Analytic Expressions of Some Statistics in Radar Target Recognition Based on Late Time Representation

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Abstract **─** In this paper, we consider the projection of the late time response of an unknown radar target onto the column space and the left null space of the matrix whose entries are from the natural frequencies of the specific radar target. We get explicit expressions for the projection onto the column space, the projection onto the left null space, the square of projection onto the column space and the square of the projection onto the left null space. Also, we note that the norm of the defined projection onto the column space and the norm of the defined projection onto the left null space are Ricean distributed, and that the square of the norm of the projection onto the column space and the square of the norm of the projection onto the left null space are chi-square distributed. Accordingly, we give analytic expressions of the mean and the variance of the Ricean distribution and those of the chi-square distribution.

Index Terms **─** Chi-square distribution, late time response, projection onto the column space, projection onto the left null space, radar target recognition, Ricean distribution.

I. INTRODUCTION

There has been much research on the radar target recognition based on the natural frequencies [1-4]. In [3], the authors show the explicit expression for the mean and the variance of the square of the norm of the projection onto the left null space.

In this paper, we present explicit expressions for the defined projection onto the column space, projection onto the left null space, the square of the projection onto the column space and the square of the projection onto the left null space. In addition, we also present the analytic expressions for the mean and the variance of the norm of the defined projection onto the column space and the projection onto the left null space and those of the square of the norm of the projection onto the column space and the square of the norm of the projection onto the left null space.

The difference between this paper and [3] are the following. In [3], we only considered the square of the norm of the projection onto the left null space. In this paper, we considered the square of the norm of the projection onto the column space, the norm of the projection onto the column space and the norm of the projection onto the left null space as well as the square of the norm of the projection onto the left null space. In addition, in deriving the explicit expression of the projection, there is some difference in applying the Cramer's rule between [3] and this paper.

II. THE PROJECTION ONTO THE COLUMN SPACE AND THE PROJECTION ONTO THE LEFT NULL SPACE

It can be easily shown that, from the late time representation based on the natural frequencies, the late time response can be written as [3], $n=1,\cdots,N,$:

$$y_n = \sum_{i=1}^{M} c_i z_i^{\,n} + h_n, \tag{1}$$

where N is the number of the sampled late time response and M is the number of the natural frequencies. $h_n, n=1,\ldots,N,$ is the zero-mean Gaussian noise associated with $y_n, n=1,\ldots,N.$ $z_i, i=1,\ldots,M,$ is the z-plane natural frequencies of the target. In the noiseless case where $h_n, n=1,\ldots,N,$ is equal to zero, (1) reduces to:

$$u_n = \sum_{i=1}^{M} c_i z_i^{\,n}, \tag{2}$$

where $u_n, n=1,\ldots,N,$ is the noiseless late time response.

In matrix form, (1) can be written as:

$$\mathbf{y} = \mathbf{Z}\mathbf{c}, \tag{3}$$

where $\mathbf{Z},\mathbf{y}$ and $\mathbf{c}$ are defined as:

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$$Z_{mn} = z_n^m, \tag{4}$$

$$\mathbf{y} = \begin{bmatrix} y_1 & y_2 & \cdots & y_N \end{bmatrix}^T, \tag{5}$$

$$\mathbf{c} = \begin{bmatrix} c_1 & c_2 & \cdots & c_M \end{bmatrix}^T. \tag{6}$$

Since N is larger than M, the least squares solution of (3) can be written as

$$\hat{\mathbf{c}} = \left(\mathbf{Z}^H \mathbf{Z}\right)^{-1} \mathbf{Z}^H \mathbf{y}. \tag{7}$$

Using (7) in the right-hand side of (3), we have,

$$\mathbf{Z}\hat{\mathbf{c}} = \mathbf{Z}\left(\mathbf{Z}^H \mathbf{Z}\right)^{-1} \mathbf{Z}^H \mathbf{y}. \tag{8}$$

We note that the projection onto the column space matrix onto the column space of the matrix $\mathbf{Z}$ is defined as:

$$\mathbf{P}_Z \equiv \mathbf{Z}\left(\mathbf{Z}^H \mathbf{Z}\right)^{-1} \mathbf{Z}^H. \tag{9}$$

Using (9) in (8) enables us to write (8) as:

$$\mathbf{Z}\hat{\mathbf{c}} = \mathbf{Z}\left(\mathbf{Z}^H \mathbf{Z}\right)^{-1} \mathbf{Z}^H \mathbf{y} = \mathbf{P}_{\mathbf{Z}}\mathbf{y}. \tag{10}$$

The projection onto the left null space can be defined by subtracting the projection onto the column space of $\mathbf{y}$ onto the column space of $\mathbf{Z}$ from the noisy late time response $\mathbf{y}$:

$$\mathbf{y} - \mathbf{Z}\hat{\mathbf{c}} = \left(\mathbf{I} - \mathbf{P}_{\mathbf{Z}}\mathbf{y}\right) = \mathbf{P}_{\mathbf{Z}}^{\perp}\mathbf{y}, \tag{11}$$

where (9) is used and $\mathbf{P}_{\mathbf{Z}}^{\perp} \equiv \mathbf{I} - \mathbf{P}_{\mathbf{Z}}$ is called the projector onto the left null space of $\mathbf{Z}$.

III. EXPLICIT EXPRESSIONS FOR THE PROJECTION ONTO THE COLUMN SPACE AND THE PROJECTION ONTO THE LEFT NULL SPACE

From (4), the m-th row and the n-th column of $\mathbf{Z}^H\mathbf{Z}$ can be written as:

$$\left(\mathbf{Z}^H \mathbf{Z}\right)_{mn} = \sum_{i=1}^{N} \left(z_m^* z_n\right)^i \quad m = 1, \cdots, M \quad n = 1, \cdots, M. \tag{12}$$

If we use the cofactor expansion along the n-th column of the matrix $\mathbf{Z}^H\mathbf{Z}$, we have,

$$\det \mathbf{Z}^H \mathbf{Z} = \sum_{m=1}^{M} \left(\sum_{i=1}^{N} \left(z_m^* z_n\right)^i (-1)^{m+n} \det \mathbf{L}_{m,n} \right), \tag{13}$$

where $\mathbf{L}_{m,n}$ is the $(M-1)\times(M-1)$ matrix formed by removing from $\mathbf{Z}^H\mathbf{Z}$ its m-th row and n-th column.

From (1) and (4), $\mathbf{Z}^H\mathbf{y}$ can be written as:

$$\mathbf{Z}^H \mathbf{y} = \begin{bmatrix} \sum_{i=1}^{N} z_1^{*i} y_i & \sum_{i=1}^{N} z_2^{*i} y_i & \cdots & \sum_{i=1}^{N} z_M^{*\,i} y_i \end{bmatrix}^T. \tag{14}$$

$\mathbf{B}_n$ is defined by replacing the n-th column of the matrix $\mathbf{Z}^H\mathbf{Z}$ by $\mathbf{Z}^H\mathbf{y}$ for $n = 1, \cdots, M$:

$$\mathbf{B}_n = \begin{bmatrix} \sum_{i=1}^{N} \left(z_1^* z_1\right)^i & \cdots & \sum_{i=1}^{N} \left(z_1^* z_{n-1}\right)^i & \sum_{i=1}^{N} \left(z_1^*\right)^i y_i & \sum_{i=1}^{N} \left(z_1^* z_{n+1}\right)^i & \cdots & \sum_{i=1}^{N} \left(z_1^* z_M\right)^i \\ \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ \sum_{i=1}^{N} \left(z_M^* z_1\right)^i & \cdots & \sum_{i=1}^{N} \left(z_M^* z_{n-1}\right)^i & \sum_{i=1}^{N} \left(z_M^*\right)^i y_i & \sum_{i=1}^{N} \left(z_M^* z_{n+1}\right)^i & \cdots & \sum_{i=1}^{N} \left(z_M^* z_M\right)^i \end{bmatrix}. \tag{15}$$

The determinant of matrix $\mathbf{B}_n$ can be obtained from the cofactor expansion along the first column of the matrix $\mathbf{B}_n$:

$$\det \mathbf{B}_n = \sum_{m=1}^{M} \left(\left(\sum_{i=1}^{N} \left(z_m^*\right)^i y_i \right) (-1)^{m+n} \det \mathbf{L}_{m,n} \right). \tag{16}$$

Using the Cramer's rule, the explicit expression of the least squares solution in (7) is:

$$\hat{c}_n = \frac{\det \mathbf{B}_n}{\det \mathbf{Z}^H \mathbf{Z}} \quad n = 1, \cdots, M. \tag{17}$$

Note that, in [3], the authors applied the Cramer's rule to obtain $\left(\mathbf{Z}^H\mathbf{Z}\right)^{-1}\mathbf{Z}^H$, not to obtain $\left(\mathbf{Z}^H\mathbf{Z}\right)^{-1}\mathbf{Z}^H\mathbf{y}$. In this paper, the Cramer's rule is used to obtain $\left(\mathbf{Z}^H\mathbf{Z}\right)^{-1}\mathbf{Z}^H\mathbf{y}$. What is desirable in this new approach is that we can get expressions which are more compact, intuitive and insightful than those given in [3].

By substituting (13) and (16) in (17), we get

$$\hat{c}_n = \frac{\det \mathbf{B}_n}{\det \mathbf{Z}^H \mathbf{Z}} = \frac{\sum_{m=1}^{M} \left(\left(\sum_{i=1}^{N} \left(z_m^*\right)^i y_i \right) (-1)^{m+n} \det \mathbf{L}_{m,n} \right)}{\sum_{m=1}^{M} \left(\sum_{i=1}^{N} \left(z_m^* z_n\right)^i (-1)^{m+n} \det \mathbf{L}_{m,n} \right)}. \tag{18}$$

Using (18) in (10), we have,

$$\mathbf{Z}\hat{\mathbf{c}} = \begin{bmatrix} \sum_{n=1}^{M} z_n \dfrac{\sum_{m=1}^{M} \left(\left(\sum_{i=1}^{N} \left(z_m^*\right)^i y_i \right) (-1)^{m+n} \det \mathbf{L}_{m,n} \right)}{\sum_{m=1}^{M} \left(\sum_{i=1}^{N} \left(z_m^* z_n\right)^i (-1)^{m+n} \det \mathbf{L}_{m,n} \right)} \\ \vdots \\ \sum_{n=1}^{M} z_n{}^N \dfrac{\sum_{m=1}^{M} \left(\left(\sum_{i=1}^{N} \left(z_m^*\right)^i y_i \right) (-1)^{m+n} \det \mathbf{L}_{m,n} \right)}{\sum_{m=1}^{M} \left(\sum_{i=1}^{N} \left(z_m^* z_n\right)^i (-1)^{m+n} \det \mathbf{L}_{m,n} \right)} \end{bmatrix}. \tag{19}$$

The implicit expression of the norm of the projection onto the column space is,

$$\left\|\mathbf{P}_{\mathbf{Z}}\mathbf{y}\right\| = \left\|\mathbf{Z}\left(\mathbf{Z}^H \mathbf{Z}\right)^{-1} \mathbf{Z}^H \mathbf{y}\right\|. \tag{20}$$

From (19) and (20), the corresponding explicit expression of the norm of the projection onto the column space is,

$$\left\|\mathbf{P}_{\mathbf{Z}}\mathbf{y}\right\| = \sqrt{\sum_{j=1}^{N} \left(\sum_{n=1}^{M} z_n{}^j \frac{\sum_{m=1}^{M} \left(\left(\sum_{i=1}^{N} \left(z_m^*\right)^i y_i \right) (-1)^{m+n} \det \mathbf{L}_{m,n} \right)}{\sum_{m=1}^{M} \left(\sum_{i=1}^{N} \left(z_m^* z_n\right)^i (-1)^{m+n} \det \mathbf{L}_{m,n} \right)} \right)^2}. \tag{21}$$

Similarly, the implicit expression of the square of the norm of the projection onto the column space is,

$$\left\|\mathbf{P}_{\mathbf{Z}}\mathbf{y}\right\|^2 = \left\|\mathbf{Z}\left(\mathbf{Z}^H \mathbf{Z}\right)^{-1} \mathbf{Z}^H \mathbf{y}\right\|^2. \tag{22}$$

From (19) and (22), the explicit expression of the square of the norm of the projection onto the column space is,

$$\left\|\mathbf{P_Z y}\right\|^2=\sum_{j=1}^{N}\left(\sum_{n=1}^{M} z_n{}^j \frac{\sum_{m=1}^{M}\left(\left(\sum_{i=1}^{N}\left(z_m^*\right)^i y_i\right)(-1)^{m+n}\det\mathbf{L}_{m,n}\right)}{\sum_{m=1}^{M}\left(\sum_{i=1}^{N}\left(z_m^* z_n\right)^i(-1)^{m+n}\det\mathbf{L}_{m,n}\right)}\right)^2. \tag{23}$$

Using the same approach to get (21) and (23), the explicit expressions of $\left\|\mathbf{P_Z^\perp y}\right\|$ and $\left\|\mathbf{P_Z^\perp y}\right\|^2$ can be written as:

$$\left\|\mathbf{P_Z^\perp y}\right\|=\sqrt{\sum_{j=1}^{N}\left(y_j-\sum_{n=1}^{M} z_n{}^j \frac{\sum_{m=1}^{M}\left(\left(\sum_{i=1}^{N}\left(z_m^*\right)^i y_i\right)(-1)^{m+n}\det\mathbf{L}_{m,n}\right)}{\sum_{m=1}^{M}\left(\sum_{i=1}^{N}\left(z_m^* z_n\right)^i(-1)^{m+n}\det\mathbf{L}_{m,n}\right)}\right)^2}, \tag{24}$$

$$\left\|\mathbf{y}-\mathbf{Z}\hat{\mathbf{c}}\right\|^2=\sum_{j=1}^{N}\left(y_j-\sum_{n=1}^{M} z_n{}^j \frac{\sum_{m=1}^{M}\left(\left(\sum_{i=1}^{N}\left(z_m^*\right)^i y_i\right)(-1)^{m+n}\det\mathbf{L}_{m,n}\right)}{\sum_{m=1}^{M}\left(\sum_{i=1}^{N}\left(z_m^* z_n\right)^i(-1)^{m+n}\det\mathbf{L}_{m,n}\right)}\right)^2, \tag{25}$$

which is supposed to be Ricean-distributed and chi-square distributed, respectively [5].

IV. ANALYTIC EXPRESSIONS FOR THE MEAN AND THE VARIANCE

A. Analytic expressions for the statistics of the projection onto the column space

A sum of the squares of independent Gaussian random variables is chi-square distributed, and the degree of freedom of the chi-square random variable is the number Gaussian random variables. In $\left\|\mathbf{P_Z y}\right\|^2=\sum_{i=1}^{M} w_i^2$, $\left\|\mathbf{P}_Z\mathbf{y}\right\|^2$ is expressed as a sum of square of M Gaussian random variables, which implies that $\left\|\mathbf{P}_Z\mathbf{y}\right\|^2$ is chi-square distributed with the degree of freedom M.

In [3], the mean and the variance of $\left\|\mathbf{P_Z^\perp y}\right\|^2$ have been derived. Adopting the scheme presented in [3], the mean and variance of $\left\|\mathbf{P_Z y}\right\|^2$ are expressed as:

$$\mathrm{Mean}\left(\left\|\mathbf{Z}\left(\mathbf{Z}^H\mathbf{Z}\right)^{-1}\mathbf{Z}^H\mathbf{y}\right\|^2\right)=\mathrm{Mean}\left(\sum_{i=1}^{M} w_i^2\right)=M\sigma^2+\sum_{i=1}^{M}\mu_{w_i}^2, \tag{26}$$

$$\mathrm{Var}\left(\left\|\mathbf{Z}\left(\mathbf{Z}^H\mathbf{Z}\right)^{-1}\mathbf{Z}^H\mathbf{y}\right\|^2\right)=\mathrm{Var}\left(\sum_{i=1}^{M} w_i^2\right)=2M\sigma^4+4\sigma^2\sum_{i=1}^{M}\mu_{w_i}^2, \tag{27}$$

where w_i is the i-th entry of $\mathbf{w}=\mathbf{V}^T\mathbf{y}$ and μ_{w_i} is the expected value of w_i. Note that $\mathbf{V}$ is defined from $\mathbf{P_Z}=\mathbf{V\Lambda V}^T$.

The norm of the projection onto the column space is equal to:

$$\left\|\mathbf{P_Z y}\right\|=\sqrt{\sum_{i=1}^{M} w_i^2}. \tag{28}$$

Since $\left\|\mathbf{P_Z y}\right\|^2=\sum_{i=1}^{M} w_i^2$ is chi-square distributed, it follows that $\left\|\mathbf{P_Z y}\right\|=\sqrt{\sum_{i=1}^{M} w_i^2}$ is Ricean distributed [5].

From the moment of Ricean distribution, we can obtain the mean and the variance of (28) [5]:

$$\mathrm{Mean}(\left\|\mathbf{P_Z y}\right\|)=\sqrt{2}\sigma\exp\left(-\frac{\sum_{i=1}^{M}\mu_{w_i}^2}{2\sigma^2}\right)\frac{\Gamma\left[\frac{1}{2}(M+1)\right]}{\Gamma\left[\frac{1}{2}M\right]}{}_1F_1\left(\frac{1}{2}(M+1),\frac{1}{2}M;\frac{\sum_{i=1}^{M}\mu_{w_i}^2}{2\sigma^2}\right), \tag{29}$$

$$\mathrm{Var}(\left\|\mathbf{P_Z y}\right\|)=2\sigma^2\exp\left(-\frac{\sum_{i=1}^{M}\mu_{w_i}^2}{2\sigma^2}\right)\frac{\Gamma\left[\frac{1}{2}(M+2)\right]}{\Gamma\left[\frac{1}{2}M\right]}{}_1F_1\left(\frac{1}{2}(M+2),\frac{1}{2}M;\frac{\sum_{i=1}^{M}\mu_{w_i}^2}{2\sigma^2}\right)-\left[\mathrm{Mean}\left(\left\|\mathbf{Z}\left(\mathbf{Z}^H\mathbf{Z}\right)^{-1}\mathbf{Z}^H\mathbf{y}\right\|\right)\right]^2, \tag{30}$$

where $\mathrm{Mean}\left(\left\|\mathbf{Z}\left(\mathbf{Z}^H\mathbf{Z}\right)^{-1}\mathbf{Z}^H\mathbf{y}\right\|\right)$ is given in (29), and Γ is the gamma function. ${}_1F_1(\alpha,\beta,\gamma)$ is the hypergeometric function.

B. Analytic expressions for the statistics of the projection onto the left null space

In $\left\|\mathbf{P_Z^\perp y}\right\|^2=\sum_{i=1}^{N-M} w_i^2$, $\left\|\mathbf{P_Z^\perp y}\right\|^2$ is expressed as a sum of square of N-M Gaussian random variables, which implies that $\left\|\mathbf{P_Z^\perp y}\right\|^2$ is chi-square distributed with the degree of freedom N-M.

It is shown in [3] that the mean and the variance of $\left\|\mathbf{P_Z^\perp y}\right\|^2$ are expressed as:

$$\mathrm{Mean}\left(\left\|\left(\mathbf{I}-\mathbf{F}\left(\mathbf{F}^H\mathbf{F}\right)^{-1}\mathbf{F}^H\right)\mathbf{y}\right\|^2\right)=\mathrm{Mean}\left(\sum_{i=1}^{N-M} w_i^2\right)=(N-M)\sigma^2+\sum_{i=1}^{N-M}\mu_{w_i}^2, \tag{31}$$

$$\mathrm{Var}\left(\left\|\left(\mathbf{I}-\mathbf{F}\left(\mathbf{F}^H\mathbf{F}\right)^{-1}\mathbf{F}^H\right)\mathbf{y}\right\|^2\right)=\mathrm{Var}\left(\sum_{i=1}^{N-M} w_i^2\right)=2(N-M)\sigma^4+4\sigma^2\sum_{i=1}^{N-M}\mu_{w_i}^2, \tag{32}$$

where w_i is the i-th entry of $\mathbf{w}=\mathbf{V}^T\mathbf{y}$ and μ_{w_i} is the expected value of w_i. Note that $\mathbf{V}$ is defined from $\mathbf{P_Z^\perp}=\mathbf{V\Lambda V}^T$.

The norm of the projection onto the left null space can be written as:

$$\left\|\mathbf{P_Z^\perp y}\right\|=\sqrt{\sum_{i=1}^{N-M} w_i^2}. \tag{33}$$

Comparing (28) and (33), we can see that

$\left\|\mathbf{P}_{\mathbf{Z}}^{\perp}\mathbf{y}\right\| = \sqrt{\sum_{i=1}^{N-M} w_i^2}$ is also Ricean distributed with M in (28) replaced by $N-M$ in (33) and the corresponding mean and the variance can be written as [5]:

$$\operatorname{Mean}\left(\left\|\mathbf{P}_{\mathbf{Z}}^{\perp}\mathbf{y}\right\|\right) = \sqrt{2}\sigma \exp\left(-\frac{\sum_{i=1}^{N-M}\mu_{w_i}^2}{2\sigma^2}\right)\frac{\Gamma\left[\frac{1}{2}(N-M+1)\right]}{\Gamma\left[\frac{1}{2}(N-M)\right]}{}_1F_1\left(\frac{1}{2}(N-M+1),\frac{1}{2}(N-M);\frac{\sum_{i=1}^{N-M}\mu_{w_i}^2}{2\sigma^2}\right), \tag{34}$$

$$\operatorname{Var}\left(\left\|\mathbf{P}_{\mathbf{Z}}^{\perp}\mathbf{y}\right\|\right) = 2\sigma^2 \exp\left(-\frac{\sum_{i=1}^{N-M}\mu_{w_i}^2}{2\sigma^2}\right)\frac{\Gamma\left[\frac{1}{2}(N-M+2)\right]}{\Gamma\left[\frac{1}{2}(N-M)\right]}{}_1F_1\left(\frac{1}{2}(N-M+2),\frac{1}{2}(N-M);\frac{\sum_{i=1}^{N-M}\mu_{w_i}^2}{2\sigma^2}\right) - \left[\operatorname{Mean}\left(\left\|\left(\mathbf{I}-\mathbf{F}\left(\mathbf{F}^H\mathbf{F}\right)^{-1}\mathbf{F}^H\right)\mathbf{y}\right\|\right)\right]^2. \tag{35}$$

V. NUMERICAL RESULTS

For the square of the norm of the projection onto the column space, we validate (23), (26) and (27) by evaluating (23) and (22) using the Monte Carlo simulation, and calculate the mean of (23) and the mean of (22) to see whether they are equal to (26). Also, we evaluate the variance of (23) and the variance of (22) to see whether they are consistent with (27).

For the square of the norm of the projection onto the left null space, to show that (25), (31) and (32) are all valid we make the Monte Carlo simulation for (25) and check whether the mean of (25) is equal to (31). Also, we check whether the variance of (25) is equal to (32).

For the norm of the projection onto the column space, we validate (21), (29) and (30) by evaluating (21) and (20) using the Monte Carlo simulation, and calculate the mean of (21) and the mean of (20) for checking whether they are all equal to (29). Also, we calculate the variance of (21) and the variance of (20) to see whether they are all equal to (30).

For the norm of the projection onto the left null space, to show that (24), (34) and (35) are all correct, we evaluate (24) via the Monte Carlo simulation and check whether the mean of (24) is equal to (34). Also, we check whether the variance of (24) is consistent with (35).

The lengths of the thin wires used for the numerical simulation are equal to 1.0 meter, 0.9 meter and 0.8 meter.

In evaluating $\left\|\mathbf{P}_{\mathbf{Z}}\mathbf{y}\right\|$, $\left\|\mathbf{P}_{\mathbf{Z}}^{\perp}\mathbf{y}\right\|$, $\left\|\mathbf{P}_{\mathbf{Z}}\mathbf{y}\right\|^2$ and $\left\|\mathbf{P}_{\mathbf{Z}}^{\perp}\mathbf{y}\right\|^2$, $\mathbf{Z}$ in (4) corresponds to 1.0 meter long wire, and $\mathbf{y} = \begin{bmatrix} y_1 & y_2 & \cdots & y_N \end{bmatrix}^T$ can be the late time response of 0.8 meter, 0.9 meter or 1.0 meter. Scattering data are generated using the singularity expansion method (SEM) representation for the thin wires of 1.0 meter, 0.9 meter and 0.8 meter with incidence angle of 40 degrees. That is, the correct target is the thin wire of $L = 1.0$ m, and the wrong targets are the thin wires of $L = 0.9$ m and $L = 0.8$ m. The radii of all the wires are equal to $L/200$.

For noise simulation, each point of the SEM-based wire transient response is perturbed with Gaussian noise. We consider the various signal-to-noise ratios (SNRs) of SNR = 0, 5 and 10 dB. The details on how to make numerical simulations can be found in [3]. We use the late time transient responses of thin wires of 0.8 meter, 0.9 meter and 1.0 meter and the z-plane natural frequencies of the correct target of 1.0 m. The z-plane natural frequencies for the thin wire of 1.0 meter, $z_i, i = 1,\cdots,6$, are $0.9617 \pm j1.5464, 0.9445 \pm j2.4621$ and $0.9321 \pm j3.8925$ [3].

In Fig. 1, the line with legend 'Analytic' are from (26). The lines with legends 'Simulation: Implicit' and 'Simulation: Explicit' are given from the averages of the Monte Carlo simulation of (22) and (23). The line with legend 'Analytic' are from (26), and those with legends 'Simulation: Implicit' and 'Simulation: Explicit' are given from the variances of (22) and (23) via the Monte Carlo simulation in Fig. 2.

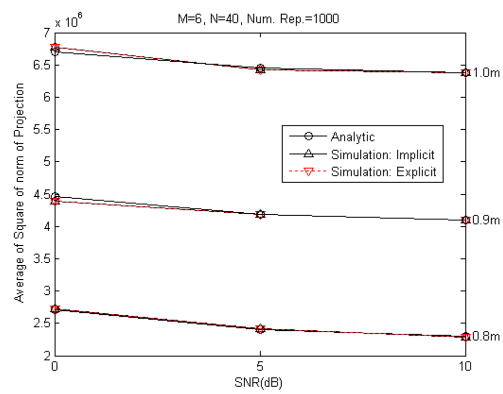

Fig. 1. Average of square of norm of projection onto the column space ($M = 6, N = 40$).

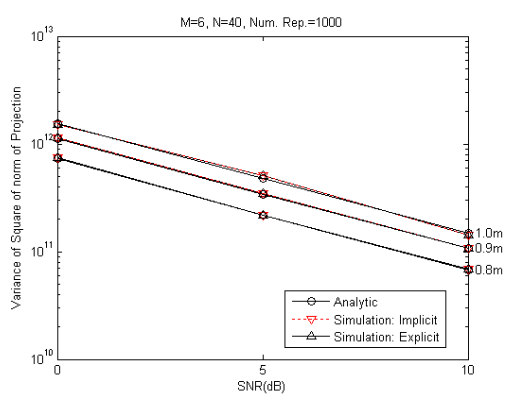

Fig. 2. Variance of square of norm of projection onto the column space ($M = 6, N = 40$).

In Fig. 3, the result with legend 'Analytic' are from (31), and those with legends 'Simulation: Implicit' and 'Simulation: Explicit' are given from the averages of the Monte Carlo simulation of (25). In Fig. 4, the line with legend 'Analytic' are given by (32), and those with legends 'Simulation: Implicit' and 'Simulation: Explicit' are given from the variances of the Monte Carlo simulation of (25).

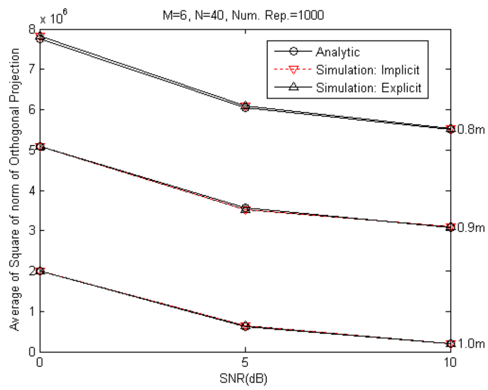

Fig. 3. Average of square of norm of projection onto the left null space ($M = 6, N = 40$).

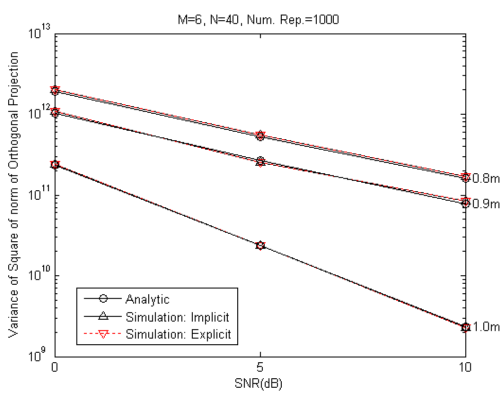

Fig. 4. Variance of square of norm of projection onto the left null space ($M = 6, N = 40$).

In Fig. 5, the line with legend 'Analytic' are from (29), and those with legends 'Simulation: Implicit' and 'Simulation: Explicit' are given from the averages of the Monte Carlo simulation corresponding to (20) and (21). In Fig. 6, the line with legend 'Analytic' are from (30), and those with legends 'Simulation: Implicit' and 'Simulation: Explicit' are given from the variances of the Monte Carlo simulation of (20) and (21).

In Fig. 7, the line with legend 'Analytic' are from (34), and those with legends 'Simulation: Implicit' and 'Simulation: Explicit' are given from the averages of the Monte Carlo simulation of (24). The result with legend 'Analytic' are from (35), and those with legends 'Simulation: Implicit' and 'Simulation: Explicit' are calculated from the variances of the Monte Carlo simulation of (24) in Fig. 8.

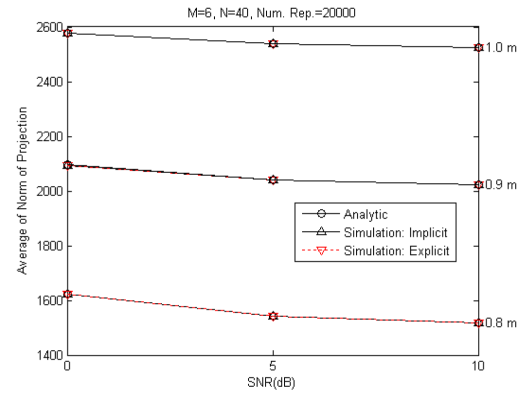

Fig. 5. Average of norm of projection onto the column space ($M = 6, N = 40$).

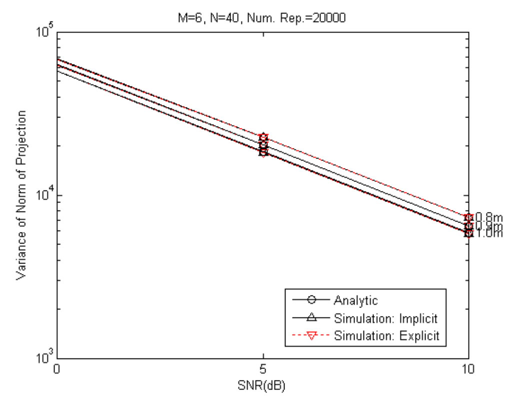

Fig. 6. Variance of norm of projection onto the column space ($M = 6, N = 40$).

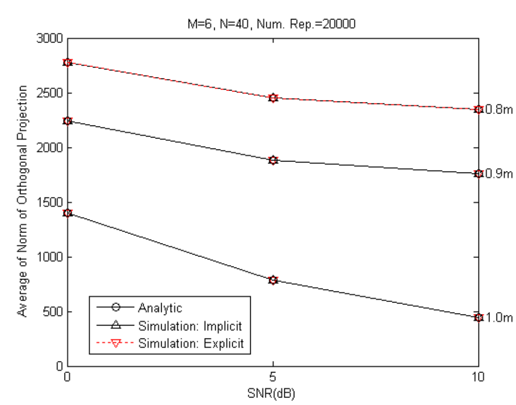

Fig. 7. Average of norm of projection onto the left null space ($M = 6, N = 40$).

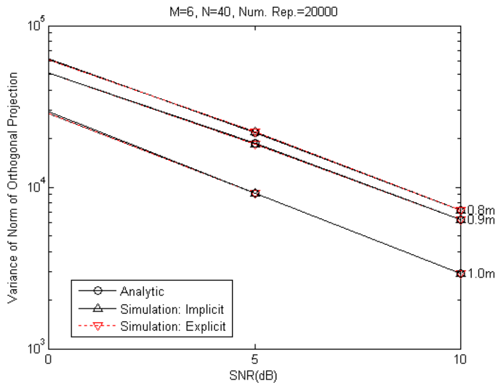

Fig. 8. Variance of norm of projection onto the left null space ($M = 6, N = 40$).

In Fig. 5, we can observe the following:

- In Fig. 5, the norm of the projection onto the column space of the correct target of 1.0 m is small enough to recognize the correct target compared with that for the wrong targets of 0.9 m and 0.8 m.
- As SNR increases, the norm of the projection onto the column space of the correct target decreases significantly, which is favorable for the recognition of the correct target.
- As the number of natural frequencies increases, the norm of the projection onto the column space of the correct target increases and those for the wrong target decrease, which improves the performance of the radar target recognition with an increase of the number of the natural frequencies. Since the results in Fig. 1 are for the square of the norm of the projection onto the column space of $\mathbf{Z}$ of the late time response and those in Fig. 5 are the norm of the projection onto the column space of $\mathbf{Z}$ of the late time response, the observation for Fig. 5 is also true for Fig. 1.

In Fig. 7, we can see the following:

- The norm of the projection onto the left null space of the matrix $\mathbf{Z}$ corresponding to the correct target of length 1.0 meter is large enough to recognize the correct target compared with that for the wrong targets of length 0.9 meter and 0.8 meter.
- As SNR increases, the norm of the projection onto the left null space for the correct target decreases significantly, which is favorable for the recognition of the correct target.
- As the number of natural frequencies increases, the norm of the projection onto the left null space for the correct target decrease and those for the wrong target increase, which improves the performance of the radar target recognition with an increase of the number of the natural frequencies. Since the results in Fig. 3 are for the square of the norm of the projection onto the left null space of $\mathbf{Z}$ of the late time response and those in Fig. 7 are the norm of the projection onto the left null space of $\mathbf{Z}$ of the late time response, the observation for Fig. 7 is also true for Fig. 3.

VI. CONCLUSION

The mean and the variance of the square of the norm of the projection onto the column space are given in (26) and (27), respectively, and those of the square of the norm of the projection onto the left null space are given in (31) and (32), respectively.

The mean and the variance of the norm of the projection onto the column space are given in (29) and (30), respectively, and those of the square of the norm of the projection onto the left null space are given in (34) and (35), respectively.

(23), (26) and (27) are verified in Figs. 1-2, and (25), (31) and (32) are verified in Figs. 3-4. In Figs. 5-6, the validity of (21), (29) and (30) is shown. Figures 7-8 show the validity of (24), (34) and (35).

From the numerical results, for the square of the norm of the projection onto the column space, it has been confirmed that the mean and the variance of the norm of the projection onto the column space can be available from the derived expressions in (26) and (27) without actually performing the Monte Carlo simulations which require many evaluations of (22) or (23). In addition to the square of the norm of the projection onto the column space, it is also true for the norm of the projection onto the column space, the square of the norm of the projection onto the left null space, and the norm of the projection onto the left null space.

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