition of element interactions.

Up to now, for the torque components, the calculations were first realized for a system of two magnets with parallel polarization direction by Allag [1] and Janssen [2]. For the perpendicular case the results have been recently published [3].

In this paper, we develop the calculation for systems with two magnets with inclined polarization direction. The torque expressions are valid for any given point in the space, not only around the center of the moving magnet. The expressions of the torque components are obtained using the Lorentz force method [4]. A comparison with numerical results using the commercial software Flux3D validates our analytical calculation of the torque exerted between two permanent magnets.

II. MATHEMATICAL MODEL

We study the interaction between two parallelepiped magnets, as presented in Fig. 1. The polarizations J and J' are supposed to be rigid and uniform in each magnet. The difference is that J' are arbitrary oriented in the YZ plane. The model can be replaced by distributions of magnetic charges on the poles, generally called coulombian approach. For simplifying calculation, the polarization J' will be decomposed into parallel

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component $J'_{//}$ and perpendicular one $J'_{\perp}$ (Fig. 2).

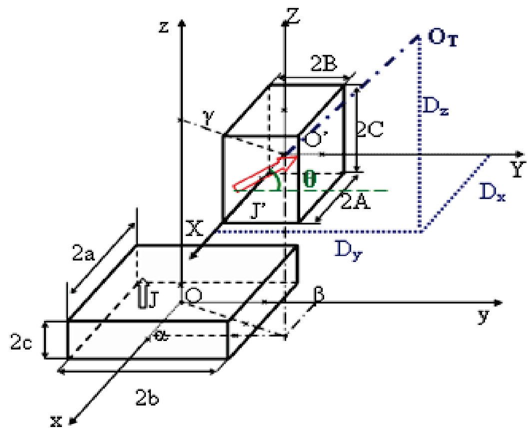

Fig. 1. System with two magnets.

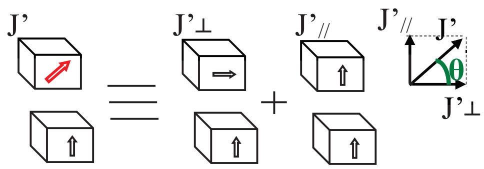

Fig. 2. Polarization decomposition.

A. Parallel polarizations

The first 3-D fully analytical expressions of the energy and force were presented at the 1984 INTERMAG Conference, Hamburg, Germany [5]. The forces were analytically calculated for two cuboidal magnets with parallel polarization directions (Fig. 3).

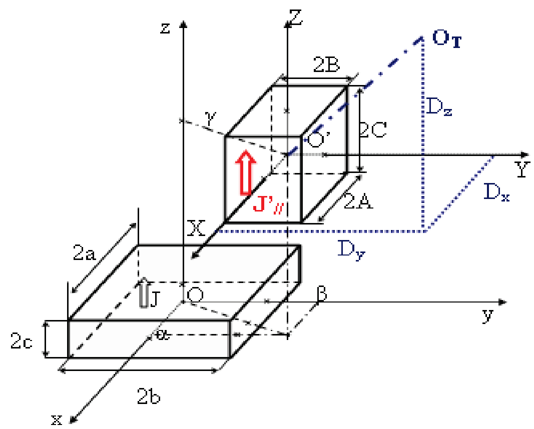

Fig. 3. Parallel polarizations.

The energy expressions are:

$$E_{//} = \frac{J.J_{//}'}{4\pi\mu_0}\sum_{p=0}^{1}\sum_{q=0}^{1}(-1)^{p+q}\int_{-C}^{C}dY\int_{-A}^{A}dX\int_{-b}^{b}dy\int_{-a}^{a}\frac{1}{r}dx, \quad (1)$$

with

$$r=\sqrt{(\alpha+X-x)^2+(\beta+Y-y)^2+(\gamma+(-1)^l C-(-1)^p c)^2}. \quad (2)$$

The obtained expressions of the interaction energy are:

$$E_{//} = \frac{J\cdot J_{//}'}{4\pi\mu_0}\sum_{i=0}^{1}\sum_{j=0}^{1}\sum_{k=0}^{1}\sum_{l=0}^{1}\sum_{p=0}^{1}\sum_{q=0}^{1}(-1)^{i+j+k+l+p+q}.\psi(U_{ij},V_{kl},W_{pq},r), \quad (3)$$

with

$$\psi_{//}(U,V,W,r)=\frac{U\left(V^2-W^2\right)}{2}\ln\left(r-U\right)+\frac{V\left(U^2-W^2\right)}{2}\ln\left(r-V\right)+UVW\cdot tg^{-1}\left(\frac{UV}{rW}\right)+\frac{r}{6}\left(U^2+V^2-2W^2\right). \quad (4)$$

The secondary variables are:

$$U_{ij}=\alpha+(-1)^j A-(-1)^i a,$$
$$V_{kl}=\beta+(-1)^l B-(-1)^k b, \quad (5)$$

where:

$$W_{pq}=\gamma+(-1)^q C-(-1)^p c,$$

and

$$r=\sqrt{U_{ij}^2+V_{kl}^2+W_{pq}^2}.$$

From the interaction energy, the force components can be obtained by $\vec{F}=-gr\vec{a}d\ E$. Consequently, the force components are:

$$F_{//} = \frac{J\cdot J_{//}'}{4\pi\mu_0}\sum_{i=0}^{1}\sum_{j=0}^{1}\sum_{k=0}^{1}\sum_{l=0}^{1}\sum_{p=0}^{1}\sum_{q=0}^{1}(-1)^{i+j+k+l+p+q}.\varphi(U_{ij},V_{kl},W_{pq},r), \quad (6)$$

with

$$\varphi_{//X}(U,V,W,r)=\frac{\left(V^2-W^2\right)}{2}\ln\left(r-U\right)+UV\ln(r-V)+VW\cdot tg^{-1}\left(\frac{UV}{W\cdot r}\right)+\frac{1}{2}U\cdot r,$$

$$\varphi_{//Y}(U,V,W,r)=\frac{\left(U^2-W^2\right)}{2}\ln\left(r-V\right)+UV\ln(r-U)+UW\cdot tg^{-1}\left(\frac{UV}{W\cdot r}\right)+\frac{1}{2}V\cdot r,$$

$$\varphi_{//Z}(U,V,W,r)=-UW\ln\left(r-U\right)-VW\ln(r-V)+UW\cdot tg^{-1}\left(\frac{UV}{W\cdot r}\right)-W\cdot r. \quad (7)$$

For the torque calculation, the first magnet is supposed to be fixed and the second magnet is free to move in any direction. The torque is calculated for a movement around the point O_T. The O_T position is defined by its coordinates (D_x, D_y, D_z) in the reference axes of the second magnet OXYZ. View from O, the centre of the fixed permanent magnet, the O_T position is defined by ($D_x+\alpha$, $D_y,+\beta$, $D_z+\gamma$).

The torque exerted in the second magnet at O_T is calculated by Lorentz formulas [2, 3, 4]:

$$\vec{\Gamma}_{//}=\frac{J.J_{//}'}{4\pi\mu_0}\iint_S\begin{pmatrix}r'_Y B_z-r'_Z B_y\\ r'_Z B_x-r'_X B_z\\ r'_X B_y-r'_Y B_x\end{pmatrix}dS=\frac{J.J_{//}'}{4\pi\mu_0}\int_Z\int_X\begin{pmatrix}r'_Y B_z-r'_Z B_y\\ r'_Z B_x-r'_X B_z\\ r'_X B_y-r'_Y B_x\end{pmatrix}dXdY, \quad (8)$$

with

$$\vec{r}'=r'_X\vec{i}+r'_Y\vec{j}+r'_Z\vec{k}=\left(X-D_X\right)\vec{i}+\left(Y-D_Y\right)\vec{j}+\left(Z-D_Z\right)\vec{k}. \quad (9)$$

The torque can be also written as:

$$\vec{\Gamma}_{//} = \frac{J.J_{//}'}{4\pi\mu_0}\iint_{Z\,X}\begin{pmatrix} (Y-D_Y)\frac{\partial}{\partial z}\iint_{y\,x}\frac{1}{r}dxdy-(Z-D_Z)\frac{\partial}{\partial y}\iint_{y\,x}\frac{1}{r}dxdy \\ (Z-D_Z)\frac{\partial}{\partial x}\iint_{y\,x}\frac{1}{r}dxdy-(X-D_X)\frac{\partial}{\partial z}\iint_{y\,x}\frac{1}{r}dxdy \\ (X-D_x)\frac{\partial}{\partial y}\iint_{y\,x}\frac{1}{r}dxdy-(Y-D_y)\frac{\partial}{\partial x}\iint_{y\,x}\frac{1}{r}dxdy \end{pmatrix}dXdY. \quad (10)$$

The distance ***r*** is always the same (see equation (2)), and D_X, D_Y and D_Y are the projections of the distance between the centre of the moving magnet and the point of torque calculation O_T.

After the analytical integrations, the torque is given by:

$$\Gamma_{//} = \frac{J\cdot J_{//}'}{4\pi\mu_0}\sum_{i=0}^{1}\sum_{j=0}^{1}\sum_{k=0}^{1}\sum_{l=0}^{1}\sum_{p=0}^{1}\sum_{q=0}^{1}(-1)^{i+j+k+l+p+q}.\tau_{//}(U_{ij},V_{kl},W_{pq},r)\,. \quad (11)$$

And the functions τ are respectively:

For $\Gamma_{//X}$:

$$\tau_{//X} = \left(\frac{\left(U^2-W^2\right)}{2}\ln\left(r-V\right)+UV\ln(r-U)+UW\cdot tg^{-1}\left(\frac{UV}{W\cdot r}\right)+\frac{1}{2}V\cdot r\right)\cdot\left(C\left(-1\right)^q-\frac{W}{2}\right)$$
$$-\left(-UW\ln\left(r-U\right)-VW\ln(r-V)+UV\cdot tg^{-1}\left(\frac{UV}{W\cdot r}\right)-W\cdot r\right)\cdot\left(B\left(-1\right)^l-\frac{V}{2}\right),$$

For $\Gamma_{//Y}$:

$$\tau_{//Y} = \left(-UW\ln\left(r-U\right)-VW\ln(r-V)+UV\cdot tg^{-1}\left(\frac{UV}{W\cdot r}\right)-W\cdot r\right)\cdot\left(A\left(-1\right)^j-\frac{U}{2}\right)$$
$$-\left(\frac{\left(V^2-W^2\right)}{2}\ln\left(r-U\right)+UV\ln(r-V)+VW\cdot tg^{-1}\left(\frac{UV}{W\cdot r}\right)+\frac{1}{2}U\cdot r\right)\cdot\left(C\left(-1\right)^q-\frac{W}{2}\right),$$

For $\Gamma_{//Z}$:

$$\tau_{//Z} = \left(\frac{\left(V^2-W^2\right)}{2}\ln\left(r-U\right)+UV\ln(r-V)+VW\cdot tg^{-1}\left(\frac{UV}{W\cdot r}\right)+\frac{1}{2}U\cdot r\right)\cdot\left(B\left(-1\right)^l-\frac{V}{2}\right)$$
$$-\left(\frac{\left(U^2-W^2\right)}{2}\ln\left(r-V\right)+UV\ln(r-U)+UW\cdot tg^{-1}\left(\frac{UV}{W\cdot r}\right)+\frac{1}{2}V\cdot r\right)\cdot\left(A\left(-1\right)^j-\frac{U}{2}\right).$$

(12)

It is easy to identify the link between the expressions of the torque (12) and the force components $\phi_{//X}$, $\phi_{//Y}$, $\phi_{//Z}$ from (7). Therefore we can write:

$$\tau_{//X} = \phi_{//Y}\cdot\left(C(-1)^q-\frac{W}{2}\right)-\phi_{//Z}\cdot\left(B(-1)^l-\frac{V}{2}\right),$$
$$\tau_{//Y} = \phi_{//Z}\cdot\left(A(-1)^j-\frac{U}{2}\right)-\phi_{//X}\left(C(-1)^q-\frac{W}{2}\right),$$
$$\tau_{//Z} = \phi_{//X}\cdot\left(B(-1)^l-\frac{V}{2}\right)-\phi_{//Y}\left(A(-1)^j-\frac{U}{2}\right). \quad (13)$$

B. Perpendicular polarizations

For the perpendicular polarizations case, the chosen system is presented in Fig. 4, in which the polarization of a second magnet is collinear with the Y axis.

The analytical expressions of the interaction energy and the forces components for this system were previously developed [6, 7]. The difference is in the Z integration:

$$E_{\perp} = \frac{JJ_{\perp}'}{4\pi\mu_0}\sum_{l=0}^{1}\sum_{p=0}^{1}\int_{-C}^{C}dZ\int_{-A}^{A}dX\int_{-b}^{b}dy\int_{-a}^{a}\frac{1}{r}dx\,. \quad (14)$$

The distance ***r*** is given by:

$$r=\sqrt{(\alpha+X-x)^2+(\beta+(-1)^l B-y)^2+(\gamma+Z-(-1)^p c)^2}\,. \quad (15)$$

After analytical integration, the energy is given by:

$$E_{\perp} = \frac{JJ_{\perp}'}{4\pi\mu_0}\sum_{i=0}^{1}\sum_{j=0}^{1}\sum_{k=0}^{1}\sum_{l=0}^{1}\sum_{p=0}^{1}\sum_{q=0}^{1}(-1)^{i+j+k+l+p+q}.\psi_{\perp}(U_{ij},V_{kl},W_{pq},r)\,. \quad (16)$$

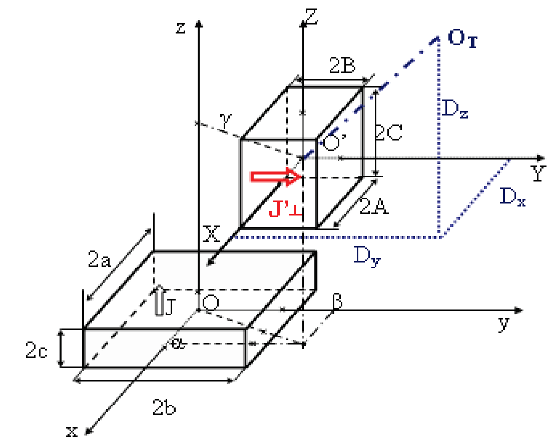

Fig. 4. System with perpendicular polarizations.

The $\psi_{\perp}$ function depends on the geometrical parameters (U, V, W, r):

$$\psi_{\perp} = \frac{V(V^2-3U^2)}{6}\ln(W+r)+\frac{W(W^2-3U^2)}{6}\ln(V+r)+UVW\cdot\ln(-U+r)$$
$$+\frac{U}{6}\left(3V^2\mathrm{tg}^{-1}\left(\frac{UW}{V\cdot r}\right)+3W^2\mathrm{tg}^{-1}\left(\frac{UV}{W\cdot r}\right)+U^2\mathrm{tg}^{-1}\left(\frac{VW}{U\cdot r}\right)\right)+\frac{V\cdot W\cdot r}{3}\quad .(17)$$

The variables *U*, *V*, *W* are given by (5).

The force components can be calculated from the gradient of energy:

$$F_{\perp} = \frac{JJ_{\perp}'}{4\pi\mu_0}\sum_{i=0}^{1}\sum_{j=0}^{1}\sum_{k=0}^{1}\sum_{l=0}^{1}\sum_{p=0}^{1}\sum_{q=0}^{1}(-1)^{i+j+k+l+p+q}.\phi_{\perp}(U_{ij},V_{kl},W_{pq},r)\quad .(18)$$

For F_x, the function $\phi_{\perp x}$ is given by:

$$\phi_{\perp x} = -VW\ln(r-U)+VU\ln(r+W)+WU\ln(r+V)$$
$$-\frac{U^2}{2}\mathrm{tg}^{-1}\left(\frac{VW}{U\cdot r}\right)-\frac{V^2}{2}\mathrm{tg}^{-1}\left(\frac{UW}{V\cdot r}\right)-\frac{W^2}{2}\mathrm{tg}^{-1}\left(\frac{UV}{W\cdot r}\right).$$

For F_y and F_z:

$$\phi_{\perp y} = \frac{\left(U^2-V^2\right)}{2}\ln\left(r+W\right)-UW\ln(r-U)-UV\cdot\mathrm{tg}^{-1}\left(\frac{UW}{V\cdot r}\right)-\frac{1}{2}W\cdot r\,,$$
$$\phi_{\perp z} = \frac{\left(U^2-W^2\right)}{2}\ln\left(r+V\right)-UV\ln(r-U)-UW\cdot\mathrm{tg}^{-1}\left(\frac{UV}{W\cdot r}\right)-\frac{1}{2}V\cdot r\,. \quad (19)$$

Similarly to the parallel polarization case, the torque exerted on the second magnet at O_T (Fig. 4) is expressed by:

$$\vec{\Gamma}_{\perp} = \frac{J.J'}{4\pi\mu_0}\iint_S \begin{pmatrix} r'_Y B_z - r'_Z B_y \\ r'_Z B_x - r'_X B_z \\ r'_X B_y - r'_Y B_x \end{pmatrix} dS = \frac{J.J'}{4\pi\mu_0}\int_Z\int_X \begin{pmatrix} r'_Y B_z - r'_Z B_y \\ r'_Z B_x - r'_X B_z \\ r'_X B_y - r'_Y B_x \end{pmatrix} dXdZ \;, \quad (20)$$

with

$$\vec{r}' = r'_X \,\vec{i} + r'_Y \,\vec{j} + r'_Z \,\vec{k} = (X - D_X)\vec{i} + (Y - D_Y)\vec{j} + (Z - D_Z)\vec{k} \;. \quad (21)$$

The torque can be also written as:

$$\vec{\Gamma}_{\perp} = \frac{J.J'}{4\pi\mu_0}\int_Z\int_X \begin{pmatrix} (Y - D_Y)\frac{\partial}{\partial z}\int_y\int_x \frac{1}{r}dxdy - (Z - D_Z)\frac{\partial}{\partial y}\int_y\int_x \frac{1}{r}dxdy \\ (Z - D_Z)\frac{\partial}{\partial x}\int_y\int_x \frac{1}{r}dxdy - (X - D_X)\frac{\partial}{\partial z}\int_y\int_x \frac{1}{r}dxdy \\ (X - D_x)\frac{\partial}{\partial y}\int_y\int_x \frac{1}{r}dxdy - (Y - D_y)\frac{\partial}{\partial x}\int_y\int_x \frac{1}{r}dxdy \end{pmatrix} dXdZ \;. \quad (22)$$

The final result is given by:

$$\Gamma_{\perp} = \frac{J \cdot J'}{4\pi\mu_0}\sum_{i=0}^{1}\sum_{j=0}^{1}\sum_{k=0}^{1}\sum_{l=0}^{1}\sum_{p=0}^{1}\sum_{q=0}^{1}(-1)^{i+j+k+l+p+q}.\tau_{\perp}(U_{ij},V_{kl},W_{pq},r) \;. \quad (23)$$

For the torque component $\Gamma\perp_x$, parallely oriented to the Ox axis, the $\tau\perp_x$ function is given by:

$$\tau_{\perp X} = \phi_{\perp y}\cdot\left(C(-1)^q - D_Z\right) - \phi_{\perp Z}\cdot\left(B(-1)^l - D_Y - V\right) + \begin{pmatrix} \frac{U(V^2+W^2)}{2}\ln(r-U) - \frac{W(U^2-V^2)}{2}\ln(r+W) \\ -\frac{V(U^2-W^2)}{2}\ln(r+V) \\ +UVW\left(\mathrm{tg}^{-1}\left(\frac{UW}{V\cdot r}\right) + \mathrm{tg}^{-1}\left(\frac{UV}{W\cdot r}\right)\right) + \frac{r^3}{6} \end{pmatrix}.$$

Function $\tau\perp_Y$ for the torque component $\Gamma\perp_Y$:

$$\tau_{\perp Y} = \varphi_{\perp Z}\cdot\left(A(-1)^j - D_x\right) - \varphi_{\perp X}\left(C(-1)^q - D_Z - W\right) + \begin{pmatrix} -\frac{U^3}{3}\ln(V+r) - \frac{V(V^2+3W^2)}{12}\ln(U+r) \\ +\frac{U^2V}{2}\ln(-U+r) \\ +\frac{W(W^2+3U^2)}{6}\cdot tg^{-1}\left(\frac{UV}{Wr}\right) + \frac{1}{12}VU\cdot r \end{pmatrix}, \quad (24)$$

Function $\tau\perp_Z$ for the torque component $\Gamma\perp_Z$:

$$\tau_{\perp z} = \varphi_{\perp X}\cdot\left(B(-1)^l - D_Y\right) - \varphi_{\perp Y}\cdot\left(A(-1)^j - D_X - U\right) + \begin{pmatrix} -\frac{U^3}{3}\ln(W+r) - \frac{W(W^2+3V^2)}{12}\ln(U+r) \\ +\frac{U^2W}{2}\ln(-U+r) \\ +\frac{V(V^2+3U^2)}{6}\cdot tg^{-1}\left(\frac{UW}{Vr}\right) + \frac{1}{12}WU\cdot r \end{pmatrix}.$$

The torque components in perpendicular case are also function of the force ones ($\phi\perp_X$, $\phi\perp_Y$ and $\phi\perp_Z$).

III. TORQUE CALCULATION FOR INCLINED POLARIZATION DIRECTION

For an inclined polarization J' as presented on Fig. 1 and Fig. 2. It can be represented as:

$$J' = J'_{//}\sin(\theta) + J'_{\perp}\cos(\theta) \,. \quad (25)$$

Therefore the total torque will be:

$$\Gamma = \Gamma_{//}\sin(\theta) + \Gamma_{\perp}\cos(\theta) \,. \quad (26)$$

Using equations (11) and (23), final expressions of the torque are:

$$\Gamma = \frac{J\cdot J_{//}'\sin(\theta)}{4\pi\mu_0}\sum_{i=0}^{1}\sum_{j=0}^{1}\sum_{k=0}^{1}\sum_{l=0}^{1}\sum_{p=0}^{1}\sum_{q=0}^{1}(-1)^{i+j+k+l+p+q}.\tau_{//}(U_{ij},V_{kl},W_{pq},r) \\ \frac{J\cdot J_{\perp}'\cos(\theta)}{4\pi\mu_0}\sum_{i=0}^{1}\sum_{j=0}^{1}\sum_{k=0}^{1}\sum_{l=0}^{1}\sum_{p=0}^{1}\sum_{q=0}^{1}(-1)^{i+j+k+l+p+q}.\tau_{\perp}(U_{ij},V_{kl},W_{pq},r) \;. \quad (27)$$

The components of $\tau_{//}$ and $\tau_{\perp}$ are given by equations (12), (13) and (24).

Expressions of the torque components:

For Γ_X:

$$\Gamma_X = \frac{J\cdot J_{//}'\sin(\theta)}{4\pi\mu_0}\sum_{i=0}^{1}\sum_{j=0}^{1}\sum_{k=0}^{1}\sum_{l=0}^{1}\sum_{p=0}^{1}\sum_{q=0}^{1}(-1)^{i+j+k+l+p+q}.\tau_{//X}(U_{ij},V_{kl},W_{pq},r) \\ \frac{J\cdot J_{\perp}'\cos(\theta)}{4\pi\mu_0}\sum_{i=0}^{1}\sum_{j=0}^{1}\sum_{k=0}^{1}\sum_{l=0}^{1}\sum_{p=0}^{1}\sum_{q=0}^{1}(-1)^{i+j+k+l+p+q}.\tau_{\perp X}(U_{ij},V_{kl},W_{pq},r) \;, \quad (28)$$

with

$$\tau_{//X} = \phi_{//Y}\cdot\left(C(-1)^q - \frac{W}{2}\right) - \phi_{//Z}\cdot\left(B(-1)^l - \frac{V}{2}\right),$$

$$\tau_{\perp X} = \phi_{\perp y}\cdot\left(C(-1)^q - D_Z\right) - \phi_{\perp Z}\cdot\left(B(-1)^l - D_Y - V\right) + \begin{pmatrix} \frac{U(V^2+W^2)}{2}\ln(r-U) - \frac{W(U^2-V^2)}{2}\ln(r+W) \\ -\frac{V(U^2-W^2)}{2}\ln(r+V) \\ +UVW\left(\mathrm{tg}^{-1}\left(\frac{UW}{V\cdot r}\right) + \mathrm{tg}^{-1}\left(\frac{UV}{W\cdot r}\right)\right) + \frac{r^3}{6} \end{pmatrix}.$$

For Γ_Y:

$$\Gamma_y = \frac{J\cdot J_{//}'\sin(\theta)}{4\pi\mu_0}\sum_{i=0}^{1}\sum_{j=0}^{1}\sum_{k=0}^{1}\sum_{l=0}^{1}\sum_{p=0}^{1}\sum_{q=0}^{1}(-1)^{i+j+k+l+p+q}.\tau_{//y}(U_{ij},V_{kl},W_{pq},r) \\ \frac{J\cdot J_{\perp}'\cos(\theta)}{4\pi\mu_0}\sum_{i=0}^{1}\sum_{j=0}^{1}\sum_{k=0}^{1}\sum_{l=0}^{1}\sum_{p=0}^{1}\sum_{q=0}^{1}(-1)^{i+j+k+l+p+q}.\tau_{\perp y}(U_{ij},V_{kl},W_{pq},r) \;, \quad (29)$$

with

$$\tau_{//Y} = \phi_{//Z}\cdot\left(A(-1)^j - \frac{U}{2}\right) - \phi_{//X}\left(C(-1)^q - \frac{W}{2}\right),$$

$$\tau_{\perp Y} = \varphi_{\perp Z}\cdot\left(A(-1)^j - D_x\right) - \varphi_{\perp X}\left(C(-1)^q - D_Z - W\right) + \begin{pmatrix} -\frac{U^3}{3}\ln(V+r) - \frac{V(V^2+3W^2)}{12}\ln(U+r) \\ +\frac{U^2V}{2}\ln(-U+r) \\ +\frac{W(W^2+3U^2)}{6}\cdot tg^{-1}\left(\frac{UV}{Wr}\right) + \frac{1}{12}VU\cdot r \end{pmatrix},$$

and finally, for Γ_Z:

$$\Gamma_Z = \frac{J\cdot J_{//}'\sin(\theta)}{4\pi\mu_0}\sum_{i=0}^{1}\sum_{j=0}^{1}\sum_{k=0}^{1}\sum_{l=0}^{1}\sum_{p=0}^{1}\sum_{q=0}^{1}(-1)^{i+j+k+l+p+q}.\tau_{//Z}(U_{ij},V_{kl},W_{pq},r) \\ \frac{J\cdot J_{\perp}'\cos(\theta)}{4\pi\mu_0}\sum_{i=0}^{1}\sum_{j=0}^{1}\sum_{k=0}^{1}\sum_{l=0}^{1}\sum_{p=0}^{1}\sum_{q=0}^{1}(-1)^{i+j+k+l+p+q}.\tau_{\perp Z}(U_{ij},V_{kl},W_{pq},r) \;, \quad (30)$$

with

$$\tau_{//Z} = \phi_{//X}\cdot\left(B(-1)^l - \frac{V}{2}\right) - \phi_{//Y}\left(A(-1)^j - \frac{U}{2}\right),$$

$$\tau_{\perp z} = \varphi_{\perp X} \cdot \left(B(-1)^l - D_Y\right) - \varphi_{\perp Y} \cdot \left(A(-1)^j - D_X - U\right)$$
$$+ \begin{pmatrix} -\dfrac{U^3}{3}\ln(W+r) - \dfrac{W\left(W^2+3V^2\right)}{12}\ln(U+r) \\ +\dfrac{U^2W}{2}\ln(-U+r) \\ +\dfrac{V\left(V^2+3U^2\right)}{6} \cdot tg^{-1}\left(\dfrac{UW}{Vr}\right) + \dfrac{1}{12}WU \cdot r \end{pmatrix}.$$

IV. APPLICATION AND RESULTS

The following example presents the torque calculation between two magnets. These magnets are identical; two cubes of 1 cm edge. The lower magnet has a vertical polarization (oriented in Z direction). For the second magnet, its polarization is inclined in the YZ plane (Fig. 5). The intensity of polarization is 1 Tesla for the two magnets. The upper magnet moves in translation along the Ox axis above the lower fixed one. The vertical distance between them (air gap when the upper magnet is above the fixed magnet) is 0.01 m (β = 0 m and γ = 0.02 m).

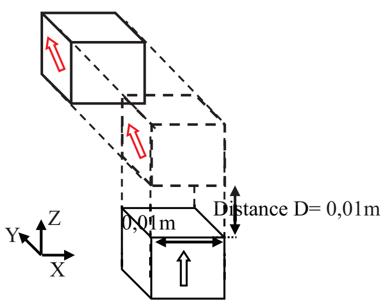

Fig. 5. Geometrical disposition of the magnets.

For the first application, the second magnet polarization is inclined (θ = 45°). The torque is calculated in the centre of the second magnet (Dx, Dy and Dz are equal to zeros). The results from analytical and numerical model using Flux3D are given in Fig. 6, proving a good accuracy of our approach.

We let the same physical and geometrical parameters as in previous example, except for the degree of inclination which is changed to θ = 30°. In this case also, the results are compared with Flux3D finite element software (Fig. 7).

In the second application, the second magnet is fixed at α = 0.0025 m, β = 0 m, γ = 0.02 m. We simulate and calculate the torque for one complete rotation of polarization (Fig. 8). The torque is computed at the centre of the magnet and its three components are presented in Fig. 9.

We can also calculate the torque components at any position of O_T, the next results concern the calculation of the torque at the position shown in Fig. 10, corresponding to Dx = -2 α, Dy = 0 and Dz = 0. The dimensions α, β and γ are the same as the last application (α = 0.0025 m, β = 0 m, γ = 0.02 m).The result in this case is presented as a function of a rotation angle θ on Fig. 11.

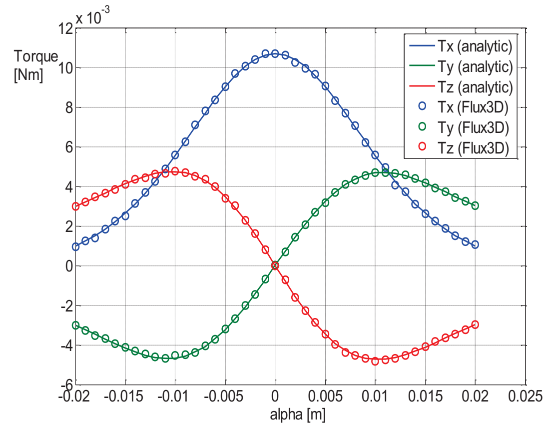

Fig. 6. Torque components for 45° inclined polarization of PM2.

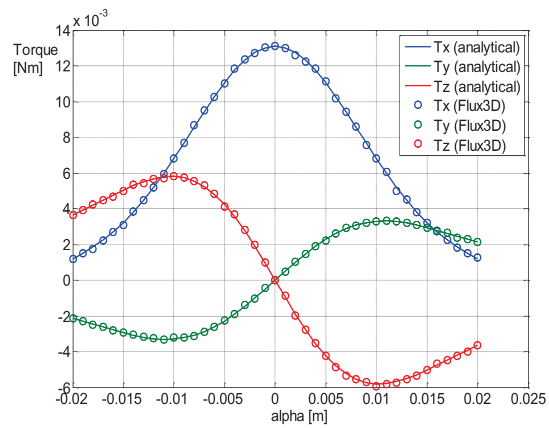

Fig. 7. Torque components for 30° inclined polarization of PM2 (second magnet).

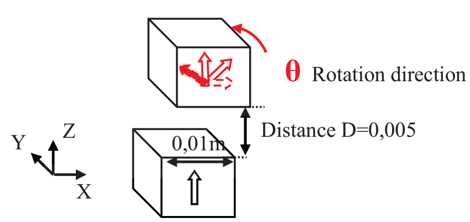

Fig. 8. Magnet position and polarization directions.

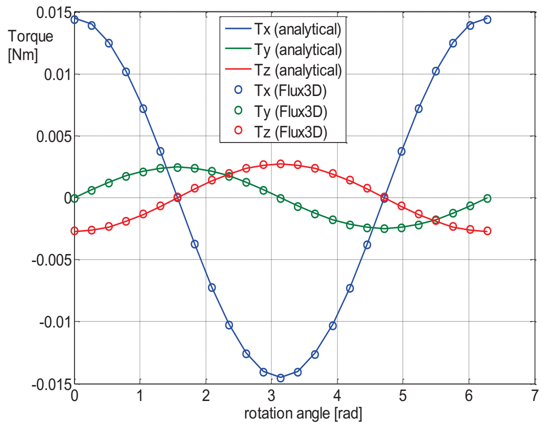

Fig. 9. Torque components calculation for one rotation of inclined polarization of PM2 ($\alpha = 0.0025$ m, $\beta = 0$ m, $\gamma = 0.02$ m).

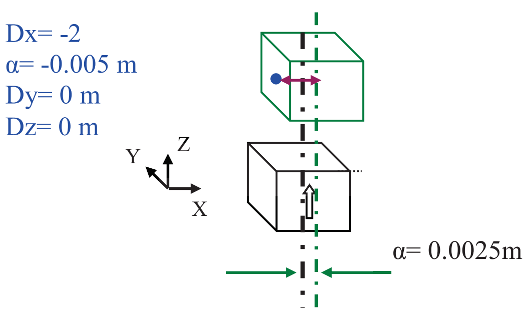

Fig. 10. Localization of the torque calculation point.

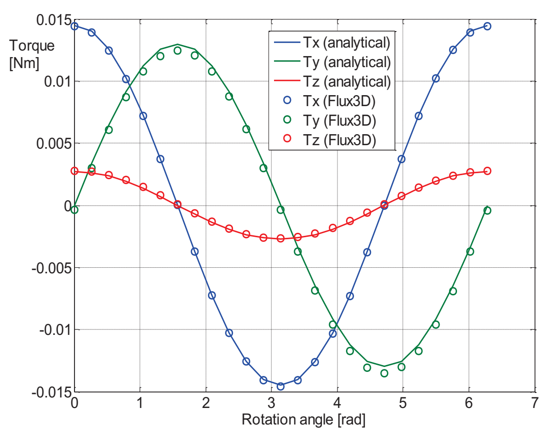

Fig. 11. Torque calculation at Dx = -0.005 m, Dy = 0 m and Dz = 0 m, as function of rotation angle θ.

V. CONCLUSION

This paper presents a new contribution in analytical torque calculations for cuboidal permanent magnets with inclined polarizations from any position. These investigations allow the direct calculation of many systems working by the forces or the torques between magnetized cuboidal elements (magnetic bearings, Halbach arrays….). These results can also be used for many other calculations, like complex shapes of magnets which can be replaced by a combination of several parallelogram ones.

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