An Accurate High-Speed Method for Mutual Inductance Calculations of Coplanar Disk Coils Using Generalized Hypergeometric Functions

Yao Luo

School of Electrical Engineering
Wuhan University, Wuhan, 430072, China
ostpreussen@qq.com

Abstract **─** Traditional method using integral of the Bessel and Struve functions is not suitable for calculating the mutual inductance between two coplanar disk coils. Considering the monotonicity of modified Bessel and Struve functions, an alternative method using these monotonic functions is applied to calculate the mutual inductance, and numerical evaluations can be accelerated considerably. Series solutions using the generalized hypergeometric functions are further obtained by solving the infinite integrations, and these series are compared with the aforementioned integral methods. The numerical results show that the series solutions are much more faster than the integral ones, and with the series method, results of very high accuracy can be obtained within a small fraction of one second in most cases. Furthermore, we point out and prove the existence of the decoupling positions at which the mutual inductance will vanish.

Index Terms **─** Disk coils, generalized hypergeometric function, modified Bessel functions, mutual inductance.

I. INTRODUCTION

Disk coils are broadly applied in the electrical instruments, especially in recently popular areas such as the wireless power transmission, in which the disk coils are essential components and the mutual inductance of the coils is very important for optimization of the efficiency of the power transmission [1-3]. However, mutual inductance calculations of the disk coils are relatively difficult compared with that of the long circular coils (*e.g.*, the thin-wall solenoids). For two coaxial thin-wall solenoids, the mutual inductance can be solved in closed-form by the complete elliptic integrals [4, 5], but for two coaxial disk coils, it seems unlikely to obtain accurate closed-form solution and so far the mutual inductance must be solved by the integral of inverse trigonometric functions [6] or that of Bessel and Struve functions [7, 8]. Only for the concentric coplanar disk coils closed-form expressions are obtained [9]. In the general non-coaxial case, the solution is given in [6] by integral of Bessel and Struve functions and it can be described as follows.

Two disk coils are located in the parallel planes with distance z_0, and their axes are separated by a distance r_0. One coil has N_1 turns and the inner and outer radii R_1, R_2, the other has corresponding parameters of N_2, R_3, R_4 (See Fig. 1), then the mutual inductance of them is given by:

$$M = \frac{\mu_0 \pi^3 N_1 N_2}{4\left(R_2 - R_1\right)\left(R_4 - R_3\right)} \int_0^\infty \left[w\left(R_2\right) - w\left(R_1\right) \right] \cdot \left[w\left(R_4\right) - w\left(R_3\right) \right] J_0\left(kr_0\right) e^{-kz_0} \frac{dk}{k^2}, \tag{1}$$

where

$$w\left(r\right) = r\left[J_1\left(kr\right)\mathbf{H}_0\left(kr\right) - J_0\left(kr\right)\mathbf{H}_1\left(kr\right) \right], \tag{2}$$

with special functions $J_n(x)$ and $\mathbf{H}_n(x)$ listed in Table 1.

Table 1: Special functions applied

Symbol	Special Function
$J_n(x)$	Bessel function of the first kind of order n
$I_n(x)$, $K_n(x)$	Modified Bessel functions of the first and second kind of order n
$\mathbf{H}_n(x)$	Struve function of order n
$\mathbf{L}_n(x)$	Modified Struve function of order n
${}_pF_q(\boldsymbol{a}; \boldsymbol{b}; x)$	Generalized hypergeometric function
$(x)_n$	Pochhammer symbol
$\Gamma(x)$	Gamma function

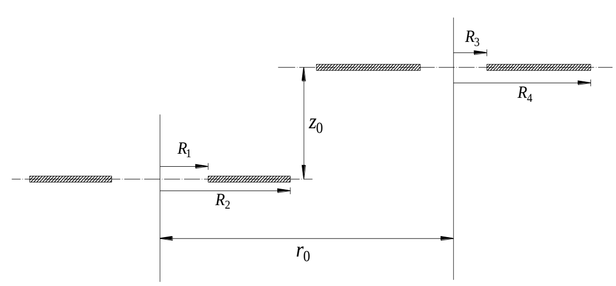

Fig. 1. Side view of two disk coils with parallel axes.

When z_0 is not too small, (1) is proper for the evaluation of the mutual inductance of disk coils since

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the exponential factor will decrease rapidly in magnitude as $k\to\infty$. However, for small values of z_0 the numerical performance of (1) will become worse, especially for the coplanar case, the efficiency of (1) is in fact doubtful for the numerical evaluations. On the other hand, if we admit the transcendental nature of (1), we can then seek the series solutions of (1) rather than the closed-form ones and we will see that this consideration is achievable for (1) with z_0=0. In this work, for the non-coaxial coplanar disk coils the mutual inductance will be given in the form of series of generalized hypergeometric functions which will lead to a high speed and accuracy method for the numerical evaluations of the mutual inductance of these coils.

In addition, alternative representations of the mutual inductance using the modified Bessel and Struve functions will be introduced, which can be derived from the alternative forms of the eigenfunction expansion of the reciprocal distance [10-13] from which the proposed series expressions will be derived. The monotonic nature of the modified Bessel and Struve functions may be beneficial to numerical integration of the expressions of mutual inductance. Without these alternative methods, it will be nearly impossible to compare the proposed method of series type with that of integral type, since when z_0=0, the evaluation of (1) is extremely time-consuming and the results of high accuracy are very difficult to obtain.

II. FORMULATIONS OF MUTUAL INDUCTANCE

A. Mutual inductance of coplanar disk coils with $r_0 \geq R_2+R_4$

According to the Neumann formula, the mutual inductance of two coplanar disk coils with dimension parameters given in Section I is (See Fig. 2):

$$M=\frac{\mu_0 N_1 N_2}{4\pi(R_2-R_1)(R_4-R_3)}\int_{R_1}^{R_2} r_1 dr_1\cdot\int_{R_3}^{R_4} r_2 dr_2\int_0^{2\pi} d\varphi_2 \cdot\int_0^{2\pi}\frac{\cos(\varphi_1-\varphi_2)}{\sqrt{r_1^2+\rho^2-2r_1\rho\cos(\varphi_1-\varphi)}}d\varphi_1, \tag{3}$$

where

$$\rho^2=r_2^2+r_0^2-2r_2r_0\cos(\varphi_2+\pi/2),$$

and

$$\rho\cos\varphi=r_2\cos\varphi_2.$$

Using the expansion of reciprocal distance in the cylindrical coordinate [10],

$$\frac{1}{\sqrt{r_1^2+\rho^2-2r_1\rho\cos(\varphi_1-\varphi)}}=\sum_{n=0}^{\infty}\varepsilon_n\cdot\cos n(\varphi_1-\varphi)\cdot f_n(r_1,\rho), \tag{4}$$

where

$$f_n(r_1,\rho)=\begin{cases}\dfrac{2}{\pi}\displaystyle\int_0^\infty I_n(kr_1)K_n(k\rho)dk, & \text{for } \rho\geq r_1,\\ \dfrac{2}{\pi}\displaystyle\int_0^\infty K_n(kr_1)I_n(k\rho)dk, & \text{for } r_1\geq\rho,\end{cases} \tag{5}$$

and

$$\varepsilon_n=\begin{cases}1, & n=0\\ 2, & n\neq 0\end{cases}, \tag{6}$$

is the Neumann's factor [14]; the following results of the mutual inductance can be obtained:

$$M=-\frac{\mu_0\pi^2 N_1N_2}{2(R_2-R_1)(R_4-R_3)}\int_0^\infty\left[u(R_2)-u(R_1)\right]\cdot\left[u(R_4)-u(R_3)\right]K_0(kr_0)\frac{dk}{k^2}, \tag{7}$$

where $r_0 \geq R_2+R_4$, and

$$M=\frac{\mu_0\pi^2 N_1N_2}{2(R_2-R_1)(R_4-R_3)}\int_0^\infty\left[u(R_2)-u(R_1)\right]\cdot\left[v(R_4)-v(R_3)\right]I_0(kr_0)\frac{dk}{k^2}, \tag{8}$$

where $0 \leq r_0 \leq R_3 - R_2$, with

$$u(r)=r\left[I_1(kr)\mathbf{L}_0(kr)-I_0(kr)\mathbf{L}_1(kr)\right], \tag{9}$$

and

$$v(r)=r\left[K_1(kr)\mathbf{L}_0(kr)+K_0(kr)\mathbf{L}_1(kr)\right]. \tag{10}$$

$I_n(x)$, $K_n(x)$ and $\mathbf{L}_n(x)$ are modified Bessel and Struve functions listed in Table 1. For the coplanar disk coils no overlap will occur.

In addition, using the same technique of (4)-(6) we can obtain the mutual inductance of two disk coils without radial overlap:

$$M=-\frac{\mu_0\pi^2 N_1N_2}{2(R_2-R_1)(R_4-R_3)}\int_0^\infty\left[u(R_2)-u(R_1)\right]\cdot\left[u(R_4)-u(R_3)\right]K_0(kr_0)\cos(kz_0)\frac{dk}{k^2}, \tag{11}$$

where $r_0 > R_2+R_4$, and

$$M=\frac{\mu_0\pi^2 N_1N_2}{2(R_2-R_1)(R_4-R_3)}\int_0^\infty\left[u(R_2)-u(R_1)\right]\cdot\left[v(R_4)-v(R_3)\right]I_0(kr_0)\cos(kz_0)\frac{dk}{k^2}, \tag{12}$$

where $0 \leq r_0 < R_3 - R_2$. Expressions (11) and (12) are suitable for the disk coils with small z_0 (the nearly coplanar coils), as the factor $\cos(kz_0)$ is slowly oscillatory in this case.

For $r_0 \geq R_2+R_4$ we solve (7) to a series form. Applying the expression:

$$u(\alpha)-u(\beta)=\frac{2k}{\pi}\int_\alpha^\beta rI_1(kr)dr, \tag{13}$$

we have

$$S_1=\int_0^\infty\left[u(R_2)-u(R_1)\right]\left[u(R_4)-u(R_3)\right]K_0(kr_0)\frac{dk}{k^2}$$
$$=\frac{4}{\pi^2}\int_{R_1}^{R_2}r_1dr_1\int_{R_3}^{R_4}r_2dr_2\int_0^\infty I_1(kr_1)I_1(kr_2)K_0(kr_0)\,dk. \tag{14}$$

Solving the infinite integral in (14) [15] we obtain:

$$S_1=\frac{1}{2\pi r_0^3}\int_{R_1}^{R_2}r_1^2dr_1\cdot\int_{R_3}^{R_4}r_2^2F_4\left(\frac{3}{2},\frac{3}{2};2,2;\frac{r_1^2}{r_0^2},\frac{r_2^2}{r_0^2}\right)dr_2. \tag{15}$$

By writing the Appell function F_4 [16] as its power series and perform the remaining radial integrations term-by-term, we obtain:

$$S_1=\frac{1}{2\pi r_0^3}\sum_{m=0}^\infty\sum_{n=0}^\infty\frac{\left(R_1^{3+2m}-R_2^{3+2m}\right)}{r_0^{2(m+n)}(3+2m)(3+2n)}\cdot\frac{\left(R_3^{3+2n}-R_4^{3+2n}\right)(3/2)_{m+n}^2}{m!n!(2)_m(2)_n}. \tag{16}$$

Then by solving the summation with respect to m we get:

$$S_1=\frac{1}{6\pi r_0^3}\sum_{n=0}^\infty\frac{\left(R_3^{3+2n}-R_4^{3+2n}\right)(3/2)_n^2}{r_0^{2n}(3+2n)n!(2)_n}\cdot\left[R_1^3f_1\left(n,\frac{R_1}{r_0}\right)-R_2^3f_1\left(n,\frac{R_2}{r_0}\right)\right], \tag{17}$$

with

$$f_1(n,x)={}_3F_2\left(\frac{3}{2},n+\frac{3}{2},n+\frac{3}{2};2,\frac{5}{2};x^2\right). \tag{18}$$

Finally we have the mutual inductance for $r_0\geq R_2+R_4$:

$$M=-\frac{\mu_0\pi^2N_1N_2}{2(R_2-R_1)(R_4-R_3)}S_1$$
$$=-\frac{\mu_0\pi N_1N_2}{12r_0^3(R_2-R_1)(R_4-R_3)}\sum_{n=0}^\infty\frac{\left(R_3^{3+2n}-R_4^{3+2n}\right)(3/2)_n^2}{r_0^{2n}(3+2n)n!(2)_n} \tag{19}$$
$$\cdot\left[R_1^3f_1\left(n,\frac{R_1}{r_0}\right)-R_2^3f_1\left(n,\frac{R_2}{r_0}\right)\right],$$

with $f_1(n, x)$ given by (18).

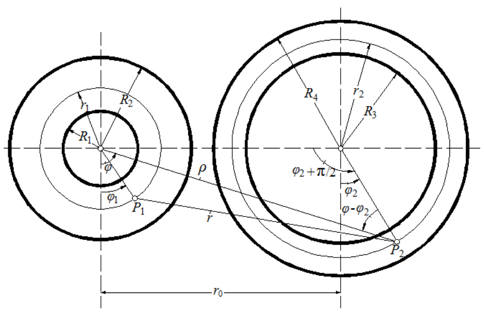

Fig. 2. Plan view of two coplanar disk coils.

B. Mutual inductance of coplanar disk coils with $0\leq r_0\leq R_3-R_2$

For the case of $0\leq r_0\leq R_3-R_2$, using (13) and the expression,

$$v(\alpha)-v(\beta)=\frac{2k}{\pi}\int_\alpha^\beta rK_1(kr)\,dr, \tag{20}$$

we have

$$S_2=\int_0^\infty\left[u(R_2)-u(R_1)\right]\left[v(R_4)-v(R_3)\right]I_0(kr_0)\frac{dk}{k^2}$$
$$=\frac{4}{\pi^2}\int_{R_1}^{R_2}r_1dr_1\int_{R_3}^{R_4}r_2dr_2\int_0^\infty I_1(kr_1)K_1(kr_2)I_0(kr_0)\,dk \tag{21}$$
$$=\frac{1}{\pi}\int_{R_1}^{R_2}r_1^2dr_1\cdot\int_{R_3}^{R_4}\frac{1}{r_2}F_4\left(\frac{3}{2},\frac{3}{2};1,2;\frac{r_0^2}{r_2^2},\frac{r_1^2}{r_2^2}\right)dr_2.$$

Then solve the remaining integrals in the similar manner we have:

$$S_2=\frac{1}{2\pi}\sum_{m=0}^\infty\sum_{n=0}^\infty C_{mn}$$
$$=\frac{1}{2\pi}\sum_{m=0}^\infty\sum_{n=0}^\infty\frac{r_0^{2m}(1/2)_{m+n}(3/2)_{m+n}}{(R_3R_4)^{2(m+n)}(3+2n)(m+n)}\cdot\frac{\left(R_1^{3+2n}-R_2^{3+2n}\right)\left(R_3^{2(m+n)}-R_4^{2(m+n)}\right)}{m!n!(1)_m(2)_n}. \tag{22}$$

It should be noticed that the general term C_{mn} of (22) has the factor $m+n$ in the denominator, hence m and n cannot vanish simultaneously, for the summation with respect to m, the lower index of n must be set to be 1, *i.e.*,

$$\sum_{m=0}^\infty\sum_{n=1}^\infty C_{mn}=\sum_{n=1}^\infty\frac{(1/2)_n(3/2)_n\left(R_2^{3+2n}-R_1^{3+2n}\right)}{(R_3R_4)^{2n}n(3+2n)n!(2)_n}\cdot\left[R_4^{2n}f_2\left(n,\frac{r_0}{R_3}\right)-R_3^{2n}f_2\left(n,\frac{r_0}{R_4}\right)\right], \tag{23}$$

with

$$f_2(n,x)={}_3F_2\left(n,n+\frac{1}{2},n+\frac{3}{2};1,n+1;x^2\right). \tag{24}$$

Then we consider the remaining terms C_{m0}. The term C_{00} is unusual and a limit process must be take:

$$C_{00}=\lim_{\substack{m\to0\\n\to0}}C_{mn}=\frac{2}{3}\left(R_1^3-R_2^3\right)\ln\frac{R_3}{R_4}. \tag{25}$$

The summation of the remaining terms C_{m0} with $m\geq1$ is:

$$\sum_{m=1}^\infty C_{m0}=\sum_{m=1}^\infty\frac{\left(R_1^3-R_2^3\right)\left(R_3^{2m}-R_4^{2m}\right)r_0^{2m}(1/2)_m(3/2)_m}{3(R_3R_4)^{2m}m(m!)^2}$$
$$=\frac{r_0^2\left(R_1^3-R_2^3\right)\left[R_3^2f_3\left(\frac{r_0}{R_4}\right)-R_4^2f_3\left(\frac{r_0}{R_3}\right)\right]}{4(R_3R_4)^2}, \tag{26}$$

with

$$f_3(x) = {}_4F_3\left(1,1,\frac{3}{2},\frac{5}{2};2,2,2;x^2\right). \tag{27}$$

Then combining (8) and (22)-(27) gives the mutual inductance for $0 \le r_0 \le R_3 - R_2$:

$$\begin{aligned} M &= \frac{\mu_0 \pi^2 N_1 N_2}{2(R_2-R_1)(R_4-R_3)} S_2 \\ &= \frac{\mu_0 \pi N_1 N_2}{4(R_2-R_1)(R_4-R_3)} \left(\sum_{m=0}^{\infty}\sum_{n=1}^{\infty} C_{mn} + \sum_{m=0}^{\infty} C_{m0} + C_{00} \right) \\ &= \frac{\mu_0 \pi N_1 N_2}{2(R_2-R_1)(R_4-R_3)} \left\{ \frac{1}{2}\sum_{n=1}^{\infty} \left\{ \left(R_2^{3+2n} - R_1^{3+2n}\right) \right.\right. \\ &\cdot \frac{(1/2)_n (3/2)_n}{(R_3R_4)^{2n} n(3+2n) n! (2)_n} \left[R_4^{2n} f_2\left(n, \frac{r_0}{R_3}\right) - R_3^{2n} f_2\left(n, \frac{r_0}{R_4}\right) \right] \Bigg\} \\ &\left. + \frac{r_0^2\left(R_1^3 - R_2^3\right)\left[R_3^2 f_3\left(\frac{r_0}{R_4}\right) - R_4^2 f_3\left(\frac{r_0}{R_3}\right) \right]}{8(R_3R_4)^2} + \frac{1}{3}\left(R_1^3 - R_2^3\right)\ln\frac{R_3}{R_4} \right\}. \end{aligned} \tag{28}$$

Expression (28) may be a little complicated, and in fact a concise form can be found if we write (21) in another way. Using the formula [17]:

$$\begin{aligned} &\Gamma(\mu+1)\Gamma(\nu+1) I_\mu(ax) I_\nu(bx) = (ax/2)^\mu (bx/2)^\nu \\ &\cdot \sum_{n=0}^{\infty} \frac{(ax/2)^{2n}}{n!(\mu+1)_n} {}_2F_1\left(-n, -\mu-n; \nu+1; b^2/a^2\right), \end{aligned} \tag{29}$$

and

$$\int_0^\infty k^{2n+1} K_1(kr_2)\, dk = \frac{\pi}{2r_2^2}\left(\frac{2}{r_2}\right)^{2n} \left(\frac{1}{2}\right)_n \left(\frac{3}{2}\right)_n, \tag{30}$$

we have

$$\begin{aligned} S_2 &= \frac{1}{6\pi}\sum_{n=0}^{\infty} C_n \\ &= \frac{1}{6\pi}\sum_{n=0}^{\infty} \frac{\left(\frac{1}{2}\right)_n \left(\frac{3}{2}\right)_n r_0^{2n} (R_3R_4)^{-2n}\left(R_3^{2n} - R_4^{2n}\right)}{n(n!)^2} \\ &\cdot \left[R_1^3 f_4\left(n, \frac{R_1}{r_0}\right) - R_2^3 f_4\left(n, \frac{R_2}{r_0}\right) \right], \end{aligned} \tag{31}$$

where

$$f_4(n,x) = {}_3F_2\left(\frac{3}{2}, -n, -n; 2, \frac{5}{2}; x^2\right). \tag{32}$$

When n=0 the term C_n of (31) must be treated with a limit process as well:

$$C_0 = \lim_{n\to 0} C_n = 2\left(R_1^3 - R_2^3\right)\ln\frac{R_3}{R_4}. \tag{33}$$

Combining (8) and (31)-(33) gives an alternative form of the mutual inductance for $0 \le r_0 \le R_3 - R_2$:

$$\begin{aligned} M &= \frac{\mu_0 \pi}{6(R_2-R_1)(R_4-R_3)} \left\{ \sum_{n=1}^{\infty} \frac{\left(\frac{1}{2}\right)_n \left(\frac{3}{2}\right)_n r_0^{2n} \cdot \left(R_3^{2n} - R_4^{2n}\right)}{2n(n!)^2 (R_3R_4)^{2n}} \right. \\ &\left. \cdot \left[R_1^3 f_4\left(n, \frac{R_1}{r_0}\right) - R_2^3 f_4\left(n, \frac{R_2}{r_0}\right) \right] + \left(R_1^3 - R_2^3\right)\ln\frac{R_3}{R_4} \right\}. \end{aligned} \tag{34}$$

Expression (34) cannot be applied to the concentric case r_0=0 but it can converge faster than (28) especially when r_0 is very close to the value $R_3 - R_2$. Letting z_0=0 in (1) and performing the integrations in a similar manner of (14) or (21), (19) and (28) can also be obtained. Letting r_0=0 in (28), the term (26) will vanish and (23) can be solved to a closed-form of ${}_{q+1}F_q(\boldsymbol{a}; \boldsymbol{b}; x)$ and the result is:

$$\begin{aligned} M &= \frac{\mu_0 \pi N_1 N_2}{288(R_2-R_1)(R_4-R_3)} \\ &\cdot \left\{ 16\left\{ R_1^3\left[f_5\left(\frac{R_1}{R_3}\right) - f_5\left(\frac{R_1}{R_4}\right) \right] + R_2^3\left[-f_5\left(\frac{R_2}{R_3}\right) + f_5\left(\frac{R_2}{R_4}\right) \right] \right\} \right. \\ &+ 9\left\{ R_1^5\left[-\frac{1}{R_3^2} f_6\left(\frac{R_1}{R_3}\right) + \frac{1}{R_4^2} f_6\left(\frac{R_1}{R_4}\right) \right] \right. \\ &\left.\left. + R_2^5\left[\frac{1}{R_3^2} f_6\left(\frac{R_2}{R_3}\right) - \frac{1}{R_4^2} f_6\left(\frac{R_2}{R_4}\right) \right] \right\} + 48\left(R_1^3 - R_2^3\right)\ln\left(\frac{R_3}{R_4}\right) \right\}, \end{aligned} \tag{35}$$

with

$$f_5(x) = {}_3F_2\left(\frac{1}{2}, \frac{3}{2}, \frac{3}{2}; 2, \frac{5}{2}; x^2\right), \tag{36}$$

and

$$f_6(x) = {}_4F_3\left(1,1,\frac{3}{2},\frac{5}{2};2,2,3;x^2\right). \tag{37}$$

This result of concentric coplanar case coincides with that given in [9].

III. DECOUPLING POSITIONS OF DISK COILS WITH PARALLEL AXES

It is interesting to give some additional discussions for the contents described above. When $r_0 \ge R_2 + R_4$, from (7), (9) and (13) we have the mutual inductance of two coplanar disk coils:

$$\begin{aligned} M &= -\frac{2\mu_0 N_1 N_2}{(R_2-R_1)(R_4-R_3)} \int_0^\infty K_0(kr_0)\, dk \\ &\cdot \int_{R_1}^{R_2} r_1 I_1(kr_1)\, dr_1 \int_{R_3}^{R_4} r_2 I_1(kr_2)\, dr_2. \end{aligned} \tag{38}$$

Considering $K_0(x)$ and $I_1(x)$ are always positive throughout $0<x<\infty$, it can be concluded that (38) is always negative for any pair of disk coils. On the other side, for the general case of disk coils with $z_0\neq 0$, from (1) and,

$$w(\alpha)-w(\beta)=\frac{2k}{\pi}\int_{\alpha}^{\beta} rJ_1(kr)\,dr, \tag{39}$$

we have

$$M=\frac{\mu_0\pi N_1N_2}{(R_2-R_1)(R_4-R_3)}\int_{R_1}^{R_2} r_1dr_1\int_{R_3}^{R_4} r_2dr_2 \cdot\int_0^{\infty} J_1(kr_1)J_1(kr_2)J_0(kr_0)e^{-kz_0}dk. \tag{40}$$

For sufficient large z_0 the following asymptotic relation holds [18]:

$$\int_0^{\infty} J_1(kr_1)J_1(kr_2)J_0(kr_0)e^{-kz_0}dk \sim \sum_{s=0}^{\infty}\frac{q^{(s)}(0)}{z_0^{s+1}}, \tag{41}$$

where

$$q^{(s)}(0)=\lim_{k\to 0}q^{(s)}(k), \tag{42}$$

and

$$q(k)=J_1(kr_1)J_1(kr_2)J_0(kr_0). \tag{43}$$

Hence, for very large z_0 we have:

$$\int_0^{\infty} J_1(kr_1)J_1(kr_2)J_0(kr_0)e^{-kz_0}dk \sim \frac{r_1r_2}{2z_0^3}-\frac{3r_1r_2\left(r_0^2+r_1^2+r_2^2\right)}{4z_0^5} +\frac{15r_1r_2\left(3r_0^4+r_1^4+3r_1^2r_2^2+r_2^4+6r_0^2\left(r_1^2+r_2^2\right)\right)}{16z_0^7}-..., \tag{44}$$

by omitting the terms of infinitesimal of higher order we can write:

$$\int_0^{\infty} J_1(kr_1)J_1(kr_2)J_0(kr_0)e^{-kz_0}dk \sim \frac{r_1r_2}{2z_0^3}, \tag{45}$$

for $z_0\to+\infty$. Hence, for very large z_0, (45) is always positive. As a corollary, (40) must also be positive when $z_0\to+\infty$. Noticing the mutual inductance is the continuous function of z_0, the following assertion can be obtained immediately.

For any given pair of disk coils with parallel axes and $r_0\geq R_2+R_4$, there is at least one zero point of z_0, at which the mutual inductance M will vanish, *i.e.*, the disk coils will be decoupled magnetically in this decoupling position.

IV. NUMERICAL EVALUATIONS

A. Coplanar disk coils with $r_0\geq R_2+R_4$

In case of $r_0\geq R_2+R_4$, the numerical validation of (7) and (19) will be implemented by using the following dimension parameters of the disk coils: R_1=0.2 m, R_2=0.8 m, R_3=2 m, R_4=3.5 m. The turns of both coils are irrelevant, since the normalized value $M/(N_1N_2)$ will be calculated. The results are given in Table 2. t_a and t_b are the computation time of (7) and (19), respectively, to obtain the same values in the second column of Table 2. The calculations were coded in Mathematica and implemented on a personal computer with a 3.4-GHz processor. As r_0 increases, both t_a and t_b decrease, and this is just contrary to the nature of (1). As long as the computation time is less than 1ms (which is the default minimum time interval of the timing program), it will be omitted automatically by the program and we will record it as "t<0.001s". We can see that t_b is always less than 1 second except for the case of $r_0\leq 4.4$ m. The superior performance of the series expression (19) can be proved sufficiently. When $r_0= R_2+R_4$=4.3 m, both (7) and (19) are not efficient enough and we just give a result of 4 significant figures evaluated by (7): $M/(N_1N_2)$=−65.08nH, with the computation time of 1228.929s, and that of 7 significant figures evaluated by (19): $M/(N_1N_2)$=−65.08078nH, with the computation time of 18.658s. In addition, the evaluations of (1) with z_0=0 is extremely time-consuming so the computation time of it is not included in Table 2, but we can give a example here: for r_0=4.8m, it takes about 1253s to get a result of 5 significant figures: $M/(N_1N_2)$=−35.273nH.

Table 2: Performance of the mutual inductance for coplanar disk coils of $R_0\geq R_2+R_4$ evaluated with (7) and (19)

r_0 (m)	$M/(N_1N_2)$ (nH)	t_a (s)	t_b (s)	t_a/t_b
4.4	−56.064144480280	1941.713	1.404	1382.986
4.5	−49.190311789750	402.061	0.608	661.285
4.6	−43.669129789264	265.592	0.234	1135.009
4.7	−39.109839095754	187.217	0.156	1200.109
4.8	−35.273252555243	5.803	0.109	53.239
5.2	−24.550774873203	5.772	0.062	93.097
5.5	−19.428031335091	4.181	0.047	88.957
5.7	−16.840560836502	4.009	0.047	85.298
6.0	−13.806335361406	3.947	0.031	127.323
6.5	−10.251166170897	3.869	0.031	124.806
7.0	−7.861716936199	3.838	0.016	239.875
8.0	−4.962362649715	2.418	0.016	151.125
9.0	−3.353222796361	2.168	<0.001	>2168
10.0	−2.380250924534	2.153	<0.001	>2153

B. Coplanar disk coils with $0\leq r_0\leq R_3-R_2$

In the case of $0\leq r_0\leq R_3-R_2$, the numerical validation of (8), (28) and (34) will be implemented by using the same dimension parameters as before: R_1=0.2 m, R_2=0.8 m, R_3=2 m, R_4=3.5 m. The results of the normalized value $M/(N_1N_2)$ are also given in Table 3. t_a and t_b are the computation time of (8) and (34), respectively, to obtain the same values in the second column of Table 3 (t_b is the computation time of (28) when r_0=0). We can see that t_b is always less than 0.1s, and for most values of r_0 in Table 3, it needs only less than 50 terms of (34) to converge to the results of 15 significant figures. The superior performance of (34) can be proved sufficiently. In addition, the computation time of (28) is slightly slower than that of (34) but it is still less than 1s. When $r_0=R_3-R_2$=1.2 m, (8), (28) and (34) are all inefficient and we just give a result of 5 significant figures evaluated by (8): $M/(N_1N_2)$=0.26645mH, with the computation time of 812.172s, and a result of 7 significant figures evaluated by (34): $M/(N_1N_2)$=0.2664547mH, with the computation time of 0.452s. The convergence rate of (1) is still very slow and it will not be discussed further.

Table 3: Performance of the mutual inductance for coplanar disk coils of $0\le r_0\le R_3-R_2$ evaluated with (8), (28) and (34)

r_0 (m)	$M/(N_1N_2)$ (mH)	t_a (s)	t_b (s)	t_a/t_b
0	0.210962364718285	3.307	<0.001	>3307
0.01	0.210965011812987	3.229	<0.001	>3229
0.1	0.211227575213022	3.214	<0.001	>3214
0.2	0.212029338411955	6.412	<0.001	>6412
0.3	0.213386502485260	6.443	<0.001	>6443
0.4	0.215332068455948	6.599	<0.001	>6599
0.5	0.217915867666375	6.380	<0.001	>6380
0.6	0.221209010694124	6.365	<0.001	>6365
0.7	0.225311129537439	22.932	0.016	1433.250
0.8	0.230362442894204	40.014	0.016	2500.875
0.9	0.236565023271967	134.910	0.016	8431.875
1.0	0.244224093537914	502.261	0.031	16200.516
1.1	0.253843100876854	1505.862	0.094	16019.809

C. Decoupling positions

For the numerical validation of the existence of decoupling positions, we plot the curves of mutual inductance with respect to z_0 for given values of r_0 and vice versa, using these parameters of coils: R_1=1 m, R_2=2 m, R_3=3 m, R_4=4 m. The curves are shown in Figs. 3, 4, which illustrate the decoupling positions clearly. In Fig. 3, for r_0=7 m, r_0=8.5 and r_0=10 m, the corresponding decoupling positions are z_0=3.947 m, z_0=5.274 m, and z_0=6.408 m; in Fig. 4, for z_0=4.5 m, z_0=5.5 and z_0=6.5 m, the corresponding decoupling positions are r_0=7.639 m, r_0=8.852 m, and r_0=10.118 m, respectively.

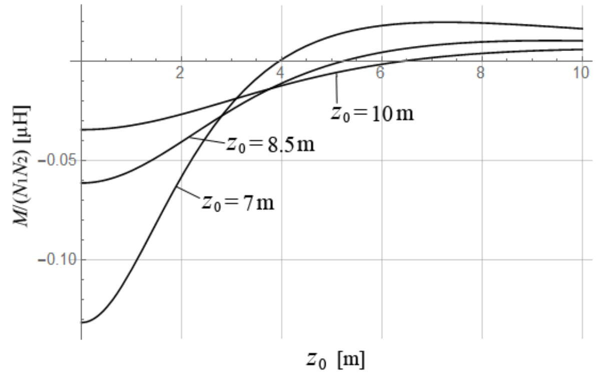

Fig. 3. The normalized mutual inductance of the disk coils with parallel axes, plotted with respect to z_0 for given r_0.

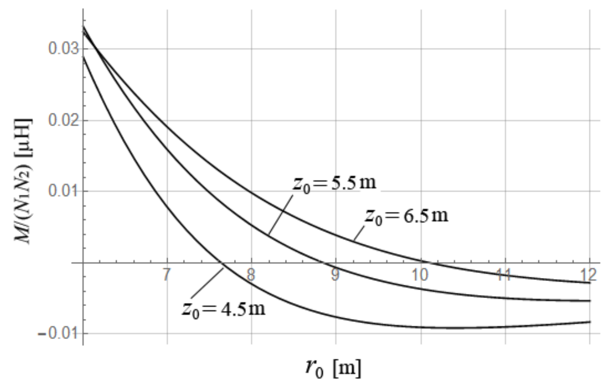

Fig. 4. The normalized mutual inductance of the disk coils with parallel axes, plotted with respect to r_0 for given z_0.

V. CONCLUSION

The integral expression using Bessel and Struve function is extremely time-consuming for the mutual inductance calculations of the coplanar disk coils. The method using modified Bessel and Struve functions is introduced to improve the numerical performance of the integral expressions, from which the series expressions using the generalized hypergeometric functions have been obtained and these expressions can be easily coded in the common mathematical packages such as Mathematica or Matlab. The numerical calculations show that the series expressions are much more faster than the expressions of integral type to get the results with the same accuracy. In most cases, it only takes less than 1 second to obtain a result of 15 significant figures by using the series expressions. In addition, the decoupling positions of the mutual inductance in the case of $r_0\ge R_2+R_4$ are noticed and we have proved formally that these positions always exist for the disk coils with parallel axes.

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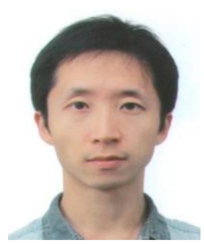

Yao Luo received the B.S. degree in Communication Engineering and the Ph.D. degree in Electrical Engineering from the Wuhan University (WHU), Wuhan, China, in 2006 and 2012, respectively.

From 2012 to 2013, he was a Lecturer in Electrical and Electronic Engineering, Hubei University of Technology. From 2013 to 2015, he was a Postdoctoral Researcher with Power and Mechanical Engineering, WHU. He is currently a Lecturer in Electrical Engineering, WHU. His current research interests include analytical calculation methods of electromagnetic fields, and the practical applications of the theory of special functions.