# Novel Mathematical Formulation of the Antenna Array Factor for Side Lobe Level Reduction

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***Abstract*** **─** In this article a new approach is used to improve the performance of antenna arrays. The antenna array performance is improved when its directivity is increased and its side lobes are decreased. To do this, a concept of array hybridization (mixing two distinct arrays) is presented and applied to uniform arrays to generate a new array for satisfying the requirement. Two new arrays are generated using the proposed principle. The first is obtained from two arrays with different number of elements (UUDNH). The second generated array is based on the use of two arrays with different spacing between their elements (UUDdH).

The obtained arrays parameters (array factor, side lobe levels, directivity and excitation coefficients) are given in closed form expressions. Furthermore, performances of the proposed arrays exceed that of Tschebyscheff arrays with the same number of elements.

***Index Terms*─** Antenna array, array factor, convolution, directivity, distance conversion, feeding currents, first null bandwidth, Fourier transform, hybridization, side lobe.

## I. INTRODUCTION

In this modern era, the telecommunications become an important research field because all the new physical cosmological phenomena are happening in the far field. To collect the electromagnetic signals coming from these phenomena without any interference and any noise a high directive antenna with very low side lobes is needed. A single element antenna cannot be used to obtain the needed radiation pattern and the required results. For this reason an antenna array is used in order to have large number of controllable parameters to obtain the needed pattern that satisfies the desired specifications [1-2]. The uniform array is the simpler and the well-known array that offers high directivity, but unfortunately with very high side lobes levels. Previously, many methods have been used to improve the antenna array performance [3]. In [3], the non-uniform arrays have been studied and the obtained distributions are quite complicated for practical implementation. Other works also exist for the same purpose and most of them are based on the use of Genetic Algorithm (GA) or other optimization techniques [4-9]. The results obtained by using optimization techniques are reasonably satisfactory but the feeding currents and the elements positions are randomly distributed and the implementation of the array using these distributions is very hard and costly.

The present work is based on mathematical derivation of the array factor of linear antenna array based on linear systems techniques [10]. The idea may be viewed as hybridization of two distinct uniform antenna arrays; in which the current excitation for each element is viewed as one term that belongs to an overall discrete sequence in given coordinate system. By applying the Fourier transform and convolution properties the hybridization (mixing) can be performed between two different arrays yielding a new array (Fig. 1) having high directivity with very low side lobe levels. Furthermore, the feeding currents are given by closed form expression that can easily be practically implemented.

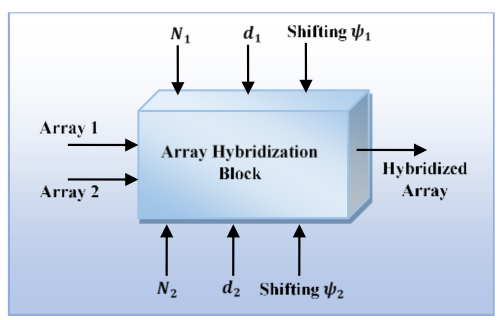


Fig. 1. The Hybridization block diagram where $(N_1, d_1, \psi_1)$ and $(N_2, d_2, \psi_2)$ are the controllable parameters of the array 1 and the array 2 respectively; $N_i$ is the number of the elements, $d_i$ is the spacing distance, and $\psi_i$ is the array shifting $\left(A(\psi - \psi_i)\right)$ of array *i* (*i* = 1 or 2).

## II. MATHEMATICAL THEORY OF HYBRIDIZATION

As already stated, the hybridization is performed between two distinct uniform arrays. The array factor of

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a linear array is given by the following relation:

$$A(\psi) = \sum_n a_n e^{j\psi n}\,, \tag{1}$$

where $a_n$ is complex sequence that represent the feeding current (excitation coefficients) of each antenna element, and $\psi$ is the array digital wave number ([2] at chapter 18). A linear antenna aligned along the z-axis has a digital wave number $\psi = kd\cos(\theta)$; where $d$ is the spacing distance and $\theta$ is the elevation angle.

Now the proposed method starts by replacing the term $a_n$ by the term $x[n]$, which is more general than $a_n$, because $n$ can be negative, and it indicates the existence of coordinate. So, the array factor is written in the following form:

$$A(\psi) = \sum_n x[n] e^{j\psi n}. \tag{2}$$

The relation (2) represents the discrete Fourier transform $\mathcal{DF}$:

$$A(\psi) = \mathcal{DF}\{x[n]\}_{\psi \to -\psi}. \tag{3}$$

The property of convolution given in the following equation is applied in the coming sections:

$$\mathcal{DF}\{x_1[n] * x_2[n]\} = A_1(\psi)A_2(\psi). \tag{4}$$

### A. Application of convolution on the array elements

We consider an array $A_1(\psi)$ with five identical elements positioned along z-axis with equal spacing $d$ between them. The total electric field $\vec{E}_{T_1}$ is given as ($\vec{E}_0$ is the electric field of an individual element):

$$\vec{E}_{T_1} = \vec{E}_0 A_1(\psi), \tag{5}$$

where the array factor is given by:

$$A_1(\psi) = \sum_{n=-2}^{+2} x_1[n] e^{j\psi n}. \tag{6}$$

Let's take a step further by considering the array as one antenna and we repeat the arraying process (using the array factor $A_2(\psi)$) over the first array and we obtain the following results:

The total electric field would be $\vec{E}_{T_2} = \vec{E}_{T_1} A_2(\psi)$, where $A_2(\psi)$ is the array factor of the second array. Consequently the following result is obtained:

$$\vec{E}_{T_2} = A_1(\psi)A_2(\psi)\vec{E}_0. \tag{7}$$

The resulted array factor is $A_T(\psi) = A_1(\psi)A_2(\psi)$.

The resulted feeding currents are simply obtained by convolution between the currents of the first and the second arrays. This process may be seen as the shift of the first array along the second one as represented by Fig. 2 (for illustration the second array has three elements). The application of the shifting process produces superposition of some elements on each other as shown in Fig. 3.

The superposed elements in one given position are replaced by one element antenna. This element is fed by all the currents of the superposed antennas. When the first array is translated along the second array elements positions, the first array is scaled by the second array coefficients at its corresponding position as shown in Fig. 4.

The feeding current of the obtained array is:

$$x[n] = x_1[n] * x_2[n]. \tag{8}$$

The obtained array has a total number of elements and is given by following equation:

$$N_T = N_1 + N_2 - 1, \tag{9}$$

where $\boldsymbol{N_T}$, $\boldsymbol{N_1}$**, and** $\boldsymbol{N_2}$ are the number of elements of the resulted, the first and the second arrays respectively.

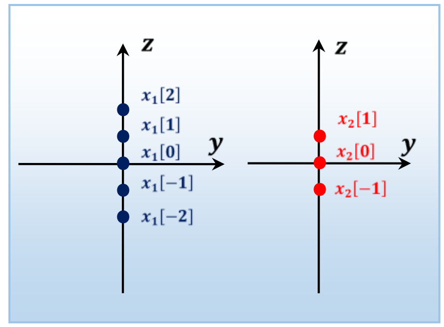


Fig. 2. The first array has five elements the second array has three elements.

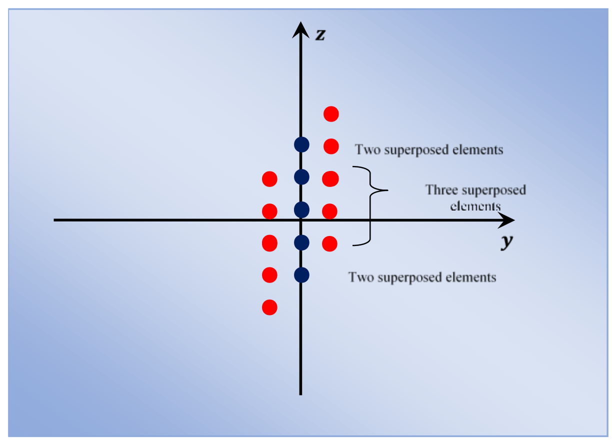


Fig. 3. Translating the first array along the positions of the second array's elements.

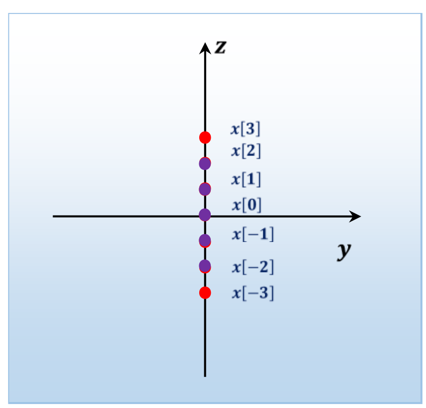


Fig. 4. There is superposed element when the translation is performed.

### B. Arrays with different spacing

In this section, the hybridization is performed between two uniform arrays which have distinct spacing between their elements. For this case the multiplication of the array factors does not lead to the convolution of the feeding currents like the case of equidistance arrays. To overcome this problem, a mathematical trick is used and it is named, *distance conversion* [10].

When two array factors with different spacing ($d_1 \neq d_2$) are multiplied then,

$$A(\psi_1, \psi_2) = A_1(\psi_1)A_2(\psi_2), \quad (10)$$

where
$$\begin{cases} \psi_1 = kd_1 cos(\theta) \\ \psi_2 = kd_2 cos(\theta) \end{cases}, \quad (11)$$

where $k$ is the wave number $k = \frac{2\pi}{\lambda}$ and $\lambda$ is the signal wavelength.

The relation between $\psi_1$ and $\psi_2$ can be written as:

$$\frac{\psi_2}{\psi_1} = \frac{d_2}{d_1} = \frac{p}{q}, \quad (12)$$

where $p$ and $q$ are positive integers different from zero.

The new value of $\psi$ and the new distance $d$ can be defined as:

$$\psi = \frac{\psi_2}{p} = \frac{\psi_1}{q}, \quad (13)$$

$$d = \frac{d_2}{p} = \frac{d_1}{q}. \quad (14)$$

The feeding currents expression is obtained by performing the following analysis.

The array factor of the first array is given by:

$$A_1(\psi_1) = \sum_n x_1[n]e^{j\psi_1 n},$$

$$\Rightarrow A_1(\psi_1) = \sum_n x_1[n]e^{j\frac{\psi_1}{q}(qn)}.$$

By putting $m = qn$, where $\psi = \frac{\psi_1}{q}$ the following result is obtained:

$$A_1(\psi) = \sum_m x_1^{\frac{d_1}{q}}[m]e^{j\psi m}, \quad (15)$$

where
$$x_1^{\frac{d_1}{q}}[m] = \begin{cases} x_1\left[\frac{m}{q}\right]; & m = kq \\ 0 \quad ; & otherwise \end{cases}. \quad (16)$$

Similar result can be obtained for the case of the second array as:

$$x_2^{\frac{d_2}{p}}[m] = \begin{cases} x_2\left[\frac{m}{p}\right]; & m = kp \\ 0 \quad ; & otherwise \end{cases}, \quad (17)$$

where the notation $x_1^{\frac{d_1}{q}}[m]$ denote the distance conversion $d_1 \xrightarrow{converted} \frac{d_1}{q}$.

The equation (10) implies that $A(\psi_1, \psi_2) = A(\psi) = A_1(\psi)A_2(\psi)$:

where
$$\begin{cases} A_1(\psi) = \sum_n x_1^{\frac{d_1}{q}}[n]e^{j\psi n} \\ A_2(\psi) = \sum_n x_2^{\frac{d_2}{p}}[n]e^{j\psi n} \end{cases}. \quad (18)$$

The obtained array feeding currents will be the convolution between $x_1^{\frac{d_1}{q}}[n]$, and $x_2^{\frac{d_1}{p}}[n]$:

$$x^d[n] = x_1^{\frac{d_1}{q}}[n] * x_2^{\frac{d_2}{p}}[n]. \quad (19)$$

The distance conversion is explained in Fig. 5 by adopting some imaginary elements with feeding currents with amplitude of zero.

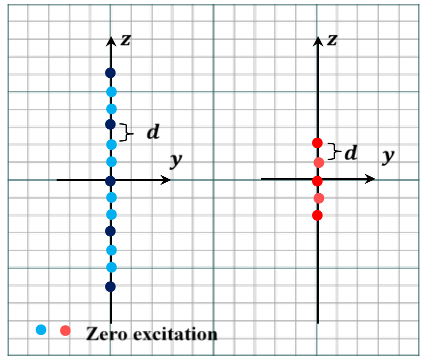


Fig. 5. Elements positions of the first and second arrays after distance conversion.

To apply the distance conversion, we propose the following definitions:

1) Elements with zero excitation are called “imaginary elements”; their number denoted by $N_i$.
2) Elements with non-zero excitation are called “real elements”; their number denoted by $N$.
3) The total number of elements is called “theoretical elements”; $N' = N_i + N$.

After applying the distance conversion, the theoretical number of elements is given by:

$$N' = q(N - 1) + 1, \quad (20)$$

where $''q''$ is the distance conversion factor.

The convolution $x^d[n] = x_1^{\frac{d_1}{q}}[n] * x_2^{\frac{d_2}{p}}[n]$ can be represented graphically as shown in Fig. 6.

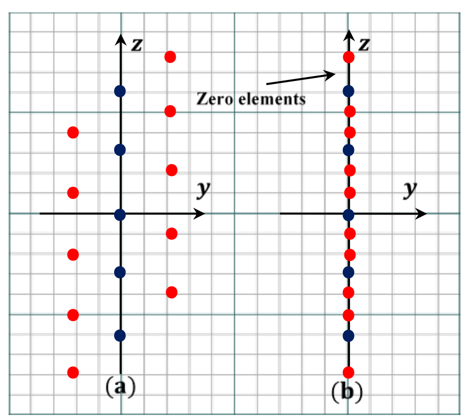


Fig. 6. (a) Translating the first array along the second array’s positions. (b) The real elements of the obtained array.

### C. Array with even number of elements

When the number of the elements in the array is even and the array is centered at the origin, the array elements are located at fractional numbers ($-\frac{3}{2}, -\frac{1}{2}, \frac{1}{2}, \frac{3}{2}$) and the notation $x[n]$ cannot be used because $n$ is an integer. Consequently, the convolution cannot be applied between even array-even array elements, even array-odd array elements. To overcome this dilemma, the distance conversion can be performed ($d \xrightarrow{converted} \frac{d}{2}$). This conversion makes the notation $x[n]$ applicable and the convolution can be used.

Let's take $x[n]$, the relative feeding current with even number array elements. By applying the distance conversion, the theoretical number of the array elements will be odd as shown in Fig. 7. The expression of conversion is shown below:

$$x^{\frac{d}{2}}[n] = \begin{cases} x\left[\frac{n}{2}\right]; & n \text{ is odd} \\ 0\ ; & n \text{ is even} \end{cases}. \quad (21)$$

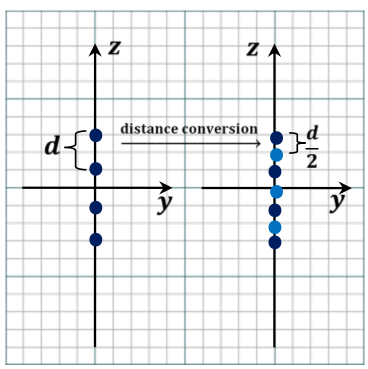


Fig. 7. The distance conversion is applied to array with even number of elements to obtain an array with odd number of elements.

## III. APPLICATION OF THE HYBRIDIZATION CONCEPT

Now the theory of hybridization discussed in the forgoing sections is applied to two uniform arrays with specified criteria to obtain new arrays with improved parameters.

### A. Array factor multiplication with different number of elements

The hybridization will be performed by multiplying two uniform arrays. Each array has its own number of elements ($N_1$ and $N_2$) where the two arrays have the same spacing distance. To design the hybridized array with desired specification the ratio, $N_1/N_2$ should be chosen as explained below:

$$A_T(\psi) = A_1(\psi)A_2(\psi),$$

where $A_1(\psi) = \frac{\sin\left(\frac{N_1}{2}\psi\right)}{N_1 \sin\left(\frac{\psi}{2}\right)}$, and $A_2(\psi) = \frac{\sin\left(\frac{N_2}{2}\psi\right)}{N_2 \sin\left(\frac{\psi}{2}\right)}$.

The total array factor is given by:

$$A_T(\psi) = \frac{\sin\left(\frac{N_1}{2}\psi\right)}{N_1 \sin\left(\frac{\psi}{2}\right)} \frac{\sin\left(\frac{N_2}{2}\psi\right)}{N_2 \sin\left(\frac{\psi}{2}\right)}. \quad (22)$$

To minimize side lobes, $N_1$ and $N_2$ should be chosen so that the zeros of the first array eliminate the side lobes maxima of the second array and vice-versa as illustrated by Fig. 8. To achieve this requirement, the following condition must be satisfied:

$$\frac{2\pi}{N_1} = \frac{3\pi}{N_2}. \quad (23)$$

The condition (23) will adjust the first zero of the first array over the first minor lobe maximum of the second array yielding a hybridized array with very low side lobes. From Equation (23), the following relation can be deduced:

$$2N_2 = 3N_1 = 6N. \quad (24)$$

This imply the following:

$$\begin{cases} N_1 = 2N \\ N_2 = 3N \end{cases}. \quad (25)$$

The total array factor of the obtained array can be written under the following relation:

$$A_T(\psi) = \frac{\sin(N\psi)\sin\left(\frac{3N}{2}\psi\right)}{6N^2 \sin^2\left(\frac{\psi}{2}\right)}. \quad (26)$$

The total number of elements in the hybridized array is found by using relation (9):

$$N_T = 5N - 1. \quad (27)$$

The side lobe will be minimized after the multiplication as shown in the Fig. 8.

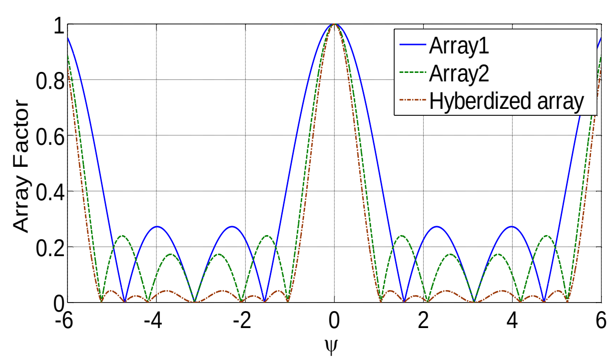


Fig. 8. Array factors in terms of $\psi$ ($N = 2, N_T = 9$).

The hybridized array parameters are given below.

*Side Lobe Level (SLL)*

The first side lobe maxima position is located between the first and the second nulls. The minor lobe maxima position is approximated as $\psi_{SLM} \cong \frac{\psi_{null1}+\psi_{null2}}{2}$, where $\psi_{null1} = \frac{2\pi}{N_1} = \frac{\pi}{N}$, and $\psi_{null2} = \frac{2\pi}{N_2} = \frac{2\pi}{3N}$. The side lobe maxima position is $\psi_{SLM} \cong \frac{5\pi}{6N}$. The side lobe level of the hybridized array is given under the following expression:

$$|A_T(\psi_{SLM})| \cong \frac{\sqrt{2}}{24\, N^2 \sin^2\left(\frac{5\pi}{12N}\right)}. \quad (28)$$

*Directivity*

The directivity of the hybridized array can be evaluated in terms of the parameter $''N''$. The radiated power is evaluated by $P_{rad} = \oiint_{\Omega} |A_T(\psi)|^2 d\Omega$. By using Equation (26) we get the following:

$$P_{rad} = 2\pi \int_0^{\pi} \left| \frac{sin(N\psi)}{2N\,sin\left(\frac{\psi}{2}\right)} \frac{sin\left(\frac{3N}{2}\psi\right)}{3N\,sin\left(\frac{\psi}{2}\right)} \right|^2 \sin(\theta) d\theta. \quad (29)$$

In the absence of grating lobes in the visible region, we can use the approximation $\frac{si\,n\left(N\frac{3\psi}{2}\right)}{N\,sin\left(\frac{\psi}{2}\right)} \cong \frac{si\,n\left(N\frac{3\psi}{2}\right)}{N\frac{\psi}{2}}$, and letting $z = Nkd\,cos(\theta)$, the radiated power is written in the following form:

$$P_{rad} = \frac{2\pi^2}{Nkd} \int_{-Nkd}^{Nkd} \left(\frac{sin(z)}{z}\right)^2 \left(\frac{sin\left(\frac{3}{2}z\right)}{\frac{3}{2}z}\right)^2 dz. \quad (30)$$

This integral can be performed by using the properties of convolution and Fourier transform and the directivity expression is obtained as:

$$D = \frac{27\,Nkd}{7\,\pi}. \quad (31)$$

When the relation (27) is used, the directivity is written in terms of the total number of element in the array, $N_T$ as:

$$D = \frac{27\,(N_T+1)kd}{35\,\pi}. \quad (32)$$

*Feeding Currents*

To implement this antenna array, the feeding current in each antenna must be determined. The hybridized array factor is found under the relation $A_T(\psi) = A_1(\psi)A_2(\psi)$, where

$$\begin{cases} A_1(\psi) = \frac{sin\left(\frac{N_1}{2}\psi\right)}{N_1\,sin\left(\frac{\psi}{2}\right)} & ;\ N_1 = 2N \\ A_2(\psi) = \frac{sin\left(\frac{N_2}{2}\psi\right)}{N_2\,sin\left(\frac{\psi}{2}\right)} & ;\ N_2 = 3N \end{cases}$$

, since the first array has always an even number of elements, the position of the array elements will be located at fractional numbers. The notation $x_1[n]$ cannot be used because $n$ must be integer. In this case the distance conversion will be applied.

$$d \longrightarrow d' = \frac{d}{2} \;\Leftrightarrow\; \psi \longrightarrow \psi' = \frac{\psi}{2}.$$

The conversion is performed by adding zero imaginary elements between the real elements.

Theoretically, the total number of elements will be, for the first array:

$$N_1{}' = N_1 + N_{i_1} = N_1 + (N_1 - 1) = 4N - 1;$$

for the second array:

$$N_2{}' = N_2 + N_{i_2} = N_2 + (N_2 - 1) = 6N - 1.$$

By applying the distance conversion the feeding current will have the following form.

When $N$ is even:

$$x_1^{\frac{d}{2}}[n] = \begin{cases} \begin{cases} \frac{1}{2N} & ;\ n\ odd \\ 0 & ;\ n\ even \end{cases} & ; -(2N-1) \le n \le (2N-1) \\ 0 & ; \quad otherwise \end{cases}, \quad (33)$$

$$x_2^{\frac{d}{2}}[n] = \begin{cases} \begin{cases} \frac{1}{3N} & ;\ n\ odd \\ 0 & ;\ n\ even \end{cases} & ; -(3N-1) \le n \le (3N-1) \\ 0 & ; \quad otherwise \end{cases}. \quad (34)$$

The feeding current can be represented in its matrix form:

$$x_1^{\frac{d}{2}}[n] = \frac{1}{2N} \overbrace{\left[ 1\ 0\ 1\ 0\ 1 \ldots \underbrace{0}_{-2,}\ \underbrace{1}_{-1,}\ \underbrace{0}_{n=0,}\ \underbrace{1}_{1,}\ \underbrace{0}_{2} \ldots 1\ 0\ 1\ 0\ 1 \right]}^{4N-1},$$

$$x_2^{\frac{d}{2}}[n] = \frac{1}{3N} \overbrace{\left[ 1\ 0\ 1\ 0\ 1 \ldots \underbrace{0}_{-2,}\ \underbrace{1}_{-1,}\ \underbrace{0}_{n=0,}\ \underbrace{1}_{1,}\ \underbrace{0}_{2} \ldots 1\ 0\ 1\ 0\ 1 \right]}^{6N-1}.$$

When $N$ is odd:

$$x_1^{\frac{d}{2}}[n] = \begin{cases} \begin{cases} \frac{1}{2N} & ;\ n\ odd \\ 0 & ;\ n\ even \end{cases} & ; -(2N-1) \le n \le (2N-1) \\ 0 & ; \quad otherwise \end{cases}, \quad (35)$$

$$x_2^{\frac{d}{2}}[n] = \begin{cases} \begin{cases} \frac{1}{3N} & ;\ n\ even \\ 0 & ;\ n\ odd \end{cases} & ; -(3N-1) \le n \le (3N-1) \\ 0 & ; \quad otherwise \end{cases}. \quad (36)$$

The feeding current can be represented in its matrix form:

$$x_1^{\frac{d}{2}}[n] = \frac{1}{2N} \overbrace{\left[ 1\ 0\ 1\ 0\ 1 \ldots \underbrace{0}_{-2,}\ \underbrace{1}_{-1,}\ \underbrace{0}_{n=0,}\ \underbrace{1}_{1,}\ \underbrace{0}_{2} \ldots 1\ 0\ 1\ 0\ 1 \right]}^{4N-1},$$

$$x_2^{\frac{d}{2}}[n] = \frac{1}{3N} \overbrace{\left[ 1\ 0\ 1\ 0\ 1 \ldots \underbrace{1}_{-2,}\ \underbrace{0}_{-1,}\ \underbrace{1}_{n=0,}\ \underbrace{0}_{1,}\ \underbrace{1}_{2} \ldots 1\ 0\ 1\ 0\ 1 \right]}^{6N-1}.$$

The feeding currents of the hybridized array is determined by performing the convolution:

$$x_T^{\frac{d}{2}}[n] = x_1^{\frac{d}{2}}[n] * x_2^{\frac{d}{2}}[n]. \quad (37)$$

By evaluating the convolution (37), the following results are obtained.

When $N$ is even:

$$x_T^{\frac{d}{2}}[n] = \frac{(-1)^n+1}{2} \begin{cases} -\frac{1}{12N^2}(|n| - 5N) & ; N \le |n| \le 5N-2 \\ \frac{1}{3N} & ; \quad |n| \le N \\ 0 & ; \quad otherwise \end{cases}, \quad (38)$$

When $N$ is odd

$$x_T^{\frac{d}{2}}[n] = \frac{(-1)^{n+1}+1}{2}\begin{cases} -\frac{1}{12N^2}(|n|-5N)\,; N \le |n| \le 5N-2 \\ \frac{1}{3N} \quad ; \quad |n| \le N \\ 0 \quad ; \quad otherwise \end{cases}, \quad (39)$$

where $N = \frac{N_T+1}{5}$, $N_T$ is the total number of the array. Note that for $N$ is even $a_n = x_T^d[n] = x_T^{\frac{d}{2}}[2n]$, for $n \ge 0$, when $N$ is odd $a_n = x_T^d[n] = x_T^{\frac{d}{2}}[2n-1]$, for $n \ge 1$.

The proposed array is named as **UUDNH,** which is the abbreviation of "**Uniform with Uniform Different-N Hybridization**".

For illustration, the feeding currents of Equations (38) and (39) are drawn for $N$ = 4 and 5 in Fig. 9 and the corresponding array factors are illustrated in Fig. 10. To avoid the appearance of the grating lobes in the visible region of $(-kd \le \psi \le kd)$, the following condition should be satisfied:

$$kd \le 2\pi - \psi_{first\ null}. \quad (40)$$

For the proposed array case, the first null is situated at $\psi_{first\ null} = \frac{2\pi}{3N}$, and this condition becomes $kd \le 2\pi\frac{3N-1}{3N}$.

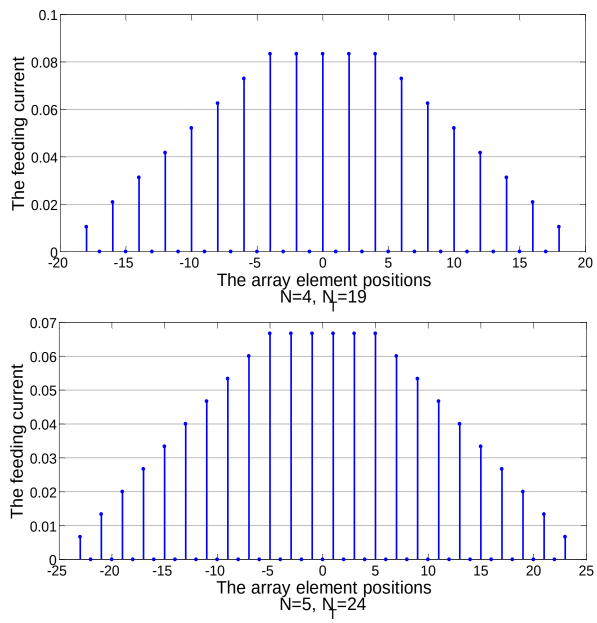


Fig. 9. Excitation coefficients of the hybridized array for $N$ = 4 and 5.

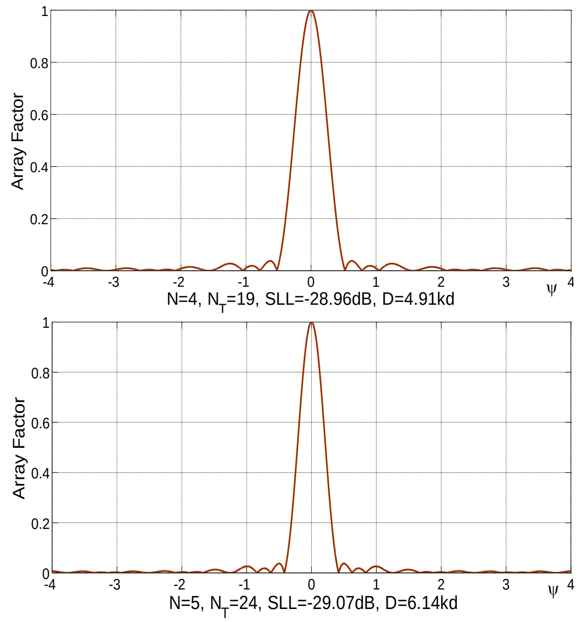


Fig. 10. Array pattern of the hybridized array (UUDNH) in terms of $\psi$ for $N$ = 4 and 5, where the directivities is given in terms of the wave number $k$ and the distance between the elements $d$.

### B. Array factor multiplication with different spacing

Previously, the hybridization between two uniform arrays with different number of elements is performed. Here, the hybridization between two uniform arrays with different spacing is proposed. To do this, a condition on the ratio of the two spacing is set as illustrated in coming sections. Similarly, the new array factor is simply given as:

$$A(\psi_1, \psi_2) = A_1(\psi_1)A_2(\psi_2),$$

where $\psi_1 = kd_1\cos(\theta), \psi_2 = kd_2\cos(\theta)$, and by letting $s = kcos(\theta)$,

then $$A_T(s) = A_1(sd_1)A_2(sd_2). \quad (41)$$

Therefore, the array factor is given by:

$$A_T(s) = \frac{sin\left(N\frac{sd_1}{2}\right)}{N\,sin\left(\frac{sd_1}{2}\right)} \times \frac{sin\left(N\frac{sd_2}{2}\right)}{N\,sin\left(\frac{sd_2}{2}\right)}. \quad (42)$$

To minimize side lobes, $d_1$ and $d_2$ should be chosen so that the zeros of the first array eliminate the side lobes maxima of the second array. To achieve this, $\frac{3\pi}{Nd_1} = \frac{2\pi}{Nd_2}$, is required. By satisfying this condition and applying the multiplication, the graph in Fig. 11 is obtained.

The condition $\frac{3\pi}{Nd_1} = \frac{2\pi}{Nd_2}$ implies that:

$$d = \frac{d_1}{3} = \frac{d_2}{2}. \quad (43)$$

By replacing it in relation (42) and $\psi = kdcos(\theta) = sd$ then,

$$A_T(\psi) = \frac{sin\left(3N\frac{\psi}{2}\right)}{N^2\, sin\left(\frac{3\psi}{2}\right)} \times \frac{sin(N\psi)}{sin(\psi)}. \quad (44)$$

Using the condition of Equation (20), the theoretical elements have a total number:

$$N_T{}' = (3(N-1)+1) + (2(N-1)+1) - 1,$$

$$\Rightarrow N_T{}' = 5N - 4.$$

Since there is two elements with zero feeding exactly as shown in Fig. 6, the actual number of elements in the array is $N_T = N_T{}' - 2$, so that,

$$N_T = 5N - 6. \quad (45)$$

For $N = 5$, the array pattern is given in Fig. 12. The visible region must be chosen to avoid the appearance of the side lobes with the highest level. This attained when $\leq \frac{2\pi}{3} - \frac{2\pi}{3N}$; where $\frac{2\pi}{3}$ is the period of the function $\frac{sin\left(3N\frac{\psi}{2}\right)}{N\, sin\left(\frac{3\psi}{2}\right)}$ and $\frac{2\pi}{3N}$ is its first zero.

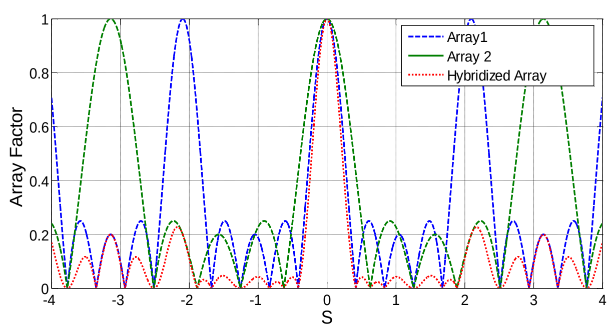


Fig. 11. Array factors and the hybridized array in terms of "s".

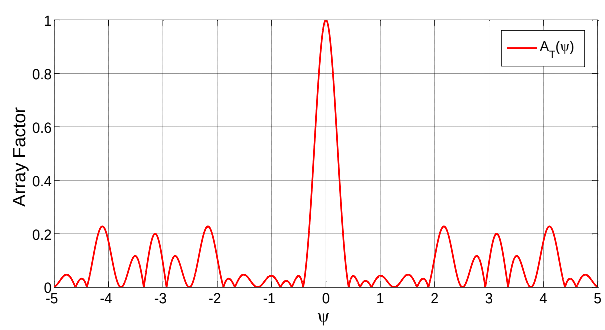


Fig. 12. Array factors in terms of "$\psi$" for $N$ = 5.

The hybridized array parameters are given below.

*Side Lobe Level (SLL)*

The side lobe maxima position is given by $\psi_{SLM} \cong \frac{\psi_{null1} + \psi_{null2}}{2}$ yielding:

$$\psi_{SLM} \cong \frac{5\pi}{6N}. \quad (46)$$

Consequently, the side lobe level is given by the following expression:

$$|A_T(\psi_{SLM})| \cong \frac{\sqrt{2}}{4N^2\left|\sin\left(\frac{5\pi}{4N}\right)\sin\left(\frac{5\pi}{6N}\right)\right|}. \quad (47)$$

*Directivity*

The radiated power is given as:

$$P_{rad} = 2\pi \int_0^{\pi} \left|\frac{sin\left(3N\frac{\psi}{2}\right)}{N^2\, sin\left(\frac{3\psi}{2}\right)} \frac{sin(N\psi)}{sin(\psi)}\right|^2 \sin(\theta)\, d\theta. \quad (48)$$

Then the directivity is given under the following expression:

$$D = \frac{27\, Nkd}{7\,\pi}. \quad (49)$$

Since $N = \frac{N_T+6}{5}$, the directivity is expressed in terms of the total number of the elements in the array as:

$$D = \frac{27\,(N_T+6)kd}{35\,\pi}. \quad (50)$$

*Feeding Currents*

Since the proposed array factor is multiplication of $A_1(\psi) = \frac{sin\left(3N\frac{\psi}{2}\right)}{N\, sin\left(\frac{3\psi}{2}\right)}$, and $A_2(\psi) = \frac{sin(N\psi)}{N\, sin(\psi)}$, and using $d = \frac{d_1}{3} = \frac{d_2}{2}$, the feeding currents may be expressed, for $N$ is odd, as:

$$x_T^d[n] = x_1^{\frac{d_1}{3}}[n] * x_2^{\frac{d_2}{2}}[n], \quad (51)$$

where

$$x_1^{\frac{d_1}{3}}[n] = \mathcal{F}^{-1}\{A_1(\psi)\}$$

$$= \begin{cases} \begin{cases} \frac{1}{N} & ;\ n = 3p \\ 0 & ;n \neq 3p \end{cases} ; -\frac{3}{2}(N-1) \leq n \leq \frac{3}{2}(N-1) \\ 0 \quad ; \quad otherwise \end{cases}, \quad (52)$$

$$\mathcal{F}^{-1}\{A_2(\psi)\} = x_2^{\frac{d_2}{2}}[n]$$

$$= \begin{cases} \begin{cases} \frac{1}{N} & ;\ n = 2p \\ 0 & ;n \neq 2p \end{cases} ; -(N-1) \leq n \leq (N-1) \\ 0 \quad ; \quad otherwise \end{cases}, \quad (53)$$

$x_1^{\frac{d_1}{3}}[n]$ and $x_2^{\frac{d_2}{2}}[n]$ can be represented in their matrix form as shown below.

$$x_1^{\frac{d_1}{3}}[n] =$$

$$\frac{1}{N}\left[\overbrace{1\ 0\ 0\ 1\ 0\ 0\ 1 \ldots \underbrace{1}_{-3,}\ \underbrace{0}_{-2,}\ \underbrace{0}_{-1,}\ \underbrace{1}_{n=0,}\ \underbrace{0}_{1,}\ \underbrace{0}_{2,}\ \underbrace{1}_{3} \ldots 1\ 0\ 0\ 1\ 0\ 0\ 1}^{3(N-1)+1}\right],$$

$$x_2^{\frac{d_2}{2}}[n] = \frac{1}{N}\left[\overbrace{1\ 0\ 1\ 0\ 1 \ldots \underbrace{1}_{-2,}\ \underbrace{0}_{-1,}\ \underbrace{1}_{n=0,}\ \underbrace{0}_{1,}\ \underbrace{1}_{2} \ldots 1\ 0\ 1\ 0\ 1}^{2(N-1)+1}\right].$$

For the case of $N$ even, the distance conversion must be applied leading to the following results (with the number of ones in each matrix remain equal to $N$):

$$x_1^{\frac{d_1}{6}}[n] = \frac{1}{N}\left[\overbrace{1\,0\,0\,0\,0\,0\,1 \ldots \underbrace{1}_{-3,}\ \underbrace{0}_{-2,}\ \underbrace{0}_{-1,}\ \underbrace{0}_{n=0,}\ \underbrace{0}_{1,}\ \underbrace{0}_{2,}\ \underbrace{1}_{3} \ldots 1\,0\,0\,0\,0\,0\,1}^{6(N-1)+1}\right],$$

$$x_2^{\frac{d_2}{4}}[n] = \frac{1}{N}\left[\overbrace{1\,0\,0\,0\,1 \ldots \underbrace{1}_{-2,}\ \underbrace{0}_{-1,}\ \underbrace{0}_{n=0,}\ \underbrace{0}_{1,}\ \underbrace{1}_{2} \ldots 1\,0\,0\,0\,1}^{4(N-1)+1}\right].$$

Consequently, the feeding currents for that case are given as:

$$x_T^{\frac{d}{2}}[n] = x_1^{\frac{d_1}{6}}[n] * x_2^{\frac{d_2}{4}}[n]. \tag{54}$$

The resulted array may be named as **UUDdH**, which is the abbreviation of "**Uniform with Uniform Different-d Hybridization**". Evaluation of these equations reveals that the excitation coefficients of the proposed array are distributed simply in a countable number of discrete levels. This number is found to be $M$ whose values is given by:

$$M = \left\lceil \frac{N}{3} \right\rceil, \tag{55}$$

where $\left\lceil \frac{N}{3} \right\rceil$ denotes the ceiling function and it is defined as:

$$\lceil x \rceil = n \quad if \quad n-1 < x \leq n. \tag{56}$$

The levels of the coefficients are given by:

$$\frac{L}{N^2} \quad ; \quad L \in \{1, 2, 3, \ldots M\}, \tag{57}$$

where $L$ denotes the level state.

Now, the values of excitation coefficients of the proposed array are determined analytically as:

$$w = \frac{5N+7}{2} - 6\left\lceil \frac{N}{3} \right\rceil.$$

When $N$ is odd:

$$x_T^d[n] = \begin{cases} f[n] & ; \quad |n| \leq w \\ \frac{\left\lceil \frac{N}{3} \right\rceil - 1}{N^2} - \frac{1}{N^2}\delta[|n| - (w+5)] & ; \ |n| \epsilon\, I_1 \\ \frac{\left\lceil \frac{N}{3} \right\rceil - 2}{N^2} - \frac{1}{N^2}\delta[|n| - (w+11)] & ; \ |n| \epsilon\, I_2 \\ \vdots & \\ \frac{\left\lceil \frac{N}{3} \right\rceil - m}{N^2} - \frac{1}{N^2}\delta[|n| - (w+6m-1)] & ; \ |n| \epsilon\, I_m \\ \vdots & \\ \frac{1}{N^2} - \frac{1}{N^2}\delta[|n| - (w+6(M-1)-1)] & ; \ |n| \epsilon I_{M-1} \end{cases}, \tag{58}$$

where $I_m = [w + 6m - 5\,; w + 6m]$, and

$$f[n] = \begin{cases} \frac{\left\lceil \frac{N}{3} \right\rceil}{N^2} - \frac{1}{N^2}\delta[|n| - (w-1)] & ; \quad N = 3P \\ \frac{1}{N^2}\left(1 - \frac{2}{\sqrt{3}}\left|sin\left(\frac{2\pi}{3}n\right)\right|\right) + \frac{\left\lceil \frac{N}{3} \right\rceil - 1}{N^2} & ; \ N = 3P+1 \\ \frac{1}{N^2}\left|\frac{2}{\sqrt{3}} sin\left(\frac{2\pi}{3}n\right)\right| + \frac{\left\lceil \frac{N}{3} \right\rceil - 1}{N^2} & ; \quad N = 3P+2 \end{cases}.$$

When $N$ is even:

$$x_T^{\frac{d}{2}}[n] = \frac{1-(-1)^n}{2}\begin{cases} g[n] & ; \quad |n| \leq 2w \\ \frac{\left\lceil \frac{N}{3} \right\rceil - 1}{N^2} - \frac{1}{N^2}\delta[|n| - (2w+10)] & ; |n| \epsilon\, I_1 \\ \frac{\left\lceil \frac{N}{3} \right\rceil - 2}{N^2} - \frac{1}{N^2}\delta[|n| - (2w+22)] & ; \ |n| \epsilon I_2 \\ \vdots & \\ \frac{\left\lceil \frac{N}{3} \right\rceil - m}{N^2} - \frac{1}{N^2}\delta[|n| - (2w+12m-2)] & ; \ |n| \epsilon I_m \\ \vdots & \\ \frac{1}{N^2} - \frac{1}{N^2}\delta[|n| - (2w+12(M-1)-2)] & ; |n| \epsilon I_{M-1} \end{cases}, \tag{59}$$

where $I_m = [2w + 12m - 11\,; 2w + 12m]$, and

$$g[n] = \begin{cases} \frac{\left\lceil \frac{N}{3} \right\rceil}{N^2} - \frac{1}{N^2}\delta[|n| - (2w-2)] & ; \quad N = 3P \\ \frac{1}{N^2}\left(1 - \frac{2}{\sqrt{3}}\left|sin\left(\frac{2\pi}{3} \times \frac{|n|+3}{2}\right)\right|\right) + \frac{\left\lceil \frac{N}{3} \right\rceil - 1}{N^2} & ; \ N = 3P+1 \\ \frac{1}{N^2}\left|\frac{2}{\sqrt{3}} sin\left(\frac{2\pi}{3} \times \frac{|n|+3}{2}\right)\right| + \frac{\left\lceil \frac{N}{3} \right\rceil - 1}{N^2} & ; \ N = 3P+2 \end{cases}, \tag{60}$$

$P$ is positive integer and $\delta[n]$ is the discrete Dirac delta function. For illustration, the feeding currents of Equations (58) and (59) are drawn for $N = 4$ and 5 in Fig. 13 with their corresponding array factors in Fig. 14. The visible region is taken for $\left(kd = \frac{2\pi(N-1)}{3N} \Leftrightarrow d = \frac{\lambda}{3}\frac{N-1}{N}\right)$, which is the maximum value of $d$ so that there is no high side lobes in the visible region.

## IV. RESULTS AND DISCUSSION

To see the evidence of the proposed arrays, a comparison with Tschebyscheff array is carried out in this section. In this comparison, the chosen total number of elements in each array is $N_T = 19$. The obtained feeding coefficients distributions for the both proposed arrays are shown in Fig. 13 (for UUDNH case) and Fig. 9 (for UUDdH case). These coefficients in addition to their calculation, which is easy, they are not highly varying; this makes the realization in the electronic circuit generating these amplitudes simple and inexpensive and economical.

The obtained array factors are also drawn in Fig. 15 together with the Tschebyscheff array factor with $SLL = -28\,dB$. The figure shows clearly that the proposed array obtained from hybridization between two uniform arrays with different spacing has performances exceeding that of Tschebyscheff array with simplicity in realization.

For deeper comparison using numerical values, in addition to the two proposed arrays, two Tschebyscheff arrays are considered. The first Tschebyscheff array with side lobe level of $-27\,dB$, whereas the second one with $-28\,dB$. The obtained numerical results are summarized in the Table 1 below. Form this table, it is clear that the proposed UUDdH has highest directivity with side lobe level comparable to Tschebyscheff. Furthermore, the very interesting result is that the excitation coefficients of UUDdH are constructed from only few levels as shown in Fig. 13; and in case of 19 elements, there are only two levels (0.0400 and 0.0800) as shown in the table and the Fig. 13. This result gives us the opportunity to implement the feeding circuit of the array using simply a digital circuit.

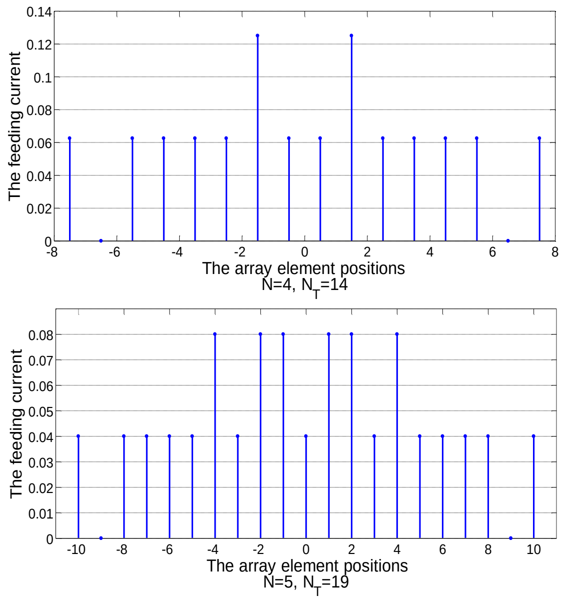


Fig. 13. Excitation coefficients of the hybridized array for $N = 4$ and 5.

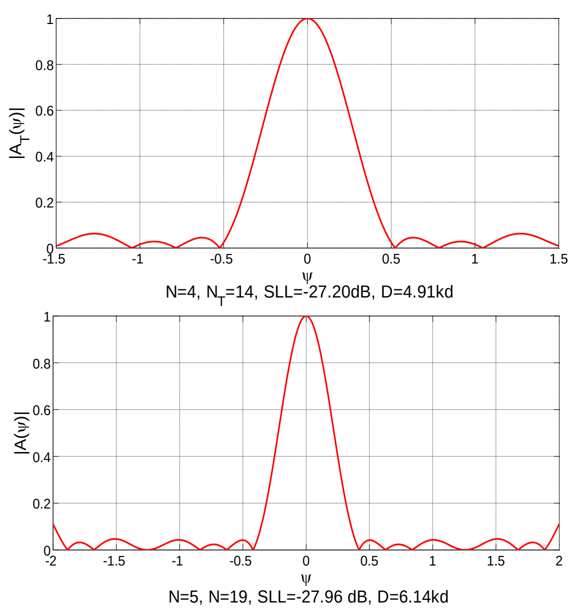


Fig. 14. Array pattern of the hybridized array (UUDdH) in terms of $\psi$ for $N = 4$ and 5 where the directivities is given in terms of the wave number $k$ and the distance between the elements $d$.

Table 1: Excitation coefficients, side lobe levels and directivity of the proposed arrays and two Tschebyscheff arrays with 19 elements in each (note that $x^d[9] = 0$ is not counted as an elements because it has zero feeding)

| | Tschebyscheff Array 1 | Tschebyscheff Array 2 |
|---|---|---|
| | $N_T = 19$ | $N_T = 19$ |
| | $x^d[0] = 0.0777$ | $x^d[0] = 0.0763$ |
| | $x^d[1] = 0.0765$ | $x^d[1] = 0.0752$ |
| | $x^d[2] = 0.0730$ | $x^d[2] = 0.0719$ |
| | $x^d[3] = 0.0675$ | $x^d[3] = 0.0666$ |
| | $x^d[4] = 0.0602$ | $x^d[4] = 0.0598$ |
| | $x^d[5] = 0.0518$ | $x^d[5] = 0.0518$ |
| | $x^d[6] = 0.0428$ | $x^d[6] = 0.0431$ |
| | $x^d[7] = 0.0337$ | $x^d[7] = 0.0343$ |
| | $x^d[8] = 0.0251$ | $x^d[8] = 0.0259$ |
| | $x^d[9] = 0.0304$ | $x^d[9] = 0.0333$ |
| | SLL = −28 dB | $SLL = -27\ dB$ |
| $\boldsymbol{d = \frac{\lambda}{4}}$ | $D = 8.50$ | $D = 8.62$ |
| $\boldsymbol{d = \frac{\lambda}{3}}$ | $D = 11.31$ | $D = 11.47$ |
| $\boldsymbol{d = \frac{\lambda}{2}}$ | $D = 16.89$ | $D = 17.10$ |
| | Our Array 1<br>UUDNH | Our Array 2<br>UUDdH |
| | $N_T = 19$ | $N_T = 19$ |
| | $x^d[0] = 0.0833$ | $x^d[0] = 0.0400$ |
| | $x^d[1] = 0.0833$ | $x^d[1] = 0.0800$ |
| | $x^d[2] = 0.0833$ | $x^d[2] = 0.0800$ |
| | $x^d[3] = 0.0729$ | $x^d[3] = 0.0400$ |
| | $x^d[4] = 0.0625$ | $x^d[4] = 0.0800$ |
| | $x^d[5] = 0.0521$ | $x^d[5] = 0.0400$ |
| | $x^d[6] = 0.0417$ | $x^d[6] = 0.0400$ |
| | $x^d[7] = 0.0313$ | $x^d[7] = 0.0400$ |
| | $x^d[8] = 0.0208$ | $x^d[8] = 0.0400$ |
| | $x^d[9] = 0.0104$ | $x^d[9] = 0.0000$<br>$x^d[10] = 0.0400$ |
| | $\boldsymbol{SLL = -28.96\ dB}$ | $\boldsymbol{SLL = -27.96\ dB}$ |
| $\boldsymbol{d = \frac{\lambda}{4}}$ | $\boldsymbol{D = 7.68}$ | $\boldsymbol{D = 9.45}$ |
| $\boldsymbol{d = \frac{\lambda}{3}}$ | $\boldsymbol{D = 10.24}$ | $\boldsymbol{D = 12.37}$ |
| $\boldsymbol{d = \frac{\lambda}{2}}$ | $\boldsymbol{D = 15.36}$ | $\boldsymbol{D = 16.90}$ |

For the case of the proposed UUDNH array, the table indicates the very low level of the side lobes compared to other arrays with relatively acceptable directivity. The importance of this array resides in the very simple expressions of the excitation coefficients that simple at the same time the calculations of these coefficients and their practical implementation.

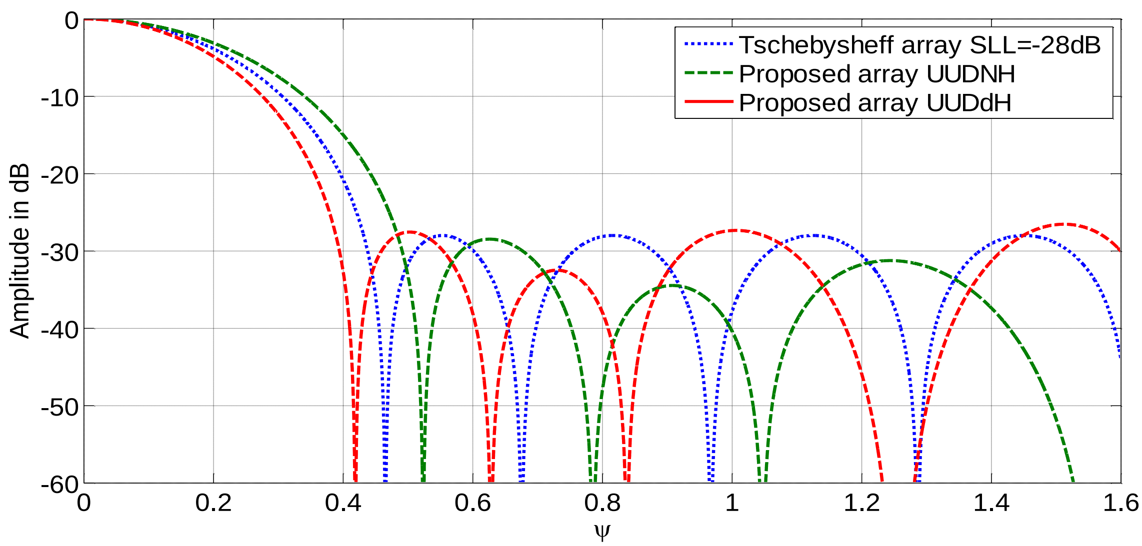


Fig. 15. Comparison between Tschebyscheff array ($SLL = -28\ dB$) and the proposed arrays for 19 elements each.

## V. CONCLUSION

In this work a concept of array hybridization (mixing two distinct arrays) is presented and applied to uniform arrays to generate a new array that presents good compromise between its directivity and the level of the secondary lobes. The choice uniform array is dictated by its high directivity; and also restricting the generation problem to only reduction of the side lobes levels. As compared to the existing techniques for solving such problem which are based on optimization approaches, the proposed ones are solely based on mathematical procedures using known analytical expressions which are presented in details.

The hybridization is applied twice on two uniform arrays. At the beginning the two hybridized arrays are chosen to have different number of elements (UUDNH). The resulted array has good compromise between the directivity and its side lobe level as required. In addition to this the excitation coefficients are not highly varying. The second generated array is based on the use of two uniform arrays with different spacing between their elements (UUDdH). The obtained array satisfies also the requirements. Furthermore, the excitation coefficients are distribution in very few levels.

The obtained arrays are compared with Tschebyscheff arrays having the same number of elements. The UUDNH generated array has comparable performances with respect to Tschebyscheff array whereas, the second, UUDdH arrays has better performances. In addition to this, the excitation coefficients of the generated arrays are not highly varying as compared to Tschebyscheff ones and this simplifies their realization.

It should also be noted that all the arrays parameters (array factor, side lobe levels, directivity and excitation coefficients) are expressed in very elegant closed form expressions leading to easy use of the proposed arrays.

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