Fast Calculation of the Filamentary Coil Impedance Using the Truncated Region Eigenfunction Expansion Method

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Abstract **─** The paper presents a mathematical model of an ideal filamentary coil with a finite number of turns, derived by means of the method called truncated region eigenfunction expansion (TREE). The proposed solution allows quick computation of the filamentary coil impedance as well as of the impedance changes caused by the presence of a two-layered conductive material. The final formulas were presented in the closed form and implemented in Matlab. The results were verified using the finite element method in the COMSOL Multiphysics package as well as by means of other mathematical models. In all cases they show a very good agreement. The obtained values of coil impedance changes were compared in terms of the time of reaching the final results. In the case of the most significant calculations, which consisted of many iterations, the proposed solution turned out to be by far the fastest one.

Index Terms **─** Eddy current testing, impedance calculation, single turn coil, truncated region eigenfunction expansion.

I. INTRODUCTION

Mathematical models of probes are applied in eddy current testing, both in the process of interpreting the results and in calculating the values of the measuring system parameters. The derivation of expressions describing a change in coil impedance makes it possible to obtain information about electrical and geometrical properties of the workpiece. Such an opportunity can be used to detect flaws in materials being examined, derive the thickness of coating or for electrical conductivity measurements.

What is highly useful for the optimum choice of the probe's geometrical dimensions or creating a scale of the measuring device are mathematical models of the ideal filamentary coil. Such a coil, shown in Fig. 1, is made of N infinitely thin turns concentrated in a circle of radius r_0 and situated at a distance h_0 from the surface of the investigated material. According to the method described in [1], any cylindrically symmetric coil used for eddy current tests can be experimentally associated with a filamentary coil with the same number of turns and with the corresponding parameters r_0 and h_0. The authors successfully apply this method to device calibration and in eliminating the influence of undesired factors on the test result. Probes of very different structures are compared using an ideal coil that has only three parameters: equivalent radius r_0, equivalent distance h_0 and the number of turns N. On the basis of the proposed mathematical model of such a coil, it is possible, for any real coil, to apply the same mechanism of calculating the measured values and eliminating the influence of factors that disturb the measurement. Complex and time-consuming calculations are replaced with much faster ones performed for the ideal filamentary coil.

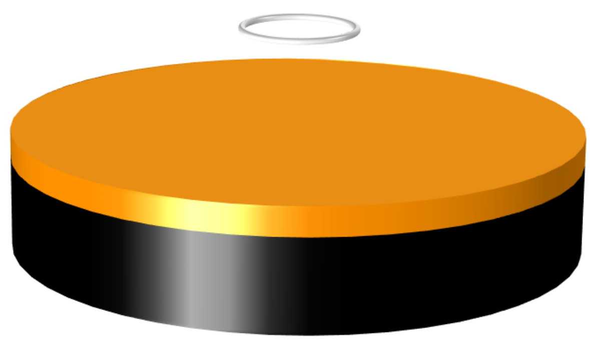

Fig. 1. Filamentary coil located above a two-layered conductive half-space.

A single turn coil situated above a conductive half-space was analyzed by Cheng [2] and then by Dodd and Deeds [3] using a computer program. In subsequent papers, the Legendre functions Q, elliptic integrals E and K [4] and the perturbation method [5] were applied. The problem of the ideal filamentary coil with N turns was presented in [6]. The final formulas describing the change in coil impedance due to the presence of a conductive half-space were derived using the Hankel transform. These expressions were verified many times,

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they were thoroughly examined and used, inter alia, in [7].

In the present paper a mathematical model, created by the aid of the Truncated Region Eigenfunction Expansion (TREE) method, of the filamentary coil with N turns situated above a two-layered conductive material, was proposed. The domain of the problem was truncated to a cylinder of radius b. The final formulas for coil impedance were presented using matrix notation not containing integrals and were implemented in Matlab. The results were verified by the finite element method (FEM) in the COMSOL Multiphysics package and by means of other mathematical models. The obtained values showed a very good agreement in all cases and the time of making calculations based on the proposed method turned out to be the shortest.

II. SOLUTION

The problem illustrated in Fig. 2 was solved by the TREE method described in detail in [8] and applied in [9-13]. The filamentary coil composed of N turns concentrated in a circle of radius r_0 was situated at a distance h_0 from the surface of a two-layered conductive material with relative permeability μ_3, μ_4 and electrical conductivity σ_3, σ_4. The conductive material has the shape of a cylinder whose radius has been truncated to the b parameter value. The problem was split into 4 regions for which the magnetic vector potential A_φ was written using a series:

$$A_1(r,z)=\sum_{i=1}^{Ns} J_1(q_i r)\ e^{-q_i z} C_{1i}, \tag{1}$$

$$A_2(r,z)=\sum_{i=1}^{Ns} J_1(q_i r)(e^{-q_i z} C_{2i}+e^{q_i z} B_{2i}), \tag{2}$$

$$A_3(r,z)=\sum_{i=1}^{Ns} J_1(q_i r)(e^{-s_{3i} z} C_{3i}+e^{s_{3i} z} B_{3i}), \tag{3}$$

$$A_4(r,z)=\sum_{i=1}^{Ns} J_1(q_i r)\ e^{s_{4i} z} B_{4i}. \tag{4}$$

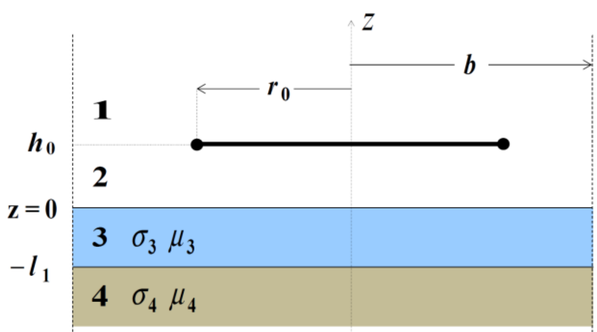

Fig. 2. Rectangular cross-sectional filamentary coil located above a two-layered conductive half-space.

Discrete eigenvalues q_i and coefficients s_{3i}, s_{4i} were computed from equations (5)-(7):

$$J_1(q_i b)=0, \qquad i=0,1,2\ldots Ns. \tag{5}$$

$$s_{3i}=\sqrt{q_i^2+j\omega\mu_3\mu_0\sigma_3}, \tag{6}$$

$$s_{4i}=\sqrt{q_i^2+j\omega\mu_4\mu_0\sigma_4}. \tag{7}$$

At the next stage, the magnetic vector potential A_φ, expressed in (1)-(4) by a series, was written for every region of the problem using matrix notation:

$$\mathrm{A}_1(r,z)=J_1(\mathbf{q}^{\mathbf{T}} r)\ e^{-\mathbf{q}z}\mathbf{C}_1, \tag{8}$$

$$\mathrm{A}_2(r,z)=J_1(\mathbf{q}^{\mathbf{T}} r)(e^{-\mathbf{q}z}\mathbf{C}_2+e^{\mathbf{q}z}\mathbf{B}_2), \tag{9}$$

$$\mathrm{A}_3(r,z)=J_1(\mathbf{q}^{\mathbf{T}} r)(e^{-\mathbf{s}_3 z}\mathbf{C}_3+e^{\mathbf{s}_3 z}\mathbf{B}_3), \tag{10}$$

$$\mathrm{A}_4(r,z)=J_1(\mathbf{q}^{\mathbf{T}} r)\ e^{\mathbf{s}_4 z}\mathbf{B}_4, \tag{11}$$

where $J_1(\mathbf{q}^{\mathbf{T}} r)$ are Bessel functions in the form of row vectors, $\mathbf{q}, \mathbf{s}_3, \mathbf{s}_4, e^{\pm\mathbf{q}z}, e^{\pm\mathbf{s}_3 z}, e^{\mathbf{s}_4 z}$ are diagonal matrices, $\mathbf{C}_i$, $\mathbf{B}_i$ are column vectors of unknown coefficients.

The continuity of the B_r and H_z components on the interfaces between neighboring regions of the problem was ensured after satisfying the following conditions for the magnetic vector potential.

$$A_m(r,z)=A_{m+1}(r,z), \qquad m=1,2,3. \tag{12}$$

$$\frac{1}{\mu_m}\frac{\partial A_m}{\partial z}-\frac{1}{\mu_{m+1}}\frac{\partial A_{m+1}}{\partial z}=-\mu_0 I\delta(r-r_0),\ m=1,2,3. \tag{13}$$

where $\mu_0 I\delta(r-r_0)$ is current density.

By solving a system of six interface equations, the $\mathbf{C}_i$ and $\mathbf{B}_i$ coefficients were derived and, subsequently, they were used to write an expression for the magnetic vector potential of the filamentary coil with N turns.

$$A(r_0,h_0)=I\mu_0 r_0 N^2\mathbf{q}^{-1}\left[\frac{J_1(\mathbf{q}r_0)}{J_0(\mathbf{q}b)\mathrm{b}}\right]^2\left(1+e^{\mathbf{q}h_0}\frac{\mathbf{C_2}}{\mathbf{B_2}}\right), \tag{14}$$

where

$$\mathbf{C_2}=\frac{1}{2}[\mathbf{C_3}(1+\frac{\mathbf{s_3}}{\mathbf{q}\mu_3})+\mathbf{B_3}(1-\frac{\mathbf{s_3}}{\mathbf{q}\mu_3})], \tag{15}$$

$$\mathbf{B_2}=\frac{1}{2}[\mathbf{C_3}(1-\frac{\mathbf{s_3}}{\mathbf{q}\mu_3})+\mathbf{B_3}(1+\frac{\mathbf{s_3}}{\mathbf{q}\mu_3})], \tag{16}$$

$$\mathbf{C_3}=\frac{1}{2}e^{-\mathbf{s_3}l_1}e^{-\mathbf{s_4}l_1}(1-\frac{\mathbf{s_4}\mu_3}{\mathbf{s_3}\mu_4}), \tag{17}$$

$$\mathbf{B_3}=\frac{1}{2}e^{\mathbf{s_3}l_1}e^{-\mathbf{s_4}l_1}(1+\frac{\mathbf{s_4}\mu_3}{\mathbf{s_3}\mu_4}). \tag{18}$$

The general formula for coil impedance can be shown in the following form:

$$Z=\frac{j\omega\,2\pi r_0\,A(r_0,h_0)}{I}. \tag{19}$$

By setting (14) in (19), an expression describing the impedance of the filamentary coil placed above the two-layered conductive material was obtained:

$$Z = j\omega\, 2\pi\, \mu_0\, r_0\, N^2 \mathbf{q}^{-1} \left[\frac{J_1(\mathbf{q}\, r_0)}{J_0(\mathbf{q}\, b)\, \mathrm{b}} \right]^2 \left(1 + e^{-2\mathbf{q} h_0} \frac{e^{-\mathbf{s}_3 l_1} \mathbf{k_1 k_3} + e^{\mathbf{s}_3 l_1} \mathbf{k_2 k_4}}{e^{-\mathbf{s}_3 l_1} \mathbf{k_1 k_4} + e^{\mathbf{s}_3 l_1} \mathbf{k_2 k_3}} \right), \tag{20}$$

where

$$\mathbf{k_1} = \mathbf{s_3}\mu_4 - \mathbf{s_4}\mu_3, \tag{21}$$

$$\mathbf{k_2} = \mathbf{s_3}\mu_4 + \mathbf{s_4}\mu_3, \tag{22}$$

$$\mathbf{k_3} = \mathbf{q}\mu_3 + \mathbf{s_3}, \tag{23}$$

$$\mathbf{k_4} = \mathbf{q}\mu_3 - \mathbf{s_3}. \tag{24}$$

The change in the filamentary coil impedance ΔZ due to the presence of the two-layered conductive material is represented by the second addend in (20) which can be written as:

$$\Delta Z = j\omega\, 2\pi\, \mu_0\, r_0\, N^2 \mathbf{q}^{-1} \left[\frac{J_1(\mathbf{q}\, r_0) e^{-\mathbf{q} h_0}}{J_0(\mathbf{q}\, b)\, \mathrm{b}} \right]^2 \frac{e^{-2\mathbf{s_3} l_1} \mathbf{k_1 k_3} + \mathbf{k_2 k_4}}{e^{-2\mathbf{s_3} l_1} \mathbf{k_1 k_4} + \mathbf{k_2 k_3}}. \tag{25}$$

In the case shown in Fig. 3 in which the conductive material consists of one layer only, we obtain: $\mathbf{s_3} = \mathbf{s_4}$, $\mu_3 = \mu_4$, $\sigma_3 = \sigma_4$, $l_1 = 0$ and equation (25) is reduced to the form:

$$\Delta Z = j\omega\, 2\pi\, \mu_0\, r_0\, N^2 \mathbf{q}^{-1} \left[\frac{J_1(\mathbf{q}\, r_0) e^{-\mathbf{q} h_0}}{J_0(\mathbf{q}\, b)\, \mathrm{b}} \right]^2 \frac{\mathbf{k_4}}{\mathbf{k_3}}. \tag{26}$$

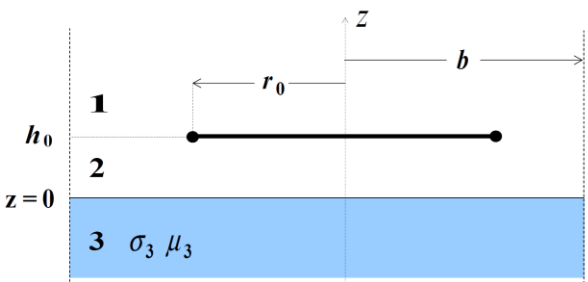

Fig. 3. Rectangular cross-sectional filamentary coil located above a conductive half-space.

III. COMPARISON WITH OTHER MODELS

The verification of obtained results was conducted by the aid of 3 mathematical models. The first one was created in the COMSOL Multiphysics package in which the finite element method is used in calculations. In region 2, between the coil and the surface of the investigated material, a mesh that consisted of around 20000 triangular elements and 400 edge elements was adaptively refined.

Calculations were made also by extending the mathematical model of a single turn coil proposed by Cheng [2]. Taking into consideration a finite number of turns N and a conductive half-space consisting of two layers, a change in the impedance of such a coil was written in the following form:

$$\Delta Z = j\omega\, \pi\, \mu_0\, r_0^2\, N^2 \int_0^{\infty} J_1^2(q\, r_0) e^{-2q h_0} \frac{e^{-2\mathbf{s_3} l_1} \mathbf{k_1 k_3} + \mathbf{k_2 k_4}}{e^{-2\mathbf{s_3} l_1} \mathbf{k_1 k_4} + \mathbf{k_2 k_3}} dq. \tag{27}$$

In the third mathematical model, described in [7], the infinite integration range was replaced with a sum of integrals whose boundaries were zeros of the Bessel function $J_1(x)$ normalized in relation to the β parameter. As a consequence, the integration in (28) is performed many times but only for relatively small intervals.

$$\Delta Z = j\omega\, \pi\, \mu_0\, r_0 N^2 \beta \sum_{k=0}^{Ns} \int_{\lambda_k}^{\lambda_{k+1}} e^{-\alpha\beta q} J_1^2(\beta q) \frac{(f_1 - f_2)(q + f_2)\mathrm{e}^{\Psi} + (f_1 + f_2)(f_2 - q)}{(f_1 - f_2)(q - f_2)\mathrm{e}^{\Psi} - (f_1 + f_2)(f_2 + q)} dq, \tag{28}$$

where

$$\alpha = \frac{2h_0}{r_0}, \tag{29}$$

$$\beta = r_0 \sqrt{\omega\, \mu_0 \sigma_3}, \tag{30}$$

$$f_1 = \sqrt{q^2 + j\frac{\sigma_4}{\sigma_3}}, \tag{31}$$

$$f_2 = \sqrt{q^2 + j}, \tag{32}$$

$$\lambda_k = \frac{q_k}{\beta}, \tag{33}$$

$$\Psi = -2 l_1\, f_2 \sqrt{\omega\, \mu_0\, \sigma_3}. \tag{34}$$

Expressions (25), (27) and (28) were implemented in Matlab where the Newton-Raphson method was applied to determine the zeros of the Bessel function $J_1(x)$. The obtained values of coil impedance change were compared with the results from the COMSOL package. The relative difference of resistance δ_R and the relative difference of reactance δ_X were used for this purpose:

$$\delta_R = \frac{\Delta R_{COMSOL} - \Delta R_{MATLAB}}{\Delta R_{MATLAB}} \cdot 100\%, \tag{35}$$

$$\delta_X = \frac{\Delta X_{COMSOL} - \Delta X_{MATLAB}}{\Delta X_{MATLAB}} \cdot 100\%. \tag{36}$$

IV. RESULTS

The calculations of the coil impedance changes $\Delta Z = \Delta R + j\,\Delta X$ were carried out using expression (25) for 50 frequency values from the range 100 Hz to 100 kHz. The parameters of the coil and of the two-layered conductive material are presented in Table 1. Calculations were also made for the second coil of radius r_0 = 12 mm. The results, normalized in relation to reactance X_0 and verified in the COMSOL package, are shown in Fig. 4 and 5. The difference between the ΔZ values obtained using the TREE and the FEM methods did not exceed in any case 0.2 %.

Table 1: Parameters of the coil and plate used in calculations

Number of turns	N	100
Coil radius	r_0	8 mm
Parameter	h_0	1 mm
Parameter	l_1	1.5 mm
Conductivity	σ_3	57 MS/m
Conductivity	σ_4	15.9 MS/m
Relative permeability	μ_3	1
Relative permeability	μ_4	1
Summation terms	Ns	150
Radius of the domain	b	10 r_0

The calculations for the filamentary coil of radius r_0 = 8 mm were performed also with expressions (27) and (28). In addition, for the TREE method, the second set of parameter values was applied, assuming Ns = 25 and $b = 5r_0$. The obtained ΔZ results for the frequency f = 1 kHz and f =100 kHz are shown in Table 2 and the times of calculations for each of the mathematical models are included in Table 3. The changes in the filamentary coil impedance were derived for 1 and 10000 different frequency values, respectively, using a computer with an Intel Pentium E2220 2.4 GHz processor equipped with the 4 GB RAM.

It results from the data shown in Tables 2 and 3 that all the mathematical models that have been used make it possible to derive changes of the filamentary coil impedance with a very high degree of accuracy. In such a situation it is the fulfillment of the requirements regarding the time of obtaining the final results that is becoming the key aspect which determines the usefulness of a given mathematical model. The calculations led to the conclusion that the model created using the TREE method turned out to be by far the fastest one. Its advantage over the other solutions is most visible with a large number of iterations. It is possible to obtain results in such a short time thanks to precomputations. In the first iteration all calculations are performed and in the subsequent ones only those that depend on the variable input parameter.

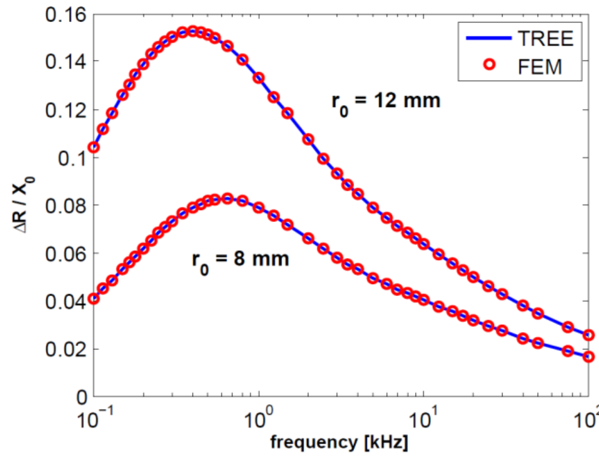

Fig. 4. Real part of the normalized impedance change as a function of frequency for filamentary coil.

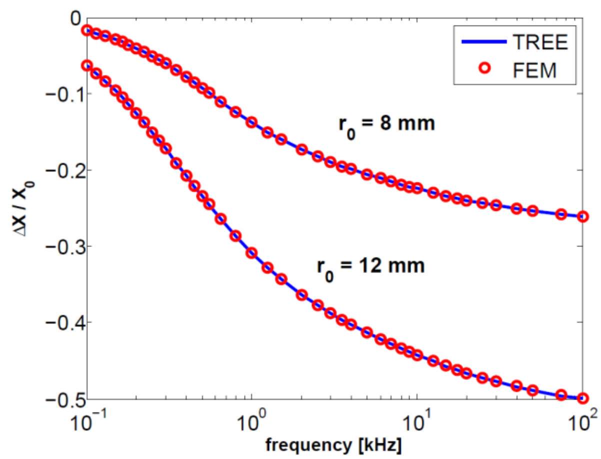

Fig. 5. Imaginary part of the normalized impedance change as a function of frequency for filamentary coil.

Table 2: Values of changes in the filamentary coil impedance

	ΔZ (Ω)					
	f = 1 kHz	δ_R [%]	δ_X [%]	f = 100 kHz	δ_R [%]	δ_X [%]
FEM	0.267 – j 0.467	---	---	5.561 – j 88.596	---	---
Eq. (27)	0.267 – j 0.468	-0.01	-0.18	5.561 – j 88.680	0.00	-0.10
Eq. (28)	0.267 – j 0.468	0.01	-0.18	5.557 – j 88.676	0.07	-0.09
TREE Eq. (25) Ns = 150 $b = 10r_0$	0.267 – j 0.467	-0.01	-0.01	5.561 – j 88.601	0.00	-0.01
TREE Eq. (25) Ns = 25 $b = 5r_0$	0.267 – j 0.462	0.03	1.21	5.503 - j 87.936	1.05	0.75

Table 3: Comparison of calculation times for different mathematical models

	Computation Time (s)	
	1 Iteration	10 000 Iterations
FEM	7	58148
Eq. (27)	0.06	233.8
Eq. (28)	0.07	70.1
TREE Eq. (25) $Ns = 150$ $b = 10r_0$	0.04	10.2
TREE Eq. (25) $Ns = 25$ $b = 5r_0$	0.03	3.2

In all the mathematical models being compared it is possible to shorten the time of making calculations at the expense of the result accuracy. In the COMSOL package the computations may be accelerated by reducing the number of mesh elements and in expressions (27) and (28) by diminishing the accuracy of the numerical integration procedure. In both cases errors in the derivation of impedance will be significantly greater. In case when the FEM software is used the computation time can be reduced as well by execution of preliminary calculations, e.g., by means of the perturbation method. In the TREE method the time of obtaining results depends primarily on the matrix size specified by the parameter *Ns*. The number of matrix elements, indeed, determines the number of arithmetic operations carried out in the computer program. Increasing the *Ns* value makes the calculations longer but at the same time it reduces the error and requires a larger solution domain defined by the parameter *b*. Both when the *Ns* value is too large and when it is too small with regard to the parameter *b*, the results are affected by significant error. The way how these parameters are selected, being usually a compromise between the computation time necessary to achieve the desired results and the acceptable error, is described in [8] more in details.

V.CONCLUSION

The paper presents a mathematical model of the ideal filamentary coil with *N* turns situated above a two-layered conductive material. An expression that describes the change in the impedance of such a coil due to the presence of the investigated material was derived by means of the TREE method. The ΔZ values calculated by applying the proposed solution were verified by means of the finite element method and the difference did not exceed in any case 0.2%. The time of obtaining the final results was compared with another 3 mathematical models. The application of precomputation and the replacement of integration with matrix operations made it possible to derive impedance changes in a significantly shorter time than using the other solutions. Such a difference was particularly visible in the case of the most relevant calculations composed of many iterations. The mathematical model shown in the present paper can be implemented directly in an eddy current device. It can be used to create a scale of the measuring device as an equivalent for real coils of any structure, too.

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