High-order Staggered Finite Difference Time Domain Method for Dispersive Debye Medium

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Abstract — In this paper, a high order accuracy Finite Difference Time Domain method was proposed for the simulation of electromagnetic waves in the Debye dispersive medium. The proposed method was based on the use of the third order Backward Differentiation scheme for the approximation of the time derivatives and the use of the fourth order Central Finite Difference scheme for the approximation of space derivatives. The stability of the present method was analyzed by using the Root-Locus method. The accuracy of the proposed method was analyzed in the case of free space and the dispersive media, in the case of plane wave and the case of a Hertzian dipole source.

The proposed method offered high performance regarding the accuracy and the stability in comparison with the other methods.

Index Terms — Accuracy, backward differentiation, central finite difference, Debye model, dispersive media, finite difference time domain, stability.

I. INTRODUCTION

Since the Finite-Difference-Time-Domain (FDTD) has been proposed [1], it has been widely used for the simulation of the electromagnetic (EM) waves behavior within frequency dependent media such as saline water [2], human tissues [3-5], and plasma [6,7]. For modeling these frequency dependent media, many dispersive models have been proposed, such as the multi-pole Debye, Lorentz, and Drude model. The use of the conventional FDTD for the simulation of these models leads to a lack of accuracy and complexity of the stability analysis, especially when multiple poles are used. Therefore, the development of more accurate and stable FDTD based schemes acquire a great interest in the scientific society.

Many techniques have been proposed for the numerical implementation of the dispersive models into the FDTD method. Those methods can be grouped into three main categories: Z-transformation methods, recursive convolution methods, and the Auxiliary Differential Equation (ADE) methods.

The Z-transform (ZT) methods are based on the digital filtering theory [8,9]. The transfer function of the dielectric permittivity is converted from the frequency domain to the Z-domain, then the actual update equations in the discrete time domain are obtained. The recursive convolution (RC) methods are based on writing the dispersion equation as a convolution product in the time domain, then using the discrete recursive integrator for the actual update equations. Among these methods, we find the trapezoidal recursive convolution technique (TRC) and the Piecewise linear recursive convolution method (PLRC) [10-12]. The ADE methods are based on writing the dispersion relations under the form of differential equations. Then using the finite difference schemes to obtain the actual update equations [13-15].

Contrary to the RC and ZT methods which are in their improved versions limited to the second order of precision, the ADE methods offer more flexibility regarding the implementation of dispersive media while using a higher order of accuracy.

Therefore, seeking for improvement of the accuracy order and the stability condition, many techniques among the ADE category have been proposed. The Alternating Direction Implicit (ADI) FDTD [16,17] have been widely used to guarantee unconditional stability. Also, it saves both the memory and time consumption. However, this technique suffers from inaccuracies especially for a high Courant–Friedrichs–Levy (CFL) number. Another method based on the fact that Maxwell’s equations can be written under the form of a symplectic integrator has been used to improve the accuracy of the FDTD method in many researches [18-20]. The symplectic method offers high performances in both stability and precision. However, it consists of repetitive loops inside one-time iteration. this leads to cumbersome the CPU use and then slows down the execution. Previously in the literature, Fang [21] proposed two high order FDTD methods for solving Maxwell’s equations. The staggered FDTD (2,4) and FDTD (4,4) (where the first and the second index refer to the time and space accuracy order, respectively) are more accurate than the previously mentioned methods. Then, the FDTD (2,4) has been widely applied for frequency dependent media [22]. However, the FDTD

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(4,4) did not get a significant focus in the simulation of the frequency dependent media, regarding its complexity.

We have proposed a high order FDTD (3,4) for the non-dispersive media [23], and in this paper, the method has been extended for the simulation of a frequency dependent media which consisted of the combination of a multi-pole Debye model with a lossy conductive model. An analysis of the accuracy and the stability proved that the proposed method offered high performance in comparison with the previously cited methods.

II. FORMULATIONS

Consider the time-dependent form of Maxwell's equations for a homogeneous Debye dispersive medium:

$$\frac{\partial \vec{D}}{\partial t} = \nabla \times \vec{H} \ , \tag{1.a}$$

$$\frac{\partial \vec{B}}{\partial t} = \frac{\partial}{\partial t}\left(\mu \vec{H}\right) = -\nabla \times \vec{E} \ . \tag{1.b}$$

Together with the dispersive permeability relationship:

$$\vec{D} = \varepsilon * \vec{E} \ , \tag{2}$$

where $\vec{E}$ and $\vec{H}$ are the electric and magnetic fields; $\vec{D}$ and $\vec{B}$ are the electric and the magnetic flux density; ε and μ are the dispersive permittivity and permeability of the medium; * denotes the convolution operator.

The permittivity of a multi-pole Debye model combined with a static conductivity factor in the frequency domain is expressed as follows:

$$\varepsilon(\omega) = \varepsilon_0 \left(\varepsilon_\infty + \sum_{k=1}^{K} \frac{\Delta\varepsilon_k}{1 + j\tau_k \omega} + \frac{\sigma_s}{j\omega\varepsilon_0} \right), \tag{3}$$

where $j = \sqrt{-1}$; ω is the angular frequency; ε_0 is the free space permittivity; ε_∞ is the permittivity at the infinite frequency; K is the number of Debye poles; $\Delta\varepsilon_k$ and τ_k are the k^{th} Debye pole's magnitude and the relaxation time, respectively; σ_s is the static conductivity.

The permittivity relation in Equation (3) can be expressed as a system of ADE as follows:

$$\vec{D}(t) = \varepsilon_0 \varepsilon_\infty \vec{E}(t) + \sum_{k=1}^{K} \vec{P_k}(t) + \vec{P_{loss}}(t), \tag{4.a}$$

$$\vec{P_k}(t) + \tau_k \frac{\partial \vec{P}}{\partial t}(t) = \varepsilon_0 \Delta\varepsilon_k \vec{E}(t), \tag{4.b}$$

$$\frac{\partial \vec{P_{loss}}}{\partial t}(t) = \sigma_s \vec{E}(t), \tag{4.c}$$

where K is the number of the Debye poles; P_k is the polarization density of the k^{th} Debye pole; P_{loss} is the conductive loss term.

The computational domain is discretized by using the Yee's staggered grid. Δx, Δy, and Δz denote the spatial step; Δt is the time increment. For a given function F which can be one of the electric or magnetic field's component, the discrete is defined form as follows:

$$F_{i,j,k}^{n} = F(i\Delta x, j\Delta y, k\Delta z, n\Delta t). \tag{5}$$

The principle of the proposed method is based on applying the fourth order Central Finite Difference scheme for the approximation of the space derivatives in the curl equations, and the third order Backward Differentiation scheme for the approximation of the time derivatives, as shown in the following equations:

$$\left.\frac{\partial F}{\partial t}\right|_{i,j,k}^{n-\frac{1}{2}} = \frac{1}{\Delta t} \sum_{q=0}^{N} a_q F_{i,j,k}^{n-q} + O\left(\Delta t^{N+1}\right), \tag{6.a}$$

$$\left.\frac{\partial F}{\partial t}\right|_{i,j,k}^{n} = \frac{1}{\Delta t} \sum_{q=0}^{N} b_q F_{i,j,k}^{n-q} + O\left(\Delta t^{N+1}\right), \tag{6.b}$$

$$\left.\frac{\partial F}{\partial x}\right|_{i,j,k}^{n} = \frac{1}{\Delta x} \sum_{p=1}^{M/2} w_p \left(F_{i+p-\frac{1}{2},j,k}^{n} - F_{i-p+\frac{1}{2},j,k}^{n} \right) + O\left(\Delta x^{M+1}\right), \tag{6.c}$$

where N and M are the accuracy order in the time and the spatial domain, respectively. The coefficients a_q, b_q, and w_p are shown in Table 1 and Table 2. Equation (6.a) is used for the approximation of the time derivative of the electric and magnetic flux density. Equation (6.b) is used to approximate the time derivatives of the polarization density P_k and the conductive lossy term P_{loss}. Equation (6.c) is used for the approximation of the spatial derivatives

Table 1: Coefficients of the N^{th} order backward time differentiation

N	$\{a_0, a_1, \ldots, a_N\}$	$\{b_0, b_1, \ldots, b_N\}$
1	{1,-1}	{1,-1}
2	{1,-1,0}	{3, -4,1}/2
3	{23, -21, -3, 1}/24	{11, -18, 9, -2}/6
4	{22, -17, -9, 5,-1}/24	{25, -48, 36, -16, 3}/12

Table 2: Coefficients of the M^{th} order central spatial differentiation

M	$\{w_1, \ldots, w_{M/2}\}$
2	{1}
4	{27, -1}/24

After applying the backward differentiation in the time domain and the central differentiation in the space domain to Maxwell's equation and the dielectric dispersive relation Equation (6), we obtain the update equations of the high order FDTD scheme which is composed of the following steps (we show only the update steps for the X-axis components. The same procedure is applied to the Y and Z-axis components):

1. Update of the electric flux density:

$$D_{x}{}_{i+\frac{1}{2},j,k}^{n} = -\frac{1}{a_0} \sum_{q=1}^{N} a_q D_{x}{}_{i+\frac{1}{2},j,k}^{n-q} + \frac{\Delta t}{a_0} \left[\frac{1}{\Delta y} \sum_{p=1}^{M/2} w_p \left(H_{z}{}_{i+\frac{1}{2},j+p-\frac{1}{2},k}^{n-\frac{1}{2}} - H_{z}{}_{i+\frac{1}{2},j-p+\frac{1}{2},k}^{n-\frac{1}{2}} \right) - \frac{1}{\Delta z} \sum_{p=1}^{M/2} w_p \left(H_{y}{}_{i+\frac{1}{2},j,k+p-\frac{1}{2}}^{n-\frac{1}{2}} - H_{y}{}_{i+\frac{1}{2},j,k-p+\frac{1}{2}}^{n-\frac{1}{2}} \right) \right]. \tag{7}$$

2. Update of the electric field:

$$E_{x\,i+\frac{1}{2},j,k}^{\;n} = \frac{1}{\varepsilon_0\left(\varepsilon_\infty + \sum_{k=1}^{K}\frac{\Delta t\Delta\varepsilon_k}{\Delta t + b_0\tau_k} + \frac{\Delta t\sigma_s}{b_0\varepsilon_0}\right)}\left[D_{x\,i+\frac{1}{2},j,k}^{\;n} + \sum_{k=1}^{K}\tau_k\sum_{q=1}^{N}\frac{b_q}{\Delta t + b_0\tau_k}P_{\boldsymbol{k},x\,i+\frac{1}{2},j,k}^{\;n-q} + \sum_{q=1}^{N}\frac{b_q}{b_0}P_{\boldsymbol{s},x\,i+\frac{1}{2},j,k}^{\;n-q}\right], \quad (8.a)$$

$$P_{k,x\,i+\frac{1}{2},j,k}^{\;n} = \varepsilon_0\frac{\Delta t\Delta\varepsilon_k}{\Delta t + b_0\tau_k}E_{x\,i+\frac{1}{2},j,k}^{\;n} - \tau_k\sum_{q=1}^{N}\frac{b_q}{\Delta t + b_0\tau_k}P_{k,x\,i+\frac{1}{2},j,k}^{\;n-q}, \quad (8.b)$$

$$P_{loss,x\,i+\frac{1}{2},j,k}^{\;n} = \frac{\Delta t\sigma_s}{b_0}E_{x\,i+\frac{1}{2},j,k}^{\;n} - \sum_{q=1}^{N}\frac{b_q}{b_0}P_{loss,x\,i+\frac{1}{2},j,k}^{\;n-q}. \quad (8.b)$$

3. Update of the magnetic field:

$$H_{x\,i,j+\frac{1}{2},k+\frac{1}{2}}^{\;n-\frac{1}{2}} = -\frac{1}{a_0}\sum_{q=1}^{N}a_q H_{x\,i,j+\frac{1}{2},k+\frac{1}{2}}^{\;n-\frac{1}{2}-q} - \frac{\Delta t}{a_0\mu}\left[\frac{1}{\Delta y}\sum_{p=1}^{M/2}w_p\left(E_{z\,i,j+p,k+\frac{1}{2}}^{\;n-\frac{1}{2}} - E_{z\,i,j-p+1,k+\frac{1}{2}}^{\;n-\frac{1}{2}}\right) + \frac{1}{\Delta z}\sum_{p=1}^{M/2}w_p\left(E_{y\,i,j+\frac{1}{2},k+p}^{\;n-\frac{1}{2}} - E_{y\,i,j+\frac{1}{2},k-p+1}^{\;n-\frac{1}{2}}\right)\right]. \quad (9)$$

III. STABILITY ANALYSIS

As stated, the Lax–Richtmyer theorem, a consistent finite difference scheme associated to a well-posed system of auxiliary differential equations is convergent to the analytical solution if and only if it is stable [24]. Therefore, the stability analysis leads also to prove the convergence of the studied scheme.

Regarding the fact that we only deal with linear equations, the Z-transformation is applicable for the update Equations (7, 8 and 9). Then by assuming a plane wave propagating in a homogeneous domain, the computed electric field can be presented as follows:

$$E_x^n(\vec{r}) = z^n E_x^0\, e^{-i\vec{k}.\vec{r}}, \quad (10)$$

where $\vec{k} = (k_x, k_y, k_z)$ is the wave vector, z is the Z-transformation variable, and $\vec{r} = (x, y, z)$ is a position vector.

By applying the Z-transformation to the update Equations (7, 8 and 9), the Z-domain wave equation is obtained:

$$\left[\delta_t^{\,2}(z)\mu_r(z)\varepsilon_r(z)I - z^{-1}c_0^2\tilde{C}\tilde{C}^T\right]E(z) = 0, \quad (11)$$

$$\tilde{C} = \begin{bmatrix} 0 & -\delta_z & \delta_y \\ \delta_z & 0 & -\delta_x \\ -\delta_y & \delta_x & 0 \end{bmatrix}, \quad (12)$$

where $c_0 = 1/\sqrt{\mu_0\varepsilon_0}$, $\delta_t = \frac{1}{\Delta t}\sum_{q=0}^{N}a_q z^{-q}$ is the Z-transform of the time backward differentiation, $\tilde{C}$ is the discrete form of the matrix curl-operator, $\delta_x = \frac{2j}{\Delta x}\sum_{p=1}^{M/2}w_p sin\left((p-1/2)k_x\Delta x\right)$ is the central difference operator for the plane wave, δ_y and δ_z are computed similarly to δ_x, and $\varepsilon_r(z)$ is obtained by applying the Z-transformation to Equation (8).

By computing the eigenvalues of Equation (11), it can be reduced to a scalar wave equation:

$$z\delta_t^{\,2}(z)\mu_r(z)\varepsilon_r(z) + c_0^{\,2}\left(\delta_x^{\,2} + \delta_y^{\,2} + \delta_z^{\,2}\right) = 0. \quad (13)$$

By consequence, the stability analysis of the proposed method is reduced to analyzing the stability of the scalar wave Equation (13).

In the automatic control theory, one of the widely used methods for analyzing the stability of discrete control systems is the Root-Locus method [25]. The Root-Locus method is based first on, writing the studied equations under the form of a linear discrete feedback system as shown Equation (14).

$$1 + K\frac{1}{z\delta_t^{\,2}(z)}\frac{1}{\mu_r(z)\varepsilon_r(z)} = 0, \quad (14)$$

where

$$K = c_0^{\,2}\sum_{\eta=x,y,z}\left[\frac{2\Delta t}{\Delta\eta}\sum_{p=1}^{M/2}w_p sin\left((p-1/2)k_\eta\Delta\eta\right)\right]^2.$$

Then, the Root-Locus analysis plots the locations of the Equation 14 roots in the complex plane as a function of the gain *K*. The stability condition of a discrete feedback system is granted when all the roots are located within the unit circle.

The maximum value of the gain K that maintains the stability criteria is evaluated. Then, assuming the worst case by letting $sin\left((p-1/2)k_\eta\Delta\eta\right) = 1$, to compute the CFL condition in the in the case of $\Delta = \Delta x = \Delta y = \Delta z$:

$$CFL = c_0\frac{\Delta t_{max}}{\Delta}. \quad (15)$$

The above procedure is applied for studying the stability condition for the High-order FDTD implementation of a Four-pole Debye model [3], which is used for modeling human muscle, fat, and skin tissues. The special resolution is selected at 25, 50, 75, or 100 points per wavelength (PPW) $\lambda_{min} = 0.05m$.

Figure 1 shows the Root-Locus plot for the free space with the implementation of different FDTD methods. The curves represent the paths traveled by the roots of Equation 14. The critical point for which the stability limit is attained and the maximal values of the gain *K* are indicated in the figure.

Figure 2 shows the Root-Locus plot for the muscle tissue with the implementation of FDTD methods with a fixed resolution at 25 PPW. Figure 3 shows the Root-Locus plot for the muscle tissue with the implementation of the FDTD (3,4) with different resolutions (25, 50, 75, and 100 PPW). Then the CFL limit for each case is computed from the maximum values of the gain *K*. Table 3 resumes a comparison of the CFL condition between the different FDTD methods. Table 4 summarizes the maximum CFL for each tissue.

Based on the results in Table 3, in the case of the free space, the Root-Locus method fits with the Von-Neumann [26] stability condition; which is independent of the spatial grid resolution. However, in the case of a dispersive media, the stability limit is inversely proportional to the grid resolution. Also, we can note that for the high grid resolution, the stability condition tends to retain that of a non-dispersive case where $\varepsilon_r = \varepsilon_\infty$.

From Table 4, regardless the type of the material, the FDTD with high accuracy order are less stable than those with low accuracy order. As a consequence, a tradeoff between the stability and the precision is recommended.

In order to validate the Root-Locus analysis results, a numerical stability test is carried out. A simple electromagnetic scenario is simulated using the FDTD (3,4). The computational space is filled with a dispersive Debye media (muscles tissue). An infinitesimal Hertzian dipole is inserted in the center of the computational space. Two current numbers (c∆t/∆x) are used, 0.1% below and above the CFL limit, respectively. Where the CFL limit of the FDTD (3,4) was found to be 1.2446 for a resolution of 25 PPW in Table 3.

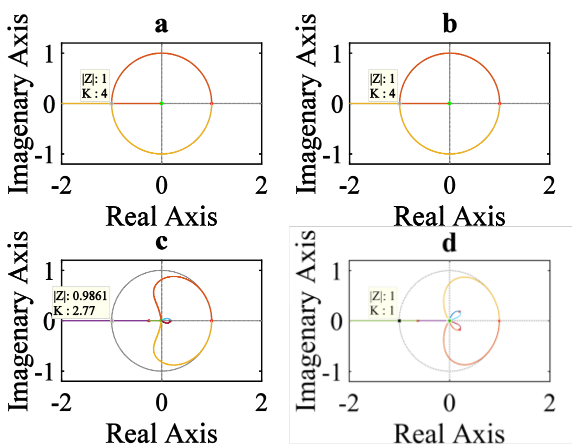

Fig. 1. Root-Locus for FDTD schemes in the free space: (a) FDTD (2,2), (b) FDTD (2,4), (c) FDTD (3,4), and (d) FDTD (4,4).

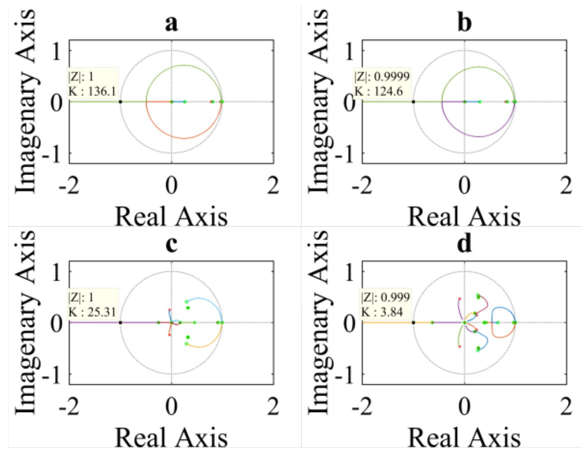

Fig. 2. Root-Locus for FDTD schemes in human muscle tissue: (a) FDTD (2,2), (b) FDTD (2,4), (c) FDTD (3,4), and (d) FDTD (4,4).

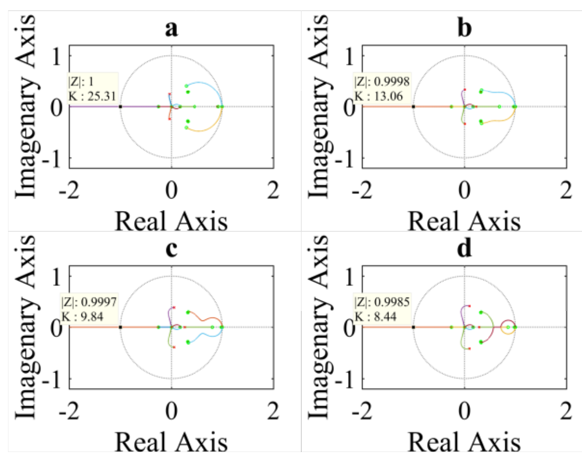

Fig. 3. Root-Locus for FDTD (3,4) schemes in human muscle tissue, for different resolutions: (a) 25 PPW, (b) 50 PPW, (c) 75 PPW, and (d) 100 PPW.

Table 3: CFL for Human muscle tissue modeled by 4-Pole Debye [3], with different PPW, with fmax = 6 GHz

Tissue	Muscle						Free Space
PPW	25	50	75	100	2000	Non-Dispersive	
FDTD(2,2)	3.3019	2.5384	2.0568	1.7565	0.8347	0.7874	0.5715
FDTD(2,4)	2.6852	2.0643	1.6726	1.4285	0.7094	0.6749	0.4899
FDTD(3,4)	1.2446	0.8942	0.7762	0.7188	0.5696	0.5621	0.4118
FDTD(4,4)	0.4849	0.4073	0.3825	0.3712	0.3384	0.3374	0.2474

Table 4: CFL for different human tissues modeled by 4-Pole Debye [3], with PPW = 25, and f_{max} = 6 GHz

Tissue	Muscle	Fate	Wet Skin	Dry Skin
FDTD(2,2)	3.3019	0.984	2.9722	2.8199
FDTD(2,4)	2.6852	0.831	2.4171	2.4171
FDTD(3,4)	1.2446	0.6328	1.3219	1.4285
FDTD(4,4)	0.4849	0.3695	0.5359	0.7493

Figure 4 (a) shows the simulation results of the first case after 20000 steps. There is no sign of instability. Figure 4 (b) shows the situation results of the second case. It is clear that the simulation, in this case, is unstable. This confirms the results obtained by the Root-Locus method.

As a conclusion of this part, the Root-Locus method offers an accurate estimation of the CFL stability condition of the FDTD methods regardless the complexity of the implementation medium and the accuracy orders of the FDTD method.

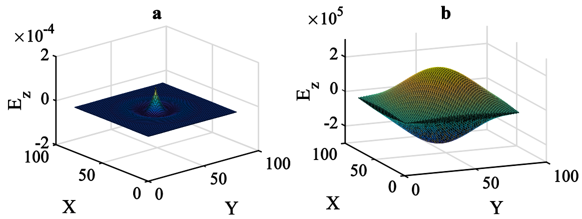

Fig. 4. Stability test in the case of muscle tissue at 25PPW: (a) cΔt/Δx = 1.2444, and (b) cΔt/Δx = 1.2458.

IV. NUMERICAL VALIDATION

In this part, the accuracy of the proposed method is compared to the original FDTD, the high order FDTD (2,4) [22], and the FDTD (4,4) [21]. The simulation of some problems for which the analytical solution is well known, to compute the numerical dispersion of electromagnetic waves traveling in the free space and the Debye dispersive media.

A. Simulation in the free space

First, we consider the scenario of a plane wave traveling in the free space in different directions. The numerical dispersion is evaluated as the relative error in the phase velocity. The multicycle sine plane wave is introduced to the computational space by using the total field scattered field (TFSF) technique [27]. The frequency of the multicycle sine pulse is a function of the PPW. The simulation runs for the necessary time for the multicycle sine wave to entirely vanish through the TFSF interface. Then, the Fourier transform is applied on the electric field to compute the phase velocity.

Figure 5 shows the phase velocity error as a function of the PPW and the propagation angle. The FDTD (4,4) [21] has the lowest phase velocity errors, then the proposed scheme comes in the second place with about 2 dB above the FDTD (4,4), but it is largely lower than both the FDTD (2,2) and FDTD (2,4).

Second, we consider the scenario of a Hertzian dipole source inserted in the center of the computational space which is 200x200x200 cells sized. The space increments are Δx = Δy = Δx = 1 mm. The CFL is 0.2. Ten cells perfectly matched layers (PML) surrounds the computational space to absorb the outgoing waves. The Hertzian dipole is fed by a wideband current pulse as shows Equation 16. The observation point is located at a distance of 20 cells from the Hertzian dipole source:

$$Src(t) = \frac{1}{s^2}(t-m)e^{-0.5\left((t-m)/s\right)^2}, \quad (16)$$

where s = 16Δt; m = 160Δt.

The obtained results are compared with the fields computed using the analytic formula of the Hertzian dipole [28]. As shown in Fig. 6, all of the FDTD (2,4), FDTD (3,4) and FDTD (4,4) fit with the analytic solution rather than the FDTD (2,2). Figure 7 shows the error of each method which is expressed by Equation 17. The proposed scheme offers almost the same precision as the FDTD (4,4) and higher than both the original FDTD (2,2) and the FDTD (2,4):

$$Err(r) = \sqrt{\frac{\sum_t |E_z(r,t) - E_{FDTD}(r,t)|^2}{\sum_t |E_z(r,t)|^2}}, \quad (17)$$

where $E_z(r,t)$ is the analytically computed electric field and $E_{FDTD}(r,t)$ is the electric field computed by the FDTD method.

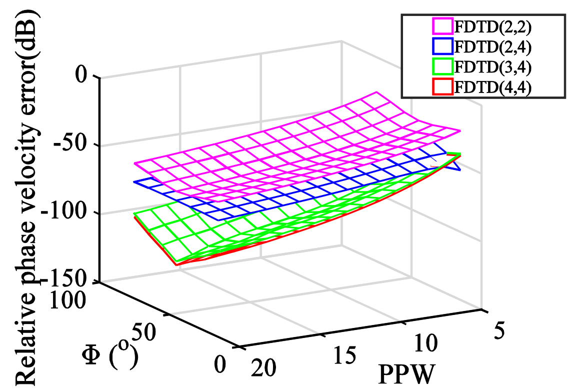

Fig. 5. Relative phase velocity errors as function of the number of points per wavelength (PPW) and the propagation angle Φ.

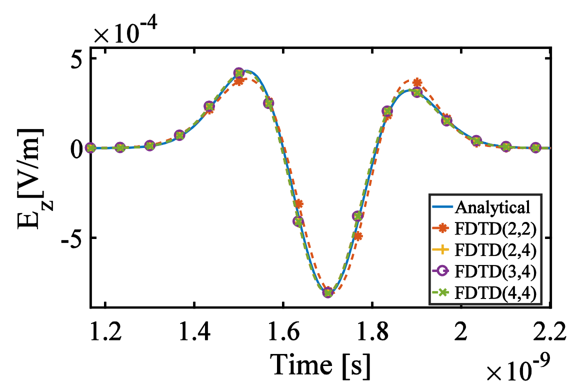

Fig. 6. Comparison of the electric field at 20Δx from the dipole source in the time domain, as computed by the FDTD methods and the theory.

From the above results, in the case of a non-dispersive media, the proposed scheme is better than the original FDTD and the FDTD (2,4), and it offers almost similar performance as the FDTD (4,4) in terms of precision. Also if we consider the low stability condition of the FDTD (4,4) discussed in Section 3, we can deduce that our scheme is more efficient regarding the stability-precision criterion.

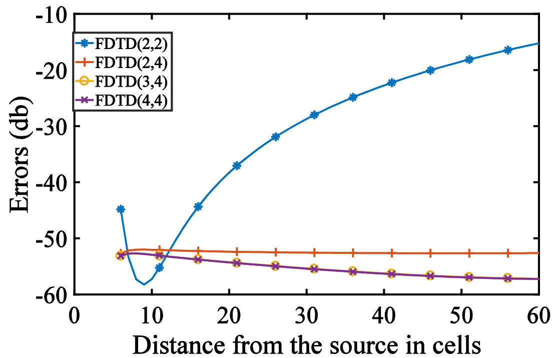

Fig. 7. Errors on the computed electric field expressed by Equation (16), as a function of the distance from the source.

B. Simulation in the Debye media

To validate the accuracy of the proposed method relating to the simulation in the dispersive media, we effectuate the following simulations using the Debye model.

The first scenario consists of the simulation of a plane wave normally incident on the interface between the free space and a homogeneous dispersive media. As an example, we take the Four-Pole Debye model of the muscles where the parameters are listed in [3]. Figure 8 shows the FDTD simulation model. The computational space is delimited by a perfect electric conductor (PEC) from z-direction boundaries, a perfect magnetic conductor (PEM) from the y-direction boundaries. At the x-direction boundaries, a polarized current source excitation surface is placed on one side, and perfectly matched layers (PML) boundary is on the other side. Figure 9 demonstrates the ability of the proposed method to model the Debye dispersive media in comparison with the others methods. It shows that at the interval above 20 PPW the reflection coefficients computed by FDTD (3,4) and FDTD (4,4) fit perfectly with the theoretical one. Figure 10 shows the errors in the computed coefficient of reflection effectuated by each scheme. The obtained results demonstrate that the errors effectuated by the proposed scheme are almost equivalent to the high order FDTD (4,4) and largely lower than those of the FDTD (2,4) and FDTD (2,2).

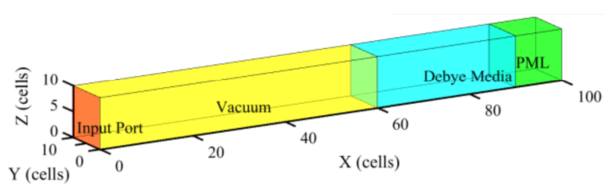

Fig. 8. Numerical FDTD model used to validate the FDTD (3,4) for modeling 4-pole Debye media.

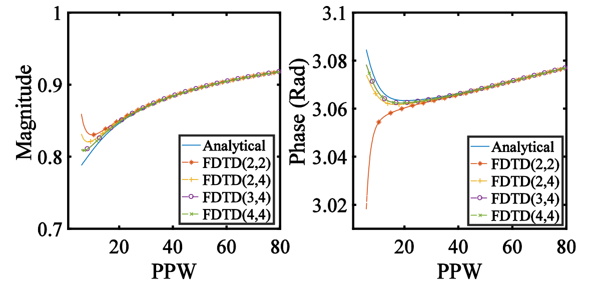

Fig. 9. Magnitude and phase of the reflection coefficient of vacuum – human muscle model interface.

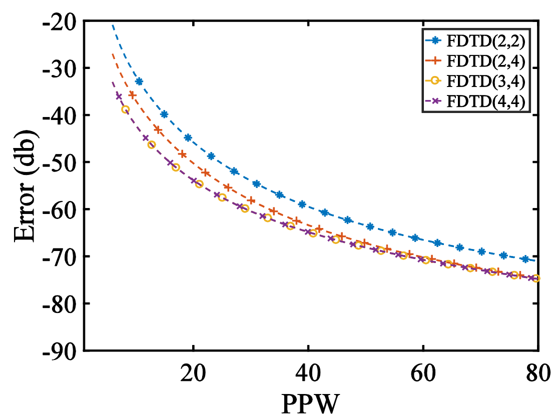

Fig. 10. Errors in the computed reflection coefficient as a function of the PPW.

Secondly, we consider a Hertzian dipole source inserted in the center of the FDTD computational space which is filled with a dispersive media. We refer to the Four-Pole Debye model of the human muscles [3]. The analytical solution is expressed in the frequency domain by including the complex dielectric conductivity and the frequency spectrum of the input source signal $S(f)$ to the general solution of the Hertzian dipole [29]. Equation 18 expresses the electric field in the frequency domain:

$$\vec{E}(r,f) = S(f)l\frac{e^{-2j\pi fr\sqrt{\mu\varepsilon_0\varepsilon_r}}}{4j\pi^2 f\varepsilon_0\varepsilon_r}\times\left(\frac{1}{r^3}+\frac{2j\pi f\sqrt{\mu\varepsilon_0\varepsilon_r}}{r^2}\right)cos(\theta)\overrightarrow{e_r} + S(f)l\frac{e^{-2j\pi fr\sqrt{\mu\varepsilon_0\varepsilon_r}}}{8j\pi^2 f\varepsilon_0\varepsilon_r}\times\left(\frac{1}{r^3}+\frac{2j\pi f\sqrt{\mu\varepsilon_0\varepsilon_r}}{r^2}-\frac{4\pi^2 f^2\mu\varepsilon_0\varepsilon_r}{r}\right)sin(\theta)\overrightarrow{e_\theta}, \quad (18)$$

where l is length of the Hertzian dipole, r is the distance of the observation point from the source, $\overrightarrow{e_r}$ is the unite vector from the source to the observation point and $\overrightarrow{e_\theta}$ is the unite vector perpendicular to $\overrightarrow{e_r}$.

The numerical dispersion of the FDTD is evaluated as follows:

$$Err(r) = \sqrt{\frac{\sum_f |E_z(r,f) - E_{FDTD}(r,f)|^2}{\sum_f |E_z(r,f)|^2}}, \quad (19)$$

where $E_z(r,f)$ is the analytically computed electric field, and $E_{FDTD}(r,f)$ is the frequency component computed by the FDTD method.

Figure 11 shows the agreement of the proposed method with the analytical solution. Also, the others methods agree with the analytical solution in varying proportion. Figure 12 shows the errors of each method as a function of the distance from the source. FDTD (3,4) and FDTD (4,4) offer almost the same precision which is higher than both the FDTD (2,4) and the original FDTD (2,2).

According to the results of the numerical experiments in both the free space and in the Debye dispersive medium, the proposed method is more accurate than both the FDTD (2,2) and the FDTD (2,4). Moreover, it offers almost the same precision as the FDTD (4,4). Hence, when considering the low stability provided by FDTD (4,4), one can conclude that the proposed method offers the best tradeoff between the accuracy and the stability.

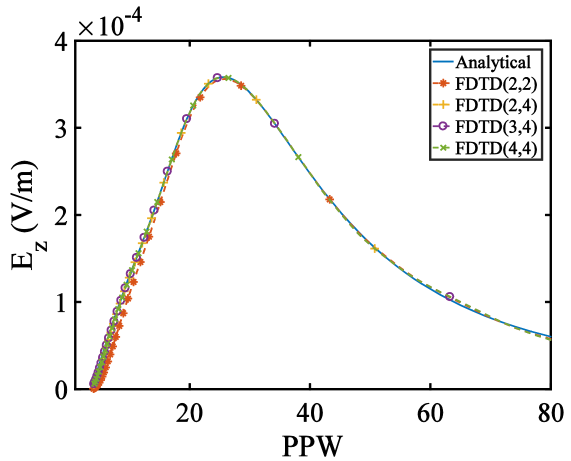

Fig. 11. Comparison of the electric field at 20Δx from the dipole source in the frequency domain, as computed by the FDTD methods and the theory as a function of the PPW.

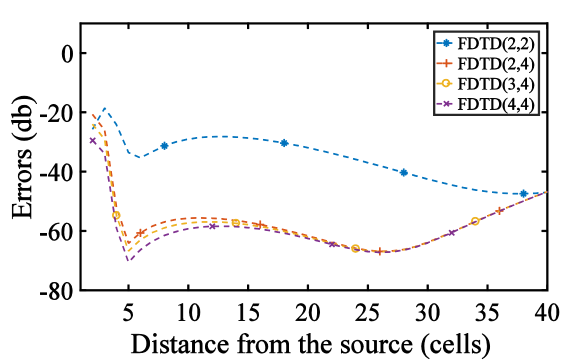

Fig. 12. Errors on the electric field expressed by Equation (18) as a function of the distance from the source.

V. CONCLUSION

A high order accuracy FDTD method for the simulation of the electromagnetic wave behavior in the dispersive media is developed. The proposed method is third order accuracy concerning the approximation of the time domain derivatives and fourth order accuracy concerning the space domain derivatives. The stability of the proposed method is analyzed by using the Root-Locus method, first in the case of the free space, then in the case of the Debye dispersive media. The proposed method comes in third place after the original FDTD method and the FDTD (2,4) which are less accurate. The numerical dispersion analysis in both cases, in the free space and the Debye dispersive media, revealed that the proposed method offered almost the same precision as the high order FDTD (4,4). However, this last has a deficient stability performance. As a tradeoff between the stability and accuracy, the proposed method offers the best performances.

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