An Exact Solution for the Generalized Spherical Orthogonality Integral of the Legendre Functions of the First and Second Kind

Amir Jafargholi

Institute of Space Science and Technology
Amirkabir University of Technology, 424 Hafez Ave., P.O. Box: 15875-4413, Tehran, Iran
jafargholi@ieee.org

Abstract **─** An exact formulation of a generalized orthogonality integral for the spherical boundary condition is proposed. This integral usually appears in the problems contained in conical and biconical antennas. The analytical results are successfully validated through a comparison with the numerical results.

Index Terms **─** Orthogonality integral, spherical boundary condition.

I. INTRODUCTION

In the analysis of electromagnetic boundary-value problems, any solution for the time-harmonic electric and magnetic fields must satisfy Maxwell's/vector wave equations as well as the appropriate boundary conditions [1]. The vector wave equations usually reduce to a number of scalar Helmholtz equations, and the general solutions can be founded in three-dimensional orthogonal coordinate systems.

Many researchers are interested in the formulation of full-wave spherical boundary value problems previously [1-9]. In order to find out unknown coefficients and to drive an exact solution for these structures, it is usually preferred to use the orthogonality properties of spherical functions to reduce the integrals to simple exact solutions. Although most of the orthogonality integrals are solved before, which may be found in the reference [10], there are some integrals which haven't been addressed yet in the mathematical or physical literatures. However, it is worth noting that the numerical solutions to these integrals are the main time consuming part of the electromagnetic boundary-value problems. The importance of the exact solutions is clearer in the problems contained to high degrees of complexity of the boundary condition, as well as mode-matching problems. In such problems, due to complexity and iterative nature of numerical solutions, it is critical to reduce all numerical integrals to their simple exact solutions.

In this paper, an exact formulation for the generalized spherical orthogonality integral of the legend functions of the first and second kind, which usually appears in the problems contained in conical and biconical antennas is proposed. The obtained analytical formulas confirm the general conclusions recently presented in [3-5]. It is demonstrated that the analytical results have been successfully validated through a comparison with the numerical results. The extracted formula is very easy to implement, essentially general and applicable to any problem, without the need to know where the singularities will take place.

II. FIELD ANALYSIS

Figure 1 (a) illustrates a slotted hollow conducting sphere of radius a, containing a Hertzian dipole $\bar{J} = \hat{z}J\delta(\bar{r})$, placed at the center $(\vec{r}=0)$; here (r,θ,φ) are the spherical coordinates and δ is a delta function. The time convention is $e^{-j\omega t}$ suppressed throughout. Due to azimuthally symmetry, the fields depend on (r,θ), and the fields are then TM waves, which can be expressed in terms of magnetic vector potentials. The total magnetic vector potential for the un-slotted sphere (first region, I) is a sum of the primary and secondary magnetic vector potentials, [3]:

$$A^i(r,\theta) = \hat{z}A_z^p(r,\theta) + \hat{r}A_r^s(r,\theta), \tag{1}$$

while the primary magnetic vector potential is a free-space Green's function as:

Submitted On: August 2, 2013
Accepted On: December 10, 2014

$$A_z^p(r,\theta)=\frac{\mu_1 J}{4\pi}\frac{e^{ikR}}{R}, \tag{2}$$

$\hat{z}$ and $\hat{r}$ are the unit vectors and $R=\sqrt{r^2+r'^2-2rr'\cos\theta}$. And the secondary magnetic vector potential is:

$$A_r^s(r,\theta)=\sum_{n=0}^{\infty}a_n\hat{J}_n(k_I r)P_n(\cos\theta), \tag{3}$$

where $\hat{J}_n(.)$ and $P_n(.)$ are the spherical Bessel and Legendre functions, respectively, and [3]:

$$a_n=\frac{\mu_1 aJ}{8\pi k_1\hat{J}_n'(k_1 a)}\frac{2n+1}{n(n+1)}\int_0^{\pi}\Omega\frac{\partial P_n(\cos\theta)}{\partial\theta}\sin^2\theta d\theta$$

$$\Omega=\left\{\left(a^2-2r'^2+ar'\cos\theta\right)\left(ik_1\tilde{R}-1\right)+k_1^2\tilde{R}^2\left(a^2-ar'\cos\theta\right)\right\}\frac{e^{ik\tilde{R}}}{\tilde{R}^5}. \tag{4}$$

Now consider a slotted conducting sphere, as shown in Fig. 1 (a). The total magnetic vector potential in region (I) consists of the incident A^i and scattered A_r^I potentials as:

$$A_r^I(r,\theta)=\sum_{n=0}^{\infty}C_n\hat{J}_n(k_I r)P_n(\cos\theta). \tag{5}$$

Here, C_n is an unknown modal coefficient.

The r-component of the magnetic vector potential in region (II) of the l-th slot is:

$$A_r^{II}(r,\theta)=\sum_{v=0}^{\infty}R_v^l(\cos\theta)\left[D_v^l\hat{J}_{\xi}(k_{II}r)+E_v^l\hat{N}_{\xi}(k_{II}r)\right], \tag{6}$$

where

$$R_v^l(\cos\theta)=\begin{cases}Q_{\xi}(\cos\theta) & v=0\\ Q_{\xi}(\cos\alpha_2^l)P_{\xi}(\cos\theta) & \\ -P_{\xi}(\cos\alpha_2^l)Q_{\xi}(\cos\theta) & v\geq 1\end{cases}, \tag{7}$$

where $Q_n(.)$ is the Legendre function of the second kind and D_v^l and E_v^l are unknown coefficients. Here $\xi_0^l=0$ and ξ_v^l satisfies $R_v^l(\cos\alpha_1^l)=0 \quad (v>1)$.

The r-component of the magnetic vector potential in region (III) is:

$$A_r^{III}(r,\theta)=\sum_{v=0}^{\infty}F_n\hat{H}_n^{(2)}(k_{III}r)P_n(\cos\theta), \tag{8}$$

where F_n is an unknown modal coefficient and $\hat{H}_n^{(2)}(.)$ is the spherical Hankel function of the second kind. To determine the modal coefficients, we enforced the field continuities, as described in details in [4-5].

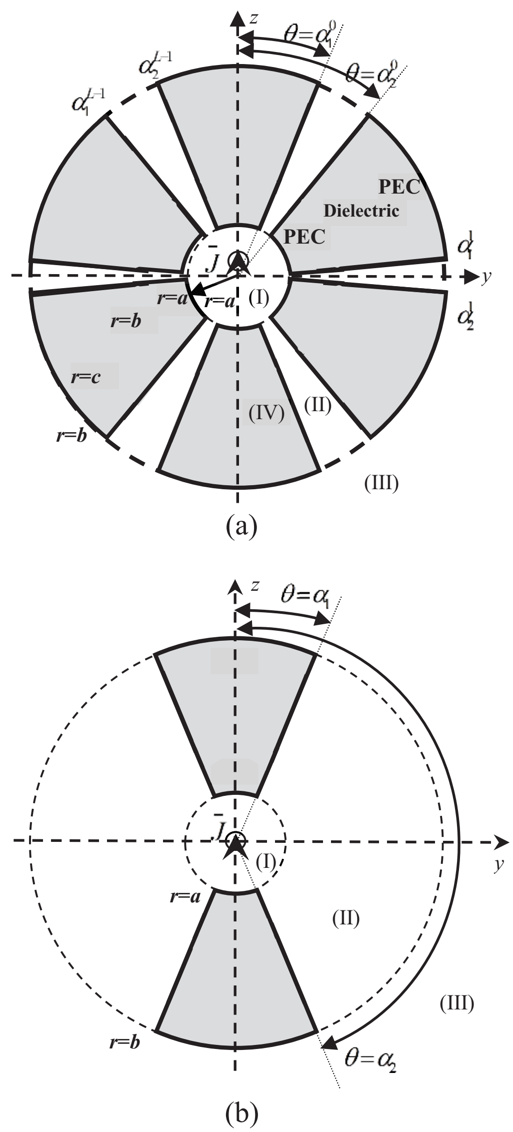


Fig. 1. (a) Multiply- and (b) single slotted conducting hollow sphere, $k_I\left(=\omega\sqrt{\mu_I\varepsilon_I}\right)$, $k_{II}\left(=\omega\sqrt{\mu_{II}\varepsilon_{II}}\right)$, and $k_{III}\left(=\omega\sqrt{\mu_{III}\varepsilon_{III}}\right)$: wave numbers of region (I) $r\leq a$, (II) $a\leq r\leq b$, and (III) $r\geq b$.

III. FINDING EXACT SOLUTION

Based on tangential electric field continuity at $r = a$, while we have:

$$E_\theta^{II}=E_\theta^I \quad \alpha_1^l<\theta<\alpha_2^l.$$

Applying Legendre function orthogonality integral, $\int_0^{\pi}(.)\left(dP_{n'}(\cos\theta)\right)/(d\theta)\sin\theta d\theta$, to this boundary condition, according to [3], $-I_{vn}$ is defined as below:

$$-I_{vn}^{l} = \int_{\alpha_1^l}^{\alpha_2^l} \frac{\partial}{\partial\theta} R_v^l(\cos\theta) \frac{\partial}{\partial\theta} P_n(\cos\theta) \sin\theta d\theta, \quad (9)$$

all required definitions are illustrated in [3-5].

The main goal of this paper is to drive an exact solution for I_{vn} integral. To start calculating the integral, first we assume that there is only a single slot configuration, so the problem is reduced to a simple biconical antenna (Fig. 1 (b)).

Integrating by part, and using Legendre function properties, some may rewrite (9) as:

$$\begin{aligned} -I_{vn}^{l} &= R_v^l(\cos\theta)\frac{\partial}{\partial\theta} P_n(\cos\theta)\sin\theta \Big|_{\alpha_1^l}^{\alpha_2^l} \\ &+ n(n+1)\int_{\alpha_1^l}^{\alpha_2^l} R_v^l(\cos\theta) P_n(\cos\theta)\sin\theta d\theta = I_1 + I_2, \end{aligned} \quad (10)$$

where

$$I_1 = \left. \begin{matrix} R_v^l(\cos\theta)(n+1)\cdot \\ \left[P_{n+1}(\cos\theta) - \cos\theta \cdot P_n(\cos\theta)\right] \end{matrix} \right|_{\alpha_1^l}^{\alpha_2^l} = (n+1) T_v T_p, \quad (11)$$

$$I_2 = M_{vn}\sin\theta \left\{ \begin{matrix} \left[R_v^l(\cos\theta) \frac{\partial}{\partial\theta} P_n(\cos\theta)\right]_{\alpha_1^l}^{\alpha_2^l} \\ -\left[P_n(\cos\theta)\frac{\partial}{\partial\theta} R_v^l(\cos\theta)\right]_{\alpha_1^l}^{\alpha_2^l} \end{matrix} \right\} = T_1 - T_2, \quad (12)$$

and

$$T_p = \left[P_{n+1}\left(\cos\alpha_1^l\right) - \cos\alpha_1^l \cdot P_n\left(\cos\alpha_1^l\right)\right], \quad (13)$$

$$\begin{aligned} T_v &= P_\xi\left(\cos\alpha_2^l\right)\cdot Q_\xi\left(\cos\alpha_1^l\right) \\ &- Q_\xi\left(\cos\alpha_2^l\right)\cdot P_\xi\left(\cos\alpha_1^l\right), \end{aligned} \quad (14)$$

$$M_{vn} = \frac{n(n+1)}{\xi_v^l\left(\xi_v^l+1\right) - n(n+1)}. \quad (15)$$

Simplification of (12) may result:

$$T_1 = (n+1) M_{vn} T_v T_p, \quad (16)$$

$$T_2 = M_{vn}\left[P_n\left(\cos\alpha_2^l\right) - P_n\left(\cos\alpha_1^l\right)\Phi_v\right], \quad (17)$$

where

$$\Phi_v = \left(\xi_v^l + 1\right)\left[\cos\alpha_1^l T_v - T_{vv}\right], \quad (18)$$

$$\begin{aligned} T_v &= P_\xi\left(\cos\alpha_2^l\right)\cdot Q_\xi\left(\cos\alpha_1^l\right) \\ &- Q_\xi\left(\cos\alpha_2^l\right)\cdot P_\xi\left(\cos\alpha_1^l\right), \end{aligned} \quad (19)$$

$$\begin{aligned} T_{vv} &= P_\xi\left(\cos\alpha_2^l\right)\cdot Q_{\xi+1}\left(\cos\alpha_1^l\right) \\ &- Q_\xi\left(\cos\alpha_2^l\right)\cdot P_{\xi+1}\left(\cos\alpha_1^l\right). \end{aligned} \quad (20)$$

Substitution (16-20) to (12) yields:

$$\begin{aligned} I_2 &= M_{vn}\left\{ \begin{matrix} (n+1) T_v T_p - \\ \left[P_n\left(\cos\alpha_2^l\right) - P_n\left(\cos\alpha_1^l\right)\Phi_v\right] \end{matrix} \right\} \\ &= M_{vn}\left\{ \begin{matrix} T_v\left[\begin{matrix} (n+1) P_{n+1}\left(\cos\alpha_1^l\right) \\ +\left(\xi_v^l - n\right)\cos\alpha_1^l \cdot P_n\left(\cos\alpha_1^l\right) \end{matrix} \right] \\ -\left[P_n\left(\cos\alpha_2^l\right) + P_n\left(\cos\alpha_1^l\right)\left(\xi_v^l+1\right) T_{vv}\right] \end{matrix} \right\}. \end{aligned} \quad (21)$$

Using (11) and (21) one obtains:

$$\begin{aligned} -I_{vn} &= (n+1)\left(M_{vn}+1\right) T_v T_p \\ &- M_{vn}\left[P_n\left(\cos\alpha_2^l\right) - P_n\left(\cos\alpha_1^l\right)\Phi_v\right]. \end{aligned} \quad (22)$$

To verify the extracted formula, a numerical evaluation of the I_{vn} integral, (9), is compared with the results of exact equation (22) in Table 1 and 2. It seems clear that the numerical estimations are in good agreement with exact formula. The commercial software Mathematica is adopted for the numerical integrations.

Table 1: Calculation comparison, for $\alpha_1=\pi/3$, $\alpha_2=2\pi/3$

Number of Modes	Eq. 22	Numerical Integration
n=2, ξ_v^l=1	0.237011470359071	0.237011470359071
n=3, ξ_v^l=1	0.612011470359070	0.612011470359070
n=4, ξ_v^l=1	-0.387356331794832	-0.387356331794832
n=1, ξ_v^l=2	-0.190747132410232	-0.190747132410232
n=1, ξ_v^l=3	0.167614963435814	0.167614963435814
n=1, ξ_v^l=4	0.265859967903335	0.265859967903335

Table 2: Calculation comparison, for n=11, ξ_v^1=3, α_2=π- α_1

Cone Angle	Eq. 22	Numerical Integration
α_1=π/3	0.034245642954503	0.034245642954479
α_1=π/6	-0.366169292285009	-0.366169292456850
α_1=π/12	0.013319388336234	0.013319365546954
α_1=π/24	-0.037508615096376	-0.037508811221276
α_1=π/48	-0.005357753392885	-0.005358297043497

IV. CONCLUSION

An exact formulation of a generalized orthogonality integral for the spherical boundary condition which is usually calculated numerically in the problems contained in conical and biconical antennas has been proposed. The analytical results have been successfully validated through a comparison with the numerical results.

REFERENCES

[1] R. E. Collin and F. J. Zucker, *Antenna Theory*, McGraw-Hill, New York, 1969.

[2] K. W. Leung, "Analysis of zonal and rectangular slots on a conducting spherical cavity," *IEEE Trans. Antennas Propag.*, vol. 49, no. 12, pp. 1739-1745, Dec. 2001.

[3] J. S. Ock and H. J. Eom, "Radiation of a hertzian dipole in a slotted conducting sphere," *IEEE Trans. Antennas Propag.*, vol. 57, no. 12, pp. 3847-3851, 2009.

[4] A. Jafargholi and M. Kamyab, *Metamaterials in Antenna Engineering, Theory and Applications*, LAP Lambert Academic Publishing, Berlin, Germany, 2011.

[5] A. Jafargholi and M. Kamyab, "Full-wave analysis of double positive/double negative loaded dipole antennas," *Electromagnetics*, 32:2, 103-116, 2012.

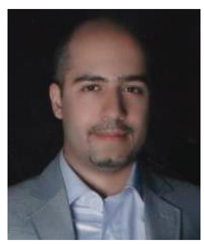
[6] A. Jafargholi and M. Kamyab, "Full-wave analysis of loaded dipole antennas using mode-matching theory," *Applied Computational Electromagnetics Society, ACES Journal*, vol. 26, no. 11, pp. 915-921, Nov. 2011.

[7] S. Saoudy and M. Hamid, "Input admittance of a biconical antenna with wide feed gap," *IEEE Trans. Antennas Propag.*, vol. 38, no. 11, pp. 1784-1790, 1990.

[8] C. Polk, "Resonance and supergain effects in small ferromagnetically or dielectrically loaded biconical antennas," *IRE Trans. Antennas Propag.*, 414-423, 1959.

[9] C. T. Tai, "On the theory of biconical antennas," *J. of Appl. Phys.*, vol. 19, 1155-1160, 1958.

[10] M. Abramowitz and I. A. Stegun, *Handbook of Mathematical Functions*, Dover, New York, 1965.



Amir Jafargholi received the Ph.D. degree in Electrical Engineering from K. N. Toosi University of Technology, Tehran, Iran, in 2011. During the first half of 2012 he was a Research Associate, and in the same year he was appointed as an Assistant Professor in the Institute of Space Science and Technology, Amirkabir University of Technology, Iran.

His research is generally in applied electromagnetics - and particularly in antennas, array and phased array antennas for applications in wireless and satellite communications. At present, his interests focus on the applications of metamaterials in the analysis and synthesis of antennas and phased array antennas. He is the author of Metamaterials in Antenna Engineering, Theory and Applications (LAP Academic Publishing, Germany, 2011). He also has authored or co-authored over 30 journal papers, 3 book chapters and 40 refereed conference papers. He has supervised or co-supervised over 15 Ph.D. and M.Sc. theses.

Jafargholi was a recipient of the Student's Best Thesis National Festival award for his BS thesis, on

May 2006. He was a recipient of the 22nd Khawarizmi International and 13th Khawarizmi Youth Award in Jan. 2009 and Oct. 2011, respectively. He was also the recipient of the Research Grant Awarded in Metamaterial 2010. Jafargholi has been a member of the IEEE Antennas and Propagation Society since 2011. He is also a member of Iranian National Elite Foundation since 2011. Currently he is the Scientific Editor of the Journal of Electrical Industries.