

# A Simple Analytical Method to Calculate Bending Loss in Dielectric Rectangular Waveguides

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**Abstract** — We present a simple analytical method to compute attenuation in bent dielectric rectangular waveguides. An approximate formulation for the attenuation constant is first derived by determining the ratio of average power loss per unit length to the average power propagating along the waveguide. Since the waveguide has been simplified into a slab in the process of derivation, losses at the four edges of the structure have been neglected. To account for these losses, the perturbation theory has been employed. The total loss is found to agree closely with that obtained via the Finite Element Method (FEM). Unlike the FEM which requires considerable computational time and power to solve, we demonstrate that the analytical method proposed here can easily be applied and it gives sufficiently accurate result.

**Index Terms** — Analytical method, attenuation constant, bending loss, perturbation theory, propagation constant, rectangular waveguides.

## I. INTRODUCTION

Dielectric waveguides, such as optical fibers, have been widely used in the fields of telecommunication and integrated optics to channel signal from one end to another. In a dielectric waveguide, the core dielectric rod is immersed in one or more layers of dielectric materials which are of lower index of refraction  $n_2$  than the core material itself  $n_1$ . This allows wave to propagate in the waveguide based on the principle of total internal reflection, described by Snell's law [1, 2]. In order to ensure that the information carried by the modulating signal is preserved, it is important to minimize losses in the waveguide during wave propagation. The losses in a dielectric waveguide can generally be classified into dielectric loss and radiation loss. In a uniformly straight waveguide, the fields are mostly confined within the core of the waveguide. Hence, radiation loss is practically negligible in the waveguide. When certain curvatures occur in the waveguide, however, wave with angles of incident exceeding the critical angle tend to radiate out from the guiding structure [3]. Because of this reason,

radiation loss or more commonly known as bending loss, in this case, can no longer be ignored. Since bent structures are inevitable when channeling the signal, both dielectric and bending losses are equally important when estimating the total loss in the waveguide. Developing mathematical expression to describe the presence of curvatures in a rectangular waveguide is inherently difficult. This is because a combination of Cartesian and cylindrical coordinates is required so as to define the cross section and the bending radius of the waveguide. Hence, analytical methods, found in most literatures [4-7], focus only on the analyses of uniformly straight waveguides. As can be seen from some of the recent literatures [8-11], computational methods such as Finite Domain Time Difference (FDTD) or Finite Element Method (FEM) are preferred when bending loss is to be accounted for in the calculation of loss. The algorithms used in computational methods discretize the solution space into meshes. The electric field in each mesh is then numerically calculated. Hence, although they produce accurate results, these methods typically consume substantial computational time and power. This is particularly true when fields are to be solved for signals with very small wavelength (such as THz or optical signal) propagating in a three dimensional structure where the number of meshes is exceptionally huge. Analytical methods, on the other hand, are simpler and require relatively less time and power to solve.

Marcatili [3] and Marcuse [12] were among some of the early researchers who had developed analytical solutions for bent rectangular waveguides. In the process of derivation, however, both of them had assumed the fields' radiation at the four corner regions of the waveguide to be negligible. Due to this reason, the loss found using their methods has been underestimated. It is worthwhile noting that, Marcatili's method is only valid when the wave is weakly guided (i.e., the difference between  $n_1$  and  $n_2$  is small), while Marcuse's method is not bounded by this limitation. Hence, Marcuse's method has the advantage of being applied in structures with arbitrary ratio of indices of refraction. In this paper, we present an improvement on the accuracy of

Marcuse's method. We consider a dielectric rectangular waveguide surrounded by homogeneous dielectric material in our study. In order to account for the loss at the corner regions, we incorporate into the formulation the correction factor developed by Deck et al. [13]. This paper shall be presented in such a way that, casual readers could appreciate the final mathematical expressions, without the need of going through the underlying mathematics.

## II. FORMULATION

Figure 1 depicts the structure of a bent rectangular waveguide with width  $a$  and height  $b$ . When deriving the attenuation constant of the waveguide, Marcuse has first assumed the fields at the vicinity of a bent waveguide to be similar to that of a straight guide. The assumption should hold valid as long as the radius of curvature  $R$  is sufficiently large. When deriving the fields' expressions, he has also assumed that there is no field variation in the  $y$  direction. This allows the propagating waves to be described as simple TE and TM waves [12]. According to the law of conservation of energy, the rate of decrease of power is to be equivalent to the power loss. Hence, power loss  $\Lambda$  can be expressed as the ratio of average power loss per unit length  $\Delta p$  to the average power propagating along the waveguide  $p$ , i.e., [1, 12]:

$$\Lambda = \frac{\Delta p}{p}. \quad (1)$$

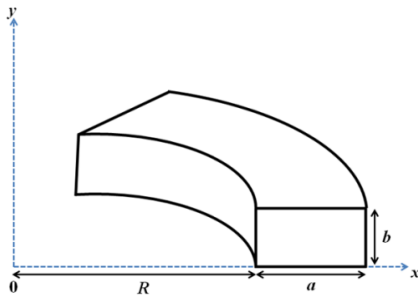


Fig. 1. A bent rectangular waveguide.

By substituting the fields' expression into (1), the loss equation  $\Lambda$  can be obtained as follows [12]:

$$\Lambda = \text{Im} \left[ \frac{2\sqrt{k_z^2 - n_2^2 k_2^2} (n_1^2 k_2^2 - k_z^2)}{k_2^2 k_z (n_1^2 - n_2^2)} \times \exp \left[ a\sqrt{k_z^2 - n_2^2 k_2^2} - \frac{2(k_z^2 - n_2^2 k_2^2)^{1.5} (R + 0.5a)}{3k_z^2} \right] \right] \frac{1}{\left[ a + \frac{1}{\sqrt{k_z^2 - n_2^2 k_2^2}} + \frac{1}{\sqrt{k_z^2 - n_1^2 k_2^2}} \right]} \quad (2)$$

where  $k_2$  is the wavenumber of the dielectric cladding

material and  $k_z$  is the propagation constant, which can be adopted from that of a straight waveguide. It is to be noted that  $k_z$  is a complex variable which comprises a phase constant  $\beta_z$  and an attenuation constant  $\alpha_z$ , i.e.,  $k_z = \beta_z - j\alpha_z$ . Here, we have applied the closed-form expression for  $k_z$ , modified from [14] as shown below:

$$k_z = \left\{ k_1^2 - \left( \frac{m\pi}{a} \right)^2 - \left( \frac{n\pi}{b} \right)^2 + 2(1-j) \frac{\delta\mu_1}{a\mu_2} \left[ \left( \frac{m\pi}{a} \right)^2 + \left( \frac{n\pi}{b} \right)^2 + k_1^2 \right] \right\}^{0.5}, \quad (3)$$

where  $k_1$  is the wavenumber of the core material,  $\mu_1$  and  $\mu_2$  are respectively the permeability of the core and cladding materials; whereas  $m$  and  $n$  are respectively the number of half cycle variations in the  $x$  and  $y$  directions. The skin depth  $\delta$  in (3) is given by [15]:

$$\delta = \frac{2Z_S}{(1+j)\omega\mu_2}, \quad (4)$$

where  $\omega$  is the angular frequency. The surface impedance of the dielectric layer  $Z_S$  can be expressed in terms of the electrical properties of the two mediums as [7, 16]:

$$Z_S = \frac{1}{j\omega b(\epsilon_{r1} - \epsilon_{r2})}, \quad (5)$$

where  $\epsilon_{rd}$  and  $\epsilon_{r0}$  are respectively the relative permittivity of the core and the cladding materials. To account for the loss at the four corner regions, we employ the formulation developed by Deck et al. [13] by means of the perturbation theory. When deriving the correction to the mode propagation and profile function, correction to the dielectric function in the four corner regions is assumed to produce changes in the squared propagation constant and fields profile function [13]. The corner field correction factor  $\Delta\Lambda$  can be expressed as [13]:

$$\Delta\Lambda = \text{Im} \left[ \left( \frac{\epsilon_1 - \epsilon_2}{2} \right) \left( \frac{\omega}{c\gamma} \right)^2 \times \left\{ 1 + \left[ \cos(k_y b) + \left( \frac{\epsilon_1}{\epsilon_2} \right) \left( \frac{\gamma}{k_y} \right) \sin(k_y b) \right]^2 \right\} \times \frac{(2\gamma b + 1)}{\left[ 1 + \left[ \cos(k_y b) + \left( \frac{\epsilon_1}{\epsilon_2} \right) \left( \frac{\gamma}{k_y} \right) \sin(k_y b) \right]^2 \right]} \right] \Lambda \quad (6)$$

where  $\epsilon_1$  and  $\epsilon_2$  are respectively the permittivity of the core and its cladding material,  $\gamma = \left( \frac{\omega}{c} \right)^2 (\epsilon_1 - \epsilon_2) - k_y^2$  and  $k_y$  is the transverse wavenumber in the  $y$  direction. For simplicity, we apply the closed-form expression of  $k_y$  in [6] as shown in (7) below:

$$k_y = \frac{\pi}{b} \left( \frac{\pi m_1^2 b \sqrt{n_1^2 - n_2^2}}{\pi m_1^2 b \sqrt{n_1^2 - n_2^2} + n_2^2 \lambda} \right). \quad (7)$$

The total bending loss  $A_T$  can therefore be determined by including the additional loss found in (6) with the loss in (2), i.e.,  $A_T = A + \Delta A$ .

### III. RESULTS AND DISCUSSION

We compute the loss in a  $2.4 \times 1.3 \text{ mm}^2$  silicon rectangular waveguide, with bending radius  $R = 1 \text{ mm}$ . The conductivities of silicon and the surrounding medium are given as  $4.33 \times 10^{-4} \text{ S/m}$  and  $8.0 \times 10^{-15} \text{ S/m}$ , respectively. To validate the closed-form formulations presented here, we compare the computed results with the S21 parameters found from the Finite Element Method (FEM). The results from FEM are simulated from Ansoft's High Frequency Structure Simulator HFSS. Since S21 accounts for the total loss in the waveguide, we have incorporated the total dielectric loss  $\alpha_z$ , i.e., the imaginary component of (3) together with the bending loss in (2) during comparison. When calculating the loss, we have set  $m = 1$  and  $n = 0$  for the dominant TE mode. It is worthwhile noting that, the loss in a practical waveguide may also be contributed from the imperfection of the waveguide structure. Since the work presented here is a theoretical exercise, such loss has therefore been neglected.

Figure 2 depicts the comparison of loss between our computed result and that obtained from HFSS. It can be seen from the figure that although the curves agree somewhat with each other, the loss from the computed result has clearly been underestimated. The average error with reference to the FEM result  $\epsilon_{ave} = 60.17\%$ . Since Marcuse has neglected the presence of the electric field in the  $x$  direction  $E_x$ , the loss of the  $E^x$  mode has not been taken into account. As shown in [6], the modes propagating in a dielectric waveguide are degenerate – both  $E^y$  and  $E^x$  modes exist concurrently and that the propagation constants of both modes are similar to each other. Figure 3 shows the total loss (i.e., the addition of dielectric and bending losses) when both  $E^x$  and  $E^y$  modes are taken into account. It can be observed from Fig. 4 that the electric fields of the  $E^x$  and  $E^y$  modes are orthogonal to each other. Despite their direction of polarizations, however, the profiles exhibited by both modes are qualitatively similar to each other [3]. Here, we have taken the bending loss exhibited by the  $E^x$  mode to be identical with that by  $E^y$ . The result turns out to be in closer agreement with that obtained from the FEM method, although discrepancy between the results is still apparent ( $\epsilon_{ave} = 44.53\%$ ). Figure 5 shows the final result when the corner field correction factor  $\Delta A$  has been included into our calculation. By considering the loss at the four edges of the waveguide, it can be observed from the figure that the result improves significantly, with

the computed result approaches that of the simulation ( $\epsilon_{ave} = 21.27\%$ ).

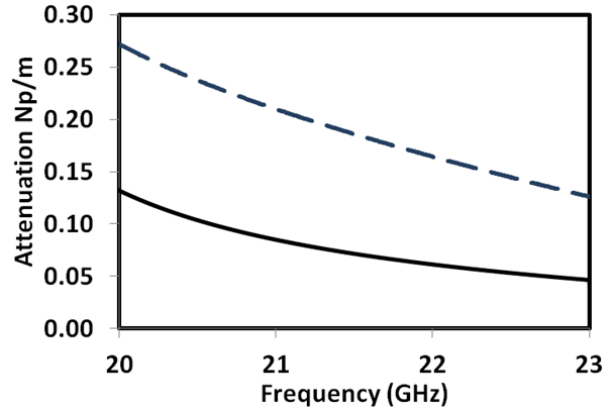


Fig. 2. Loss of a bent rectangular silicon waveguide, obtained from the analytical method proposed here (solid line) and the FEM (dashed line). The analytical method has only considered the dielectric loss and the bending loss from the  $E^y$  mode (loss at the corner regions has been neglected).

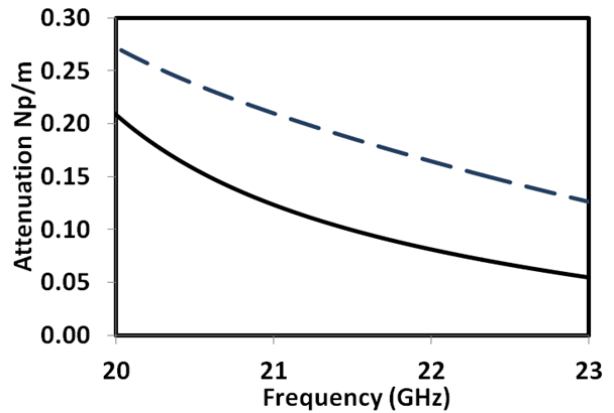


Fig. 3. Loss of a bent rectangular silicon waveguide, obtained from the analytical method proposed here (solid line) and the FEM (dashed line). The analytical method has only considered the dielectric loss and the bending loss from the  $E^y$  and  $E^x$  modes (loss at the corner regions has been neglected).

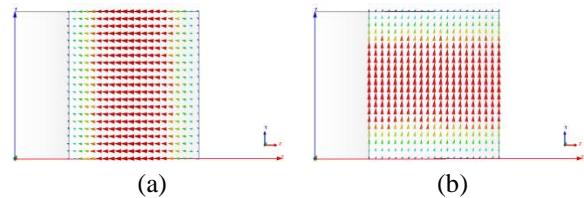


Fig. 4. Electric field lines of: (a)  $E^x$  and (b)  $E^y$  modes at the cross section of the rectangular waveguide.

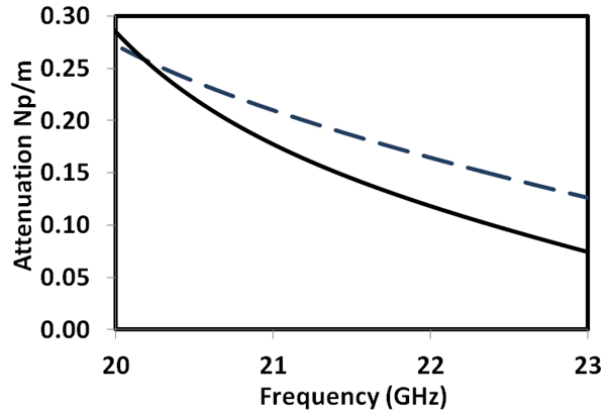


Fig. 5. Loss of a bent rectangular silicon waveguide, obtained from the analytical method proposed here (solid line) and the FEM (dashed line). The analytical method has taken into account the dielectric loss, as well as, the bending loss from the  $E^y$  and  $E^x$  modes (loss at the corner regions has been included).

#### IV. CONCLUSION

We have presented a closed-form analytical method to predict the attenuation in a bent dielectric rectangular waveguide. The dielectric loss in the waveguide can be extracted from the propagation constant obtained from a straight waveguide; whereas, the bending loss in the waveguide is determined from Marcuse's approximate method [12]. To enhance the accuracy of Marcuse's method, the correction factor in [13] has been applied to account for the loss at the corner regions. By including the bending loss exhibited by both  $E^y$  and  $E^x$  modes and the dielectric loss, the result is found to agree closely with that computed using the rigorous computational method. Since the formulations presented here are all in closed-form, it is not necessary to rely on computational intensive machines, such as a computer to calculate them. Besides being straight forward, the method also produces results which can be easily found; while at the same time, sufficiently accurate.

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