FDTD Potentials for Dispersion Analysis of Sinusoidally Modulated Media

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Abstract **─** A numerical study based on the finite difference time domain (FDTD) method is presented for the oblique incidence of TE modes with an emphasis on dispersion properties. The proposed medium has sinusoidally modulated dielectric permittivity. In order to truly address this scattering problem, total field-scattered field (TFSF) approach is suggested, which yields accurate results for the electric field distribution inside the modulated medium. A comparison between analytical plots and the FDTD results reveals the ability of FDTD in rigorous analysis of band diagrams for any arbitrary value of modulation factor. In addition, a closed form formula for numerical dispersion relation is derived for the case of small modulation.

Index Terms **─** Dispersion analysis, FDTD, inhomogeneous media, Mathieu functions, oblique incidence, permittivity-modulated media, scattering, TFSF.

I. INTRODUCTION

Modulated structures have gained a great deal of attention in recent years. It is mainly due to their band-gap behavior which allows one to control the propagation and possible radiation of electromagnetic waves off these configurations. These platforms are extensively used to realize both radiating and non-radiating devices such as leaky-wave antennas, metamaterial lenses, filters, and etc. [1-3]. In antenna applications they are particularly useful for miniaturization, bandwidth enhancement, surface-coupling reduction, and generation of holographic surfaces.

In general, two common solutions are offered to exhibit band-gap properties in materials. First is by periodic arrangement of parasitic loads, which suggests modulation of the surface impedance. The second solution is periodic variation of electromagnetic properties of the material, i.e., modulation of effective permittivity. Drilling holes in dielectrics [4] or periodic variation of the width of microstrip line [5] are among solutions proposed to change effective dielectric permittivity in terms of fabrication and manufacturing.

Over the past years surface impedance modulation has received considerable studies (see [2] for instance), whereas modulation of effective dielectric permittivity has been investigated in an intermittent way throughout the years [5-6]. It is due to the complexity involved in Maxwell's equations, once the permittivity of the medium is altered as a function of space coordinate(s). This in fact changes the whole dynamic of the wave equation, which makes it impossible to find a simple analytic solution. In this case, one has to take into consideration that the medium is no longer homogeneous and it will treat electromagnetic waves differently as they try to propagate through. Since inhomogeneous media are finding very attractive applications in electromagnetic and antennas, it is absolutely necessary to find techniques to analyze such configurations [7,8].

A lot could be said about the characteristics of wave propagation by inspecting the dispersion curve corresponding to the structure along which the wave travels. Therefore the first step in designing unit cells is to obtain the related dispersion diagram. However this is not a straightforward process even for the case of homogeneous structures. Now for the non-homogeneous media, this could be quite a challenge since it requires exact knowledge of the effective permittivity of the medium at hand. Various numbers of numerical approaches have been adopted over the years for this

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purpose. Despite the satisfactory results derived by these methods, they are still suffering from a deplorable lack of generality. Furthermore, the application of these methods is restricted by some factors like the configuration of the unit cell, the amount of loss associated with the excited modes, and etc. [2,5,9]. All these restrictions make it difficult to effectively employ the useful features of modulated structures. Hence, we were stimulated to revisit this problem in more depth. Therefore, the main motivation of this work is to develop a more general procedure capable of analyzing modulated structures in a fast and accurate way. Another aspect of this work is that it is focused on the study of an open structure, while mostly the investigations found in the literature have only considered closed structures.

Here we apply finite difference time domain FDTD method in an attempt to find both the dispersion and also field distributions for a medium whose effective permittivity has been sinusoidally modulated. The FDTD method has been selected since in addition to its generality it is robust and proved to produce reliable results for a wide number of applications [10]. However, we are aware of some negative impacts of FDTD like the numerical dispersion and try to choose the numerical parameters to avoid such effects.

The paper is structured as follows. In Section II, we begin by the wave equation inside the stratified structure. Then in Section III, we proceed to the more complicated task of finding dispersion plots. Three major methods of *graphical*, *analytical* and *numerical* are investigated and dispersion plots from each method are compared. It is proved that the proposed numerical method is capable of accurately predicting wave behavior and band gap limits inside the modulated medium and it has no restriction regarding the value of the modulation index as opposed to the analytical method. A demonstration of the electric field distribution inside modulated medium is also presented that agrees well with the theory which confirms the validity and reliability of the proposed method.

II. FORMULATION OF THE PROBLEM

A. TE wave equation

For a transverse electric, TE wave obliquely incident from free space to the semi-infinite medium of Fig. 1, under the assumption $\partial/\partial y = 0$, the wave equation is given by:

$$\frac{\partial^2 E}{\partial x^2} + \frac{\partial^2 E}{\partial z^2} = \mu_o \varepsilon_o \varepsilon_r (1 - M\cos 2\pi \frac{z}{d}) \frac{\partial^2 E}{\partial t^2}, \quad (1)$$

where $\varepsilon(z) = \varepsilon_r (1 - M\cos 2\pi \frac{z}{d})$, being the relative dielectric permittivity of the modulated medium with ε_r, d, and M identifying the average permittivity, periodicity, and modulation constant respectively.

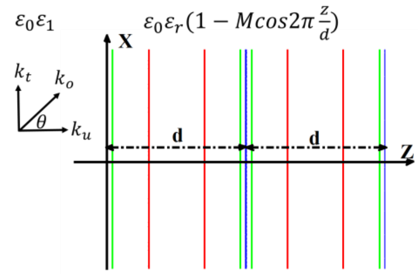


Fig. 1. Oblique incidence on a semi-infinite stratified medium. Shown to the left and right hand side of the *x*-axis, are the permittivity constants of the un-modulated and modulated media, respectively. The un-modulated medium is considered free space, hence its relative permittivity, ε_1 is equal to 1.

B. FDTD discretization

For FDTD computation we first set the following assignments:

$$E(x_i, z_m, t_n) = E(i\Delta x, m\Delta z, n\Delta t) = E^n_{i,m}, \quad (2)$$

$$\varepsilon(z_m) = \varepsilon_r (1 - M\cos 2\pi \frac{m\Delta z}{d}) = \varepsilon_m. \quad (3)$$

Now we use central-time central-space discretization scheme to form the discretized version of (1):

$$E^{n+1}_{i,m} = 2E^n_{i,m}(1 - \theta_x - \theta_z) + \theta_x\left(E^n_{i+1,m} + E^n_{i-1,m}\right) + \theta_z\left(E^n_{i,m+1} + E^n_{i,m-1}\right) - E^{n-1}_{i,m}, \quad (4)$$

where $\theta_x = \frac{1}{\mu_o \varepsilon_o \varepsilon_m}\left(\frac{\Delta t}{\Delta x}\right)^2$, and $\theta_z = \frac{1}{\mu_o \varepsilon_o \varepsilon_m}\left(\frac{\Delta t}{\Delta z}\right)^2$ with Δx, and Δz, being grid spacing and Δt designating time increment. In the next sections, we employ this scheme to calculate the electric field intensity inside the modulated medium.

III. DISPERSION ANALYSIS

A. Graphical dispersion

For the medium presented in Fig. 1 as shown in [6], solutions of wave Equation (1) can be described in the form of Mathieu functions and there is no closed form formula for the dispersion relation. In fact, dispersion in such medium can be obtained by the help of “Mathieu stability diagrams” using the graphical approach. A comprehensive study of this kind can be found in [6]. In what follows we will present a brief illustration of the method. We restrict our discussion to the case of oblique incidence in Fig. 1 considering $k_o = \omega\sqrt{\mu_0 \varepsilon_0}$ the free space wavenumber, with k_t and k_u being the wavenumbers along *x* and *z* directions respectively given by $k_t = k_o\sqrt{\varepsilon_1} sin\theta$ and $k_u^2 = k_o^2\varepsilon_r - k_t^2$, where ε_1 is the relative permittivity of un-modulated medium and θ is the angle of incidence. The last expression, i.e., $k_u^2 = k_o^2\varepsilon_r - k_t^2$, would be the dispersion relation if no modulation was present in the medium, namely $M = 0$.

However, when $M \neq 0$ the propagation constant along z, namely κ, has to be obtained by employing an iterative process to numerically solve a continued fraction expression. This leads to "stability diagrams" of Fig. 2.

It is customary to plot stability diagrams in a-q axes [5,6]. The vertical axis, q is basically an indication of modulation parameter, M which is multiplied by $\left(\sqrt{\varepsilon_r}k_o d/\sqrt{2}\pi\right)^2$ to factor in the effect of average dielectric constant and also frequency, since dispersion is the description of wavenumber variation with respect to frequency and also as modulation index, M varies, wave experiences different stratifications in the medium, and hence different dispersion effects, so modulation coefficient M has to be included in the definition of the axis as well. A more sophisticated physical meaning of parameter q is actually dependent on the specific problem at hand for which Mathieu equation arises. For example, in a problem where Mathieu equation describes vibrating modes of an elliptical membrane, q is associated with the eigenfrequencies of those vibrating modes [11]. The horizontal axis is $a = (k_u d/\pi)^2$ and it arises when the method of 'separation of variables' is used in the TE wave Equation of (1) to find solutions of Mathieu equation. (See [6] for more details).

From Fig. 2 one can distinguish two distinct regions, the shaded areas are "stable regions" where κ is real, whereas in the unshaded areas (unstable regions) κ becomes complex. Furthermore, we can draw a line through origin that will cut through all the regions of stability diagrams. Intersection points are shown with red stars in Fig. 2. The slope of this line can be written as:

$$\tan\varphi = \frac{0.5\varepsilon_r M}{\varepsilon_r - \varepsilon_1 sin^2\theta}. \qquad (5)$$

From (5), one realizes that at a specific angle θ and for constant values of , ε_1, and ε_r the slope of this line, $\tan\varphi$ is fixed and an intersection point with stability diagrams is the value of κ excited at a certain frequency. If $M = 0$, then the slope of the line is zero and it will fall on the horizontal axis which is described by k_u^2 and as stated earlier, k_u^2 is the wave number in the absence of modulation in the medium.

In a general case where $M \neq 0$, in order to construct dispersion diagram we have to find variation of κ versus frequency. Note that by changing frequency, parameters a, and q vary along the straight line yielding different intersection points each corresponding to a specific value of propagation constant at that frequency. Such diagram is shown in Fig. 3 where one can recognize a band-gap between $15.99\ GHz - 18.99\ GHz$. This is a result of the straight line of Fig. 2 passing through the first unstable region of stability diagrams and exciting waves with complex values of κ.

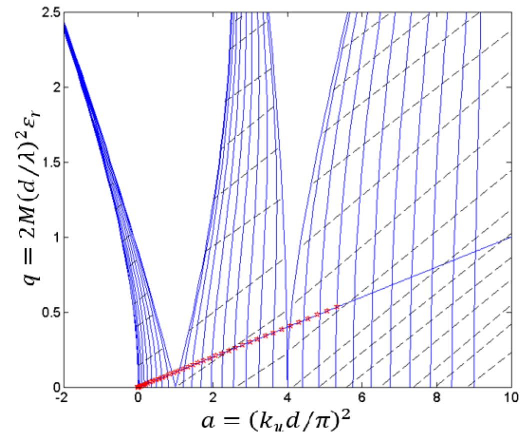


Fig. 2. Mathieu stability diagram (using the data in [12]). The intersecting points, shown with red stars on the straight line, determine the wavenumbers of modes excited in the modulated medium. Parameter λ used in the definition of q is the free space wavelength which is $1\ cm$ at the design frequency of $f = 30\ GHz$. Other parameters are as follows; $d = 1\ cm,\ M = 0.15,\ \theta = 30^o,\ \varepsilon_r = \varepsilon_1 = 1$.

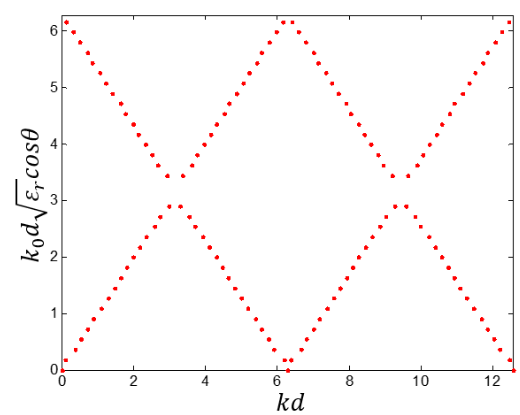


Fig. 3. Dispersion diagrams for the modulated medium shown in Fig. 1 using the graphical approach.

For the case represented in Fig. 2 based on the values given in the caption, the slope of the straight line given by (5) is equal to 0.1, and as can be seen it cuts through the stability diagrams where the appearance of the first band gap (first unshaded region) is noticeable, whereas it hardly cuts through the 2nd unstable region; hence the 2nd band gap is almost non-existent for such a small value of line slope. Therefore Fig. 3 is only focused on the first band gap. For higher values of incidence angle or modulation index the slope of the line increases and it will cut through both 1st and 2nd unshaded areas of Fig. 2 for which case both band gaps should be considered.

B. Analytical dispersion

The above approach, though accurate but is frustrating since every time a single parameter in (5) varies, the whole process needs to be repeated. A more straightforward way is to use the following analytic expression suggested in [6] which is capable of predicting dispersion curves only when the value of q is taken very small and the solution is considered inside pass bands or the “stable’ regions of the Mathieu stability diagram:

$$\frac{\kappa}{k_u}=1+\frac{1}{1-(k_u d/\pi)^2}\left(\frac{\pi}{k_u d}\frac{q}{2}\right)^2. \qquad (6)$$

Under these circumstances, (7) is a good approximation for the solution of electric field inside the modulated medium:

$$E_y(x,z)$$
$$=\left[1-\frac{q}{2}.\frac{\cos\left(\frac{2\pi z}{d}\right)-j\frac{k_u d}{\pi}\sin\left(\frac{2\pi z}{d}\right)}{1-(k_u d/\pi)^2}\right]e^{jk_t x}e^{j\kappa z}e^{j\omega t}. \qquad (7)$$

Designated with blue rhombus is the dispersion plot of (6) depicted in Figs.4 (a) and (b). One recognizes that (6) is quite accurate in pass-bands but when it comes to band-gaps its values cannot be trusted, which was expected as stated earlier.

C. Numerical dispersion

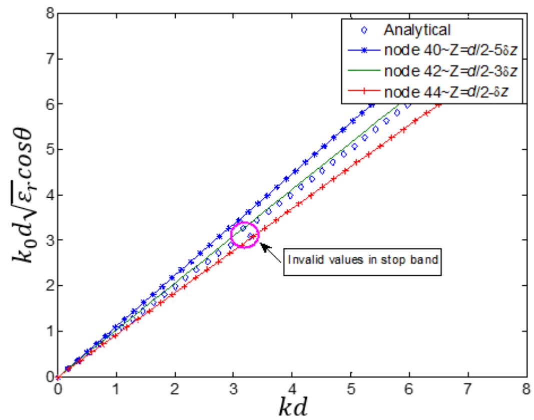
To find more satisfactory results we tried the FDTD technique. The discretized version of (7) is applied into (4) which reveals the numerical dispersion relation as in:

$$\sin^2\left(\frac{\omega\Delta t}{2}\right)=\theta_x\sin^2\left(\frac{k_t\Delta x}{2}\right)+\theta_z\sin^2\left(\frac{\kappa\Delta z}{2}\right). \qquad (8)$$

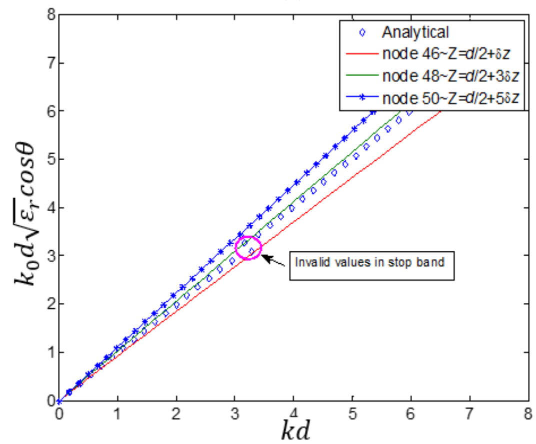
Note that the actual expression is much more complicated and what is given in (8) is derived after taking into account certain approximations [13].

To examine the amount of numerical dispersion introduced by FDTD method into the solution one may try to find the κ-ω diagram of (8) but due to θ_x and θ_z being functions of space, as shown in Figs. 4 (a) and (b), plots will change for different nodes along the axis of modulation z.

This makes it hard to understand whether or not the numerical scheme is giving valid results in terms of band diagrams. However one interesting fact concluded from comparing Figs. 4 (a) and (b), is that such trend is periodic. As a matter of fact within one period, dispersion plots for nodes symmetrically located around $z=d/2$, (line of symmetry) are the same. Another interesting observation is that for the values selected here, neither of these plots suggests the existence of “numerical” band-gaps. The only effect is a small drift of dispersion plots along the nodes. This was expected since permittivity is a function of z, which means the guided wavelength varies along the modulation direction and so does the propagation constant.



(a)



(b)

Fig. 4. Comparison of dispersion diagrams obtained by the approximate analytical solution in (6): blue rhombuses in both Figs. 4 (a) and (b), versus those obtained using (8) for: (a) sample nodes on the first half (left-hand) of the unit cell symmetry line, and (b) sample nodes on the second half (right-hand) of the unit cell line of symmetry.

These all mean that the approximations made to obtain (8) have worked out for the best since we could firstly avoid the numerical band-gaps arising from the periodic grid lattice used in the finite difference method and secondly, as suggested from Fig. 4(a) and (b), predict the dispersion plots quite accurately.

Back to our oblique incidence problem we need to find possible value(s) of κ that are excited in the modulated medium and also the band-gap frequencies. Based on the above analyses, the graphical method though accurate enough, is not necessarily the fastest way to approach the problem and (8) yields different dispersion plots for various nodes. It is also worth to emphasize that the approximations in (8) are only valid within the pass bands not to mention the restrictions on the value of modulation constant. Having that in mind it seems like the most convenient way to obtain the dispersion curve is to calculate the electric field numerically at various sample frequencies and then to determine the wavenumber at each frequency. The results of such calculations are depicted in Fig. 5.

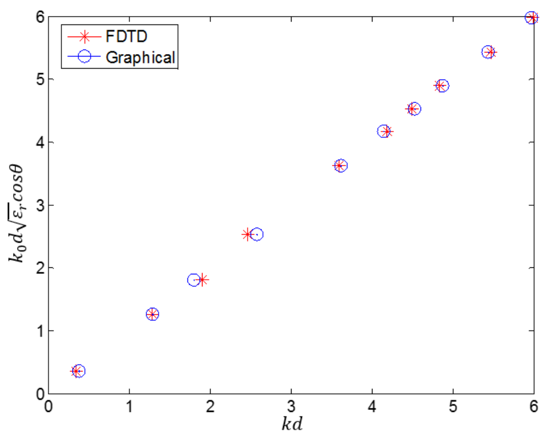


Fig. 5. Comparison between dispersion curves obtained by graphical approach and FDTD method.

A small discrepancy observed between the graphical and numerical plots is believed to be mostly due to a certain degree of approximation involved in the computation of the actual wavelength of the field. This can be explained using Fig. 6 where it can be seen that the electric field solution in the modulated medium has a slightly perturbed sinusoidal shape. One realizes this perturbation is responsible for a slight change of wavelength along the propagation direction. Hence, the results of Fig. 5 are obtained using the average value of the wavelength. There are other factors that affect the accuracy of plots in Fig. 5. First one needs to understand that the graphical approach, does not always give the exact values; to solve for the κ values we have to find the intersection points in Fig. 2 which requires the evaluation of non-integer Mathieu functions of any arbitrary order. Not enough tabulations of this kind can be found in literature. Furthermore, the intervals between two successive steps are often large not to mention the restricted range of parameters (namely q and a) for which these tables are available. Here we have used values in [12] which are obtained with an accuracy of 10^{-5}, however the intersection points, as presented in Fig. 2, do not exactly lie on the graphs available from [12] and therefore an interpolation process needs to be carried out to find the values in the intermediate steps. All these factors will introduce errors in the calculation of dispersion diagram, which then result in the small drift between the graphical and numerical plots of Fig. 5.

Lastly, we also computed the electric field solution inside the semi-infinite medium of Fig. 1, using a sinusoidal excitation at the incidence plane (at $z = 10\ cm$) chosen twenty wavelengths behind the air-dielectric interface (at $z = 30\ cm$).

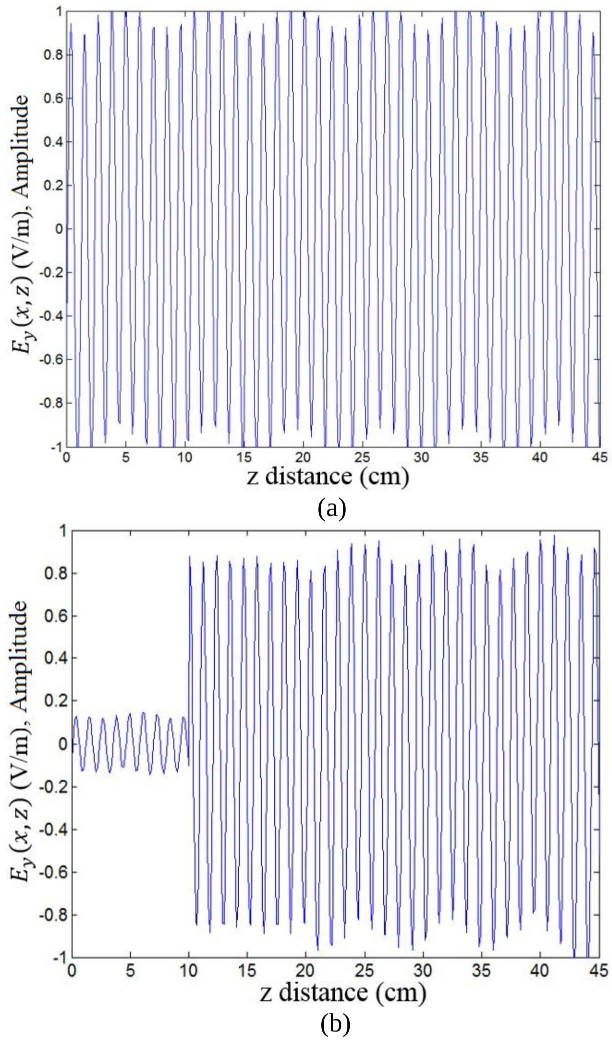


Fig. 6. Electric field distribution in modulated medium using: (a) closed form formula of (7) and (b) FDTD. As clear from the graph, $z = 30\ cm$ is the interface of air and the modulated medium whereas the incidence plane is set at $= 10\ cm$. To insure accuracy the modulated medium has extended over 15 wavelengths beyond the interface.

Using the total field-scattered field approach (TFSF), we account for multiple reflections that will happen due to continuous variation of the permittivity in the dielectric medium. It is clear from Figs. 6 (a) and (b) that the analytical and numerical results inside the modulated dielectric medium are in excellent agreement.

IV. CONCOLUSION

Realizing the challenge exist to address scattering problems involved oblique incidence, a numerical study based on FDTD has been presented to analyze wave propagation inside a modulated medium in terms of dispersion and field distribution. To avoid numerical dispersion a node-by-node band-diagram monitoring process has been carried out in order to adjust FDTD parameters to guarantee a distortion-free transmission. As an appropriate figure of merit for comparison of the numerical results, an approximate closed form formula was used. An inspection of band diagram and field distribution revealed great agreement between graphical and numerical results with the latter being less time consuming, more robust and accurate in predicting the band-limits.

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