A Novel Method to Solve 2nd Order Neumann Type Boundary Value Problems in Electrostatics

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Abstract ─ In this paper, the numerical method of non-polynomial spline approximation is used to solve 2nd order Neumann type boundary value problems (bvp's) in electrostatics. This new approach provides more accurate results than the polynomial approximations and the spectral methods. The literature contains very little on the solution of Neumann type bvp's because of the fact that a unique solution does not exist for all problems. In electrostatics, Neumann type bvp's are encountered for finding the electrostatic potential inside closed surfaces where the normal derivative of the electric potential is specified everywhere on the surface. Two examples are presented to prove the accuracy of the proposed method. In these examples, the governing differential equation is solved to find the electrostatic potential inside a region bounded by conductors that are maintained at constant voltages. The results are compared with the analytic solutions.

Index Terms ─ Boundary value problems, electrostatics, Neumann boundary conditions, numerical methods.

I. INTRODUCTION

Many problems in engineering require the solution of differential equations. If the initial conditions for the solution of the equation are given at the boundaries, these problems are called the "boundary value problems" or shortly the "bvp's". In electrostatics, there are two types of bvp's that govern the majority of problems. They are known as the "Poisson's equation" and the "Laplace equation". Both are 2nd order linear differential equations having the electric potential as the unknown variable that is a function of space coordinates. If the derivatives of the electric potential at the boundaries are specified, then these problems are named as "Neumann type bvp's" [1].

Numerical methods are used to solve bvp's, especially when analytical solutions in closed form are difficult to obtain. The finite difference method (FDM) and the finite element method (FEM) are the most frequently used numerical methods to solve electrostatic problems containing partial or ordinary differential equations [2,3]. These methods are based on discretization of the solution domain and transforming the differential equation into a system of linear equations. They are applicable to problems with non-homogenous media easily. However, one of their disadvantages is that for Neumann type bvp's, the computation matrix of the linear system of equations, also called the "stiffness matrix", is singular, and therefore a solution does not exist [4,5].

In this paper, a new numerical method, called the "Non-polynomial Spline Approximation" is introduced to solve 2nd order electrostatic bvp''s having Neumann type boundary conditions. In this method, the solution domain is sampled by n points, and for each sample, the unknown function is approximated by a non-polynomial (trigonometric) function. The approximated solution is replaced into the differential equation, and the resulting linear equation system is solved for the unknown function of the problem. The proposed method has the advantage of producing an approximate solution for Neumann type bvp's. In literature, there are many applications of FDM and FEM in electromagnetics specifically for Dirichlet or mixed type bvp's. For example, in [6], several FEM based numerical techniques are compared to existing methods in terms of their accuracy for solving boundary value problems in electromagnetics. In [7], the boundary element method (BEM) is used for solving the electromagnetic problems. In [8], the FEM is used for the solution of electromagnetic problems involving anisotropic media. In [9], FDM is analyzed as the numerical technique for non-stationary electromagnetic problems. Up to the authors' knowledge, the work in this paper is novel in the sense that the method of non-polynomial spline approximation has not been applied to electrostatic problems having Neumann type bvp's before.

Two examples are presented to prove the applicability and the accuracy of the proposed method. In the first example, the Poisson's equation is solved for the unknown electrostatic potential distribution inside a charged homogeneous dielectric medium. In the second example, the Laplace equation is solved for the

Submitted On: February 11, 2016
Accepted On: March 13, 2017

electrostatic potential in a 2-D region enclosed by conducting boundaries.

II. NON-POLYNOMIAL SPLINE APPROXIMATION

A. Formulations

The solution of the following linear 2nd order boundary value problem is considered [4]:

$$y^2 + f(x)y = g(x), \quad x \in [a, b], \tag{1}$$

subjected to the Neumann boundary conditions:

$$y^{(1)}(a) - A_1 = y^{(1)}(b) - A_2 = 0, \tag{2}$$

where A_1 and A_2 are the real constants. The functions $f(x)$ and $g(x)$ in (1) are assumed to be continuous in $[a, b]$.

The solution domain $x \in [a, b]$ is sampled by equally spaced n points as:

$$x_i = a + ih, \quad i = 0,1, \dots, n, \tag{3}$$

where $x_0 = a$, $x_n = b$, and $h = (b - a)/n$.

Now, the notation $y(x)$ is used as the exact solution of (1), and S_i as the approximate solution to $y_i = y(x_i)$ acquired by the spline function $Q_i(x)$ that fits in the points (x_i, S_i) and (x_{i+1}, S_{i+1}).

A non-polynomial spline $Q_i(x)$ is defined in the following form:

$$Q_i(x) = a_i \cos \mathrm{k}(x - x_i) + b_i \sin \mathrm{k}(x - x_i) + c_i, \quad i = 0, 1, \dots, n - 1, \tag{4}$$

where a_i, b_i and c_i are constant coefficients, and k is the frequency of the trigonometric functions. Thus, the non-polynomial spline function $S(x)$ that approximates to the exact solution can be written as:

$$S(x) = Q_i(x), \quad x \in [x_i, x_{i+1}], \quad i = 0, 1, \dots, n - 1. \tag{5}$$

The spline function $Q_i(x)$ and its derivatives are defined as:

$$Q_i(x_{i+1/2}) = S_i(x_{i+1/2}),$$
$$Q_i^{(1)}(x_i) = D_i,$$
$$Q_i^{(2)}(x_{i+1/2}) = F_{i+1/2}, \tag{6}$$

where the notation $x_{i+1/2}$ denotes the midpoint of the interval $[x_i, x_{i+1}]$. By using (4) and (6), one can obtain the following relations for the constant coefficients:

$$a_i = \frac{-1}{k^2} F_{i+1/2} \sec(\theta/2) - \frac{1}{k} D_i \tan(\theta/2)$$
$$b_i = \frac{1}{k} D_i,$$
$$c_i = S_{i+1/2} - \frac{1}{k^2} F_{i+1/2}, \tag{7}$$

where $\theta = kh$ and $i = 0, 1, \dots, n - 1$.

By using the continuity of the spline function $Q_i(x)$ at the joining nodes:

$$Q_{i-1}^{(m)}(x) = Q_i^{(m)}(x), \quad m = 0, 1, \tag{8}$$

together with (4) and (7), we obtain the following relation:

$$\left(S_{i+1/2} - 2S_{i-1/2} + S_{i-3/2}\right) = h^2(\alpha F_{i+1/2} + \omega F_{i-1/2} + \alpha F_{i-3/2}), \quad i = 2, 3, \dots, n - 1, \tag{9}$$

where

$$\alpha = \frac{\sec(\theta/2) - 1}{\theta^2}, \tag{10}$$

and

$$\omega = \frac{4\sec(\theta/2)\sin^2(\theta/2) + 2(1 - \sec(\theta/2))}{\theta^2}. \tag{11}$$

(9) gives $n - 2$ linear equations with n unknowns $S_{i+1/2}$, $i = 0, 1, \dots, n - 1$. Two more equations come from the boundary nodes as:

$$\left(-hS_0^{(1)} - S_{1/2} + S_{3/2}\right) = \frac{h^2}{24}(23F_{1/2} + F_{3/2}), \quad at\ i = 1, \tag{12}$$

and

$$\left(S_{n-3/2} - S_{n-1/2} + hS_n^{(1)}\right) = \frac{h^2}{24} 23F_{n-3/2} + 23F_{n-1/2}, \quad at\ i = n. \tag{13}$$

As a result, the linear equation solution of (9) gives the approximate solution S_i to $y_i = y(x_i)$, $i = 0, 1, \dots, n - 1$.

For $\boldsymbol{\alpha = 1/12}$ and $\boldsymbol{\omega = 10/12}$, the algorithm produces the most accurate numerical results [4], thus all of the computations in this paper are carried out using these values.

B. Error analysis

The local truncation error t_i for $i = 1,2, \dots, n$ corresponding to (9)-(13) is given as [4]:

$$t_i = \frac{-h^4}{24} y_0^{(4)} + O(h^5), \qquad i = 1,$$
$$t_i = h^2(1 - 2\alpha - \omega)y_i^{(2)} + h^3\left(\alpha + \frac{\omega}{2} - \frac{1}{2}\right)y_i^{(3)} + h^4\left(\frac{5}{24} - \frac{5}{4}\alpha - \frac{\omega}{8}\right)y_i^{(4)} - h^5\left(\frac{1}{16} - \frac{26}{48}\alpha - \frac{\omega}{48}\right)y_i^{(5)} + O(h^6), \qquad i = 2,3, \dots, n - 1$$
$$t_i = \frac{-h^4}{24} y_n^{(4)} + O(h^5), \quad i = n. \tag{14}$$

Thus, for the boundary nodes and interior nodes, the error is of the order of 5 and 6 respectively.

The truncation error associated with the algorithm can be defined as:

$$e_{i+1/2} = y_{i+1/2} - S_{i+1/2}\,, \tag{15}$$

where $y_{i+1/2} = y(x_{i+1/2})$ and $S_{i+1/2}$, $i = 0,1, \dots, n - 1$, are the exact, and the approximated solutions respectively at the $(i + 1/2)^{th}$ nodes. The total truncation error is given by [4]:

$$\|E\|_\infty \le O(h^2), \tag{16}$$

where $\|E\|_\infty$ is the maximum norm of the global error vector:

$$e_{i+1/2} = y_{i+1/2} - S_{i+\frac{1}{2}}, \qquad i = 0,1,2,..,n\,. \tag{17}$$

Thus, the error is bounded by $\boldsymbol{O(h^2)}$ which implies that the method is quadratically convergent. Also, all round-off errors in this analysis are neglected assuming high digit computations by Matlab.

III. EXAMPLES

A. Solution of 1D electrostatic potential

Finding the electrostatic potential inside a parallel plate capacitor is considered. The governing equation for this problem is the "Poisson's equation". To ensure that the equation satisfies the Neumann boundary conditions, we specify the normal derivatives of the potential at the boundaries. All numeric computations are carried out with the software program Matlab.

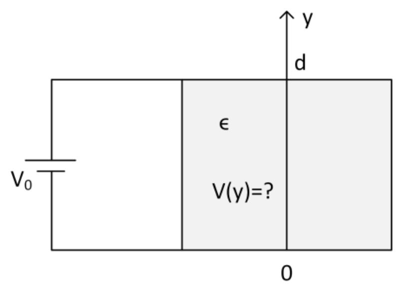

Fig. 1. Cross sectional figure of the capacitor problem.

Figure 1 shows the physical description of the problem. The parallel plate capacitor is assumed to be filled by a charged medium with the uniform charge density ρ and the electric permittivity ϵ. The capacitor has a plate separation length d, and maintained at potential V_0 volts across its plates. The fringe fields at the plate edges are assumed to be negligible, thus the voltage inside the capacitor varies only along the y direction. The Poisson's equation for this problem can be written as [10]:

$$\frac{d^2V(y)}{dy^2} = \frac{-\rho}{\epsilon}, \tag{18}$$

whose solution can be obtained by taking the integral of both sides with respect to y twice. This gives the following result:

$$V(y) = \frac{-\rho}{2\epsilon}y^2 + C_1 y + C_2, \tag{19}$$

where C_1 and C_2 are the constants to be determine by the boundary conditions. Given the following Neumann boundary conditions:

$$\frac{dV(0)}{dy} = \frac{\rho d}{2\epsilon} + \frac{V_0}{d}, \tag{20}$$

and

$$\frac{dV(d)}{dy} = \frac{-\rho d}{2\epsilon} + \frac{V_0}{d}, \tag{21}$$

the solution of the problem exists up to a constant.

The constant C_1 can be obtained by imposing (20) to the derivative of the solution in (19) as:

$$C_1 = \frac{\rho d}{2\epsilon} + \frac{V_0}{d}. \tag{22}$$

Thus, the solution becomes:

$$V(y) = \frac{-\rho}{2\epsilon}y^2 + \left(\frac{\rho d}{2\epsilon} + \frac{V_0}{d}\right)y + C_2. \tag{23}$$

Now, we consider the numeric solution of the problem using the proposed approximation. Figure 2 shows the results for the following simulation parameters: $d = 1(m)$, $V_0 = 1(V)$, $\rho = 10^{-9}(C/m^3)$, $\epsilon_r = 1$, $C_2 = 0$. The results show good accuracy especially for greater number of samples being used.

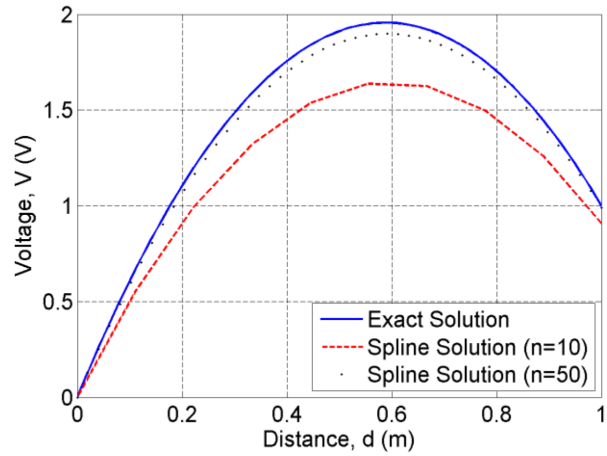

Fig. 2. Simulation results for the voltage distribution inside a parallel plate capacitor for sampling number n=10 and n=50.

The numerical stability of the proposed method is analyzed in terms of the condition number, k(A), of the solution matrix in (9). Figure 3 shows the results with respect to the number of sample points. The proportional relation between the condition number and the size of the matrix is expected [11]. In general, if the condition number $k(A) = 10^k$, then "k" digits of accuracy are lost (at most) during computation in addition to round off errors [12]. Thus, neglecting the round off errors, from Fig. 3 the method is bounded approximately by 5 digits of inaccuracy at about n=50 sampling points, and since the Matlab simulation uses 16 digits of precision, this does not disrupt convergence.

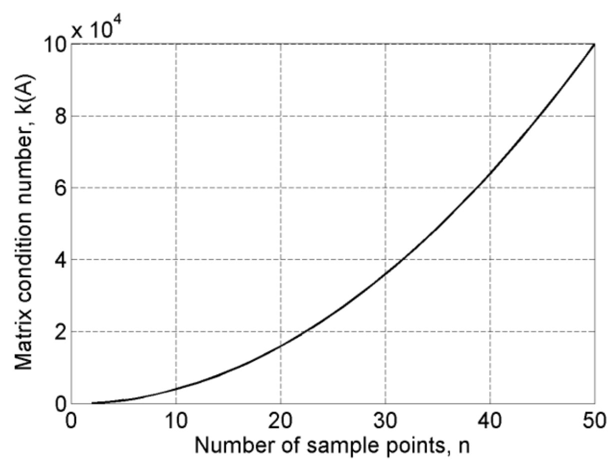

Fig. 3. Stability analysis of the proposed method.

B. Solution of 2D electrostatic potential

Finding the electrostatic potential inside a 2D charge free area is considered. The region is bounded by three conducting rods maintained at constant voltages. Figure

4 shows the physical arrangement of the problem.

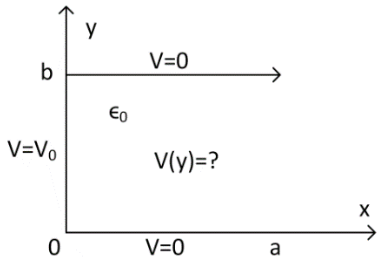

Fig. 4. Cross sectional figure of 2D electrostatic potential example.

The governing equation for this problem is the "Laplace equation", given in rectangular coordinates as:

$$\frac{\partial^2 V(x,y)}{\partial x^2} + \frac{\partial^2 V(x,y)}{\partial y^2} = 0. \tag{24}$$

The analytic solution of the problem is obtained by separating the solution into the product of functions that are only dependent on a single coordinate variable; this procedure is known as the "separation of variables". Thus, we write the solution as:

$$V(x,y) = X(x)Y(y), \tag{25}$$

where $X(x)$ and $Y(y)$ are the solutions of the following ordinary differential equations:

$$\frac{d^2 X(x)}{dx^2} + k_x^2 X(x) = 0, \tag{26}$$

and

$$\frac{d^2 Y(y)}{dy^2} + k_y^2 Y(y) = 0, \tag{27}$$

where k_x and k_y are so called the separation constants.

Let us assume that the following boundary conditions are given. In the y-direction:

$$\frac{\partial Y(0)}{\partial y} = \frac{\pi}{b}, \quad \frac{\partial Y(b)}{\partial y} = -\frac{\pi}{b}, \tag{28}$$

and in the x-direction:

$$\frac{\partial X(0)}{\partial x} = \frac{-j\pi}{b}, \quad \frac{\partial X(\infty)}{\partial x} = 0. \tag{29}$$

The analytic solutions of (26) and (27) subjected to Neumann boundary conditions (28) and (29) are given as [10]:

$$X(x) = D_2 e^{-k_x x}, \tag{30}$$

and

$$Y(y) = A_1 \sin(k_y y), \tag{31}$$

where $k_x = jk$, $k_y = k$, and $k = \frac{m\pi}{b}$ for $m = 1$.

The solution of the problem is given by:

$$V(x,y) = \frac{4V_0}{\pi} e^{-\frac{\pi x}{b}} \sin(\frac{\pi}{b} y),$$
$$x > 0, \quad 0 < y < b, \quad and\ m = 1, \tag{32}$$

where $A_1 D_2 = \frac{4V_0}{m\pi}$.

The general solution of the problem includes all the values for the constant m, and is given by:

$$V(x,y) = \sum_{m=1}^{\infty} C_m e^{-\frac{m\pi x}{b}} \sin\left(\frac{m\pi}{b} y\right),$$
$$m = 1, 3, 5, \ldots, \quad x > 0\ and\ 0 < y < b, \tag{33}$$

where $C_m = A_1 D_2 = \frac{4V_0}{m\pi}$.

Figure 5 and Fig. 6 show the solutions of (26) and (27) subjected to (28) and (29) obtained by the proposed algorithm for the variables $X(x)$ and $Y(y)$ respectively. The following parameters are used in the simulation: $a = 2(m)$, $b = 1(m)$, $V_0 = \frac{\pi}{4}(V)$, and $m = 1$. The results are in good agreement with the exact solution especially for greater number of samples.

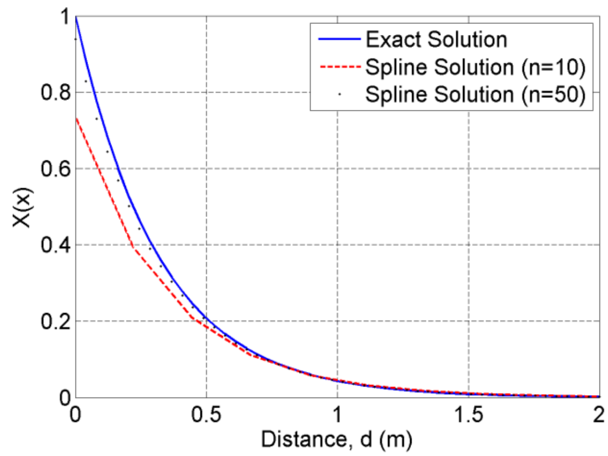

Fig. 5. Simulation results for the partial solution $X(x)$ of the voltage distribution inside a 2D area bounded by electrodes.

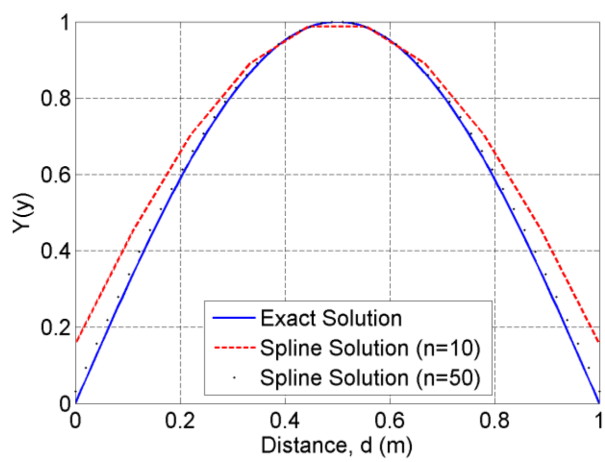

Fig. 6. Simulation results for the partial solution $Y(y)$ of the voltage distribution inside a 2D area bounded by electrodes.

Figure 7 shows the total numeric solution as in (33) obtained by the proposed algorithm for the truncated summation of the first 100 terms.

The stability analysis for the problem in (26) is given in Fig. 8. The results show that this problem can be considered to be well-posed especially for lower number of samples.

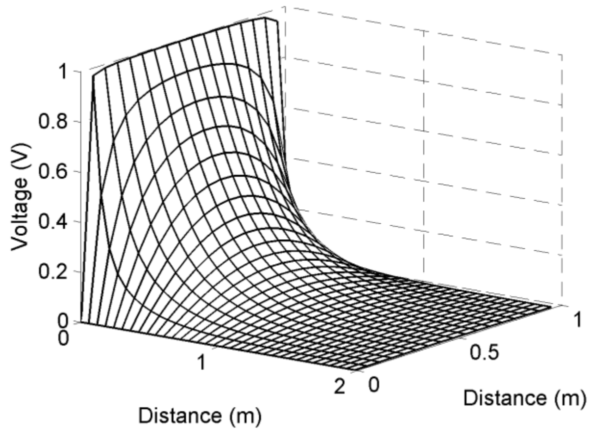

Fig. 7. Simulation results for the voltage distribution inside a 2D area bounded by electrodes.

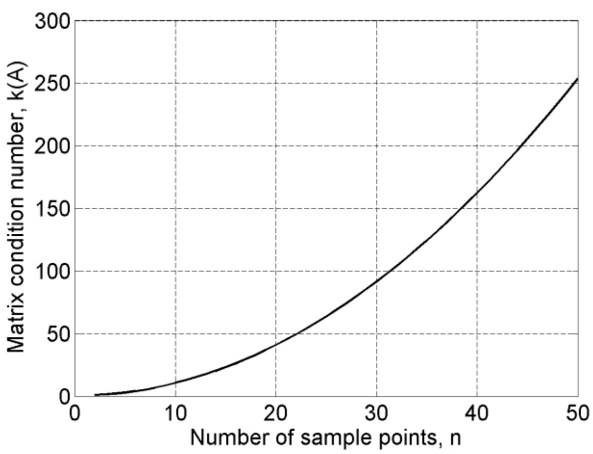

Fig. 8. Stability analysis of the proposed method.

IV. CONCLUSION

The non-polynomial spline approximation has been used to solve 2nd order Neumann type bvp's in electrostatic problems. The method is applicable to linear 2nd order bvp's given by (1), and has proven to give accurate results even for ill-conditioned problems such as the capacitor problem in Section III. The proposed method has been applied specifically to electrostatic problems, although it can also be used to solve any electromagnetic bvp's governed by 2nd order Neumann type differential equations.

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