Semi-inverse Method to the Klein-Gordon Equation with Quadratic Nonlinearity

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Abstract **─** Nonlinear electrical and mechanical systems have been widely used in the industry electronics and consumer devices. Many numerical algorithms can be employed to obtain the numerical solutions of the nonlinear dynamics or electromagnetic equations. However, it takes a lot of time and decreases the solution accuracy. In this paper, a novel method, called Semi-Inverse Method, is proposed to seek solitary solutions of nonlinear differential equations. The Klein-Gordon equation with quadratic nonlinearity is selected to illustrate the effectiveness and simplicity of the suggested method.

Index Terms **─** Dynamics equation, electromagnetic transmission, nonlinear equation, semi-inverse method, solitary solution.

I. INTRODUCTION

With the development of electrical, mechanical and control engineering, lots of nonlinear and chaotic problems and equations need to be solved [1-5]. The present numerical algorithm can obtain the numerical solutions with much time and low accuracy [6-9]. The growing recognition that the way to solving exact soliton solutions of nonlinear equations is a crucial factor in progress of nonlinear dynamics and a key access to understanding the nonlinear equations to a large extent has fueled much research on the determination of soliton solutions. In recent years, new exact solutions may help to find new phenomena. A variety of powerful methods, such as the Exp-function method [10-11], Tanh-function method [12], algebraic method [13], F-expansion method [14], auxiliary equation method [15], decomposition method [16], extended Jacobi elliptic function expansion method and others were used to develop nonlinear dispersive and dissipative problems.

The present paper is motivated by the desire to the Semi-inverse method to the Klein-Gordon equation with quadratic nonlinearity, which reads:

$$u_{tt} - \alpha^2 u_{xx} + \beta u - \gamma u^2 = 0, \tag{1}$$

where α, β and γ are known constants. Jacobi elliptic function solutions.

II. SEMI-INVERSE METHOD

As shown in Eq. (1), there are two variables, partial differential equation and strong nonlinearity. However, it should be mentioned that the solution of Eq. (1) satisfies the electromagnetic field wave equation. In order to seek its travelling wave solution, we use the following transformation:

$$u(\eta) = u(x,t), \tag{2}$$

$$\eta = x + \lambda t, \tag{3}$$

where λ is a constant to be determined later. It describes the relationship between variable x and t.

Substituting Eq. (2) and Eq. (3) into Eq. (1), we have:

$$(\lambda^2 - \alpha^2) u'' + \beta u - \gamma u^2 = 0, \tag{4}$$

where the prime expresses the derivative with respect to η. As shown in Eq. (4), the partial differential equation becomes ordinary differential equation.

According to the method and Eq. (4), we can obtain the following variation formulation:

$$J(u) = \int_0^{\infty} \left(\frac{\lambda^2 - \alpha^2}{2} u'^2 - \frac{\beta}{2} u^2 + \frac{\gamma}{3} u^3 \right) du. \tag{5}$$

III. THE KLEIN-GORDON EQUATION WITH QUADRATIC NONLINEARITY

Any function can be represented by a Fourier series or exponential expansion. According to the semi-inverse method, we assume the solution can be expressed in the

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following form:

$$u(\eta) = p\,sech^2(q\eta), \tag{6}$$

where p and q are unknown constants to be further determined.

Substituting Eq. (6) into Eq. (5), and by simple calculation, we can obtain:

$$J(u) = \frac{p^2\left[-15\beta+8p\gamma-12q^2(\alpha^2-\lambda^2)\right]}{45q}. \tag{7}$$

Making $J(u)$ stationary with respect to p and q results in:

$$\frac{\partial J}{\partial p} = \frac{8p^2\gamma}{45q} + \frac{2p\left[-15\beta+8p\gamma-12q^2(\alpha^2-\lambda^2)\right]}{45q}, \tag{8}$$

$$\frac{\partial J}{\partial q} = \frac{8p^2(\lambda^2-\alpha^2)}{15} + \frac{p^2\left[15\beta-8p\gamma+12q^2(\alpha^2-\lambda^2)\right]}{45q^2}. \tag{9}$$

From Eq. (8) and Eq. (9), the differential equations can be established to solve p and q:

$$\begin{cases} \dfrac{8p^2\gamma}{45q} + \dfrac{2p\left[-15\beta+8p\gamma-12q^2(\alpha^2-\lambda^2)\right]}{45q} = 0 \\ \dfrac{8p^2(\lambda^2-\alpha^2)}{15} + \dfrac{p^2\left[15\beta-8p\gamma+12q^2(\alpha^2-\lambda^2)\right]}{45q^2} = 0 \end{cases}. \tag{10}$$

From Eq. (10), we obtain:

$$p = \frac{3\beta}{2\gamma}, \quad q = \pm\frac{\sqrt{\beta}}{2\sqrt{\alpha^2-\lambda^2}}. \tag{11}$$

Substituting Eq. (11) into Eq. (2), Eq. (3) and Eq. (6), we have:

$$u(x,t) = \frac{3\beta}{2\gamma}\text{sech}^2\left(\frac{\sqrt{\beta}(x+t\lambda)}{2\sqrt{\alpha^2-\lambda^2}}\right). \tag{12}$$

It is the solitary solution of the Klein-Gordon equation with quadratic nonlinearity. By substituting Eq. (12) into Eq. (1), it can verify the method is effectiveness.

IV. THEORETICAL ANALYSYS

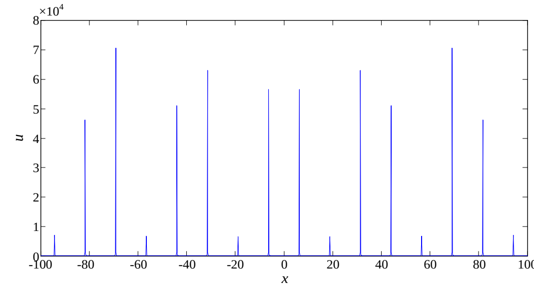
Concerned different parameters of the Klein-Gordon equation with quadratic nonlinearity, three cases have been analyzed.

CASE 1. The Solution of Equation (α=1, β=2, λ=3)

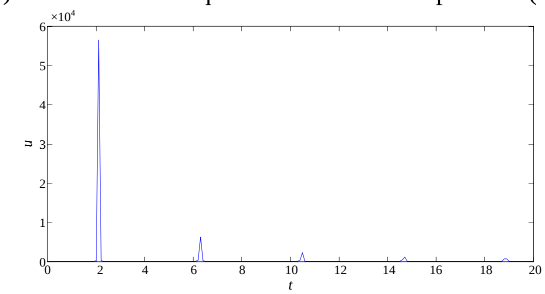
Based on the formula (11), when α=1, β=2, λ=3, the solution of equation was shown in Fig. 1, which can validate the solitary solution the Klein-Gordon equation with quadratic nonlinearity.

CASE 2. The Solution of Equation (α=1, β=5, λ=3)

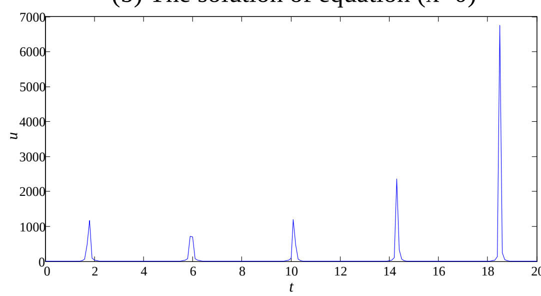
Based on the formula (11), when α=1, β=5, λ=3, the solution of equation was shown in Fig. 2, which can validate the solitary solution the Klein-Gordon equation with quadratic nonlinearity.



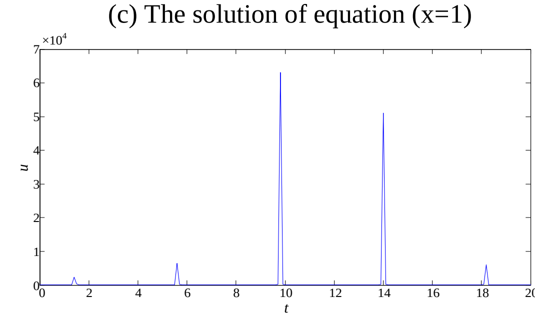
(a) The solution of equation with different position (t=0)



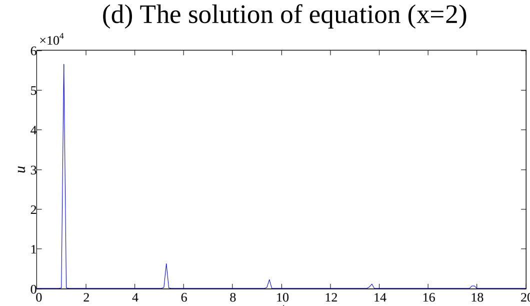
(b) The solution of equation (x=0)



(c) The solution of equation (x=1)

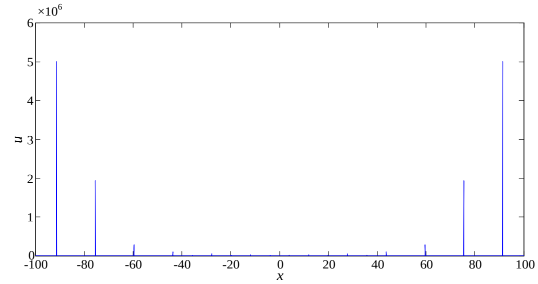


(d) The solution of equation (x=2)

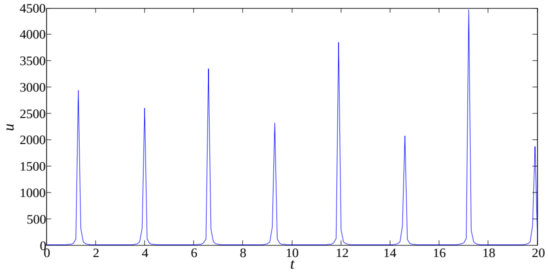


(e) The solution of equation (x=3)

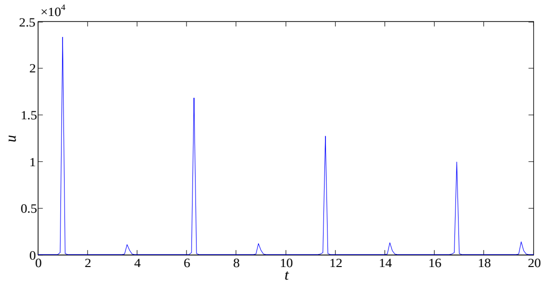
Fig. 1. The solution of equation (α=1, β=2, λ=3).



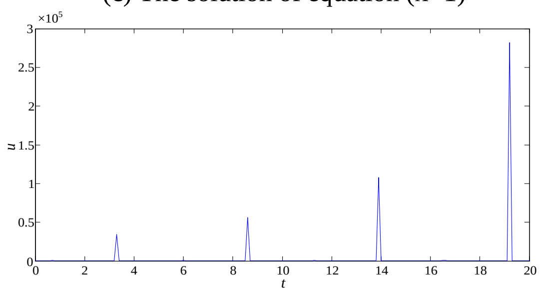
(a) The solution of equation with different position (t=0)



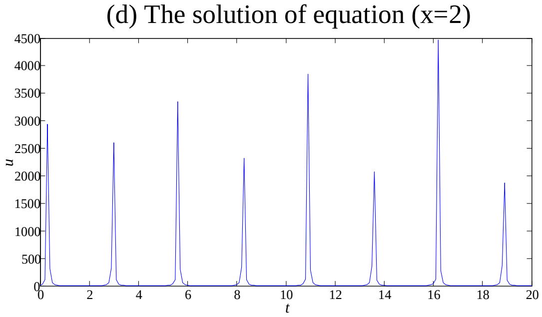
(b) The solution of equation (x=0)



(c) The solution of equation (x=1)



(d) The solution of equation (x=2)

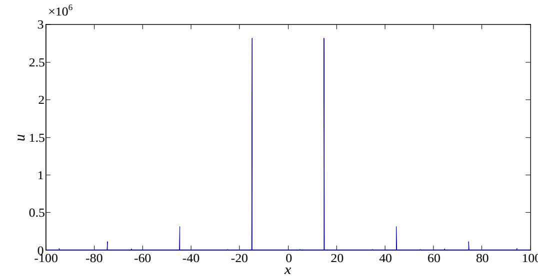


(e) The solution of equation (x=3)

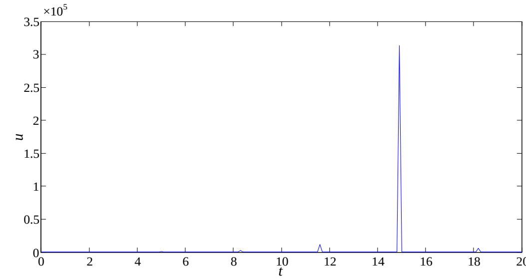
Fig. 2. The solution of equation (α=1, β=5, λ=3).

CASE 3. The Solution of Equation (α=2, β=2, λ=3)

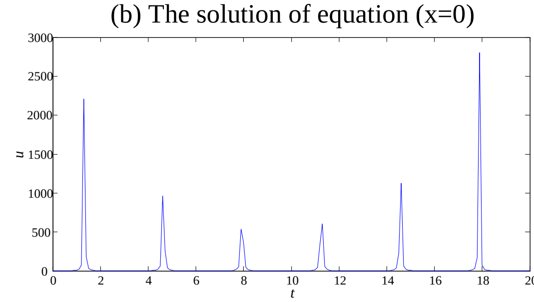
Based on the formula (11), when α=2, β=2, λ=3, the solution of equation was shown in Fig. 3, which can validate the solitary solution the Klein-Gordon equation with quadratic nonlinearity.



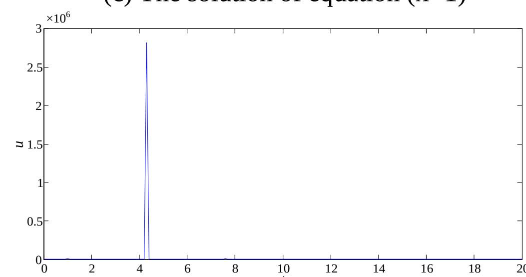
(a) The solution of equation with different position (t=0)



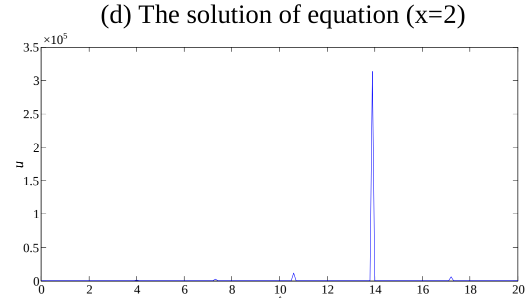
(b) The solution of equation (x=0)



(c) The solution of equation (x=1)



(d) The solution of equation (x=2)



(e) The solution of equation (x=3)

Fig. 3. The solution of equation (α=2, β=2, λ=3).

V. CONCLUSION

In summary, the Semi-inverse method with a computerized symbolic computation has been proposed to obtain the generalized solitary solutions to nonlinear evolution equations arising in mathematical physics. As a result, some new solutions for the physically important NLEEs have easily been found too. It is worthwhile to mention that the proposed Semi-inverse method is of utter straightforward and concise.

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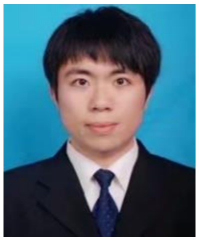

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