Hybrid Formulations for 3D Magnetostatic and Eddy Current Problems

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***Abstract*—Hybrid formulations for solving nonlinear 3D magnetostatic and eddy current problems in terms of magnetic field H are presented. In the proposed formulations integral operators are utilized as if the boundary conditions for a partial differential equation were solved within the system of equations. A good correspondence between the results of the presented methods and measurements or other methods is obtained for some test problems.**

I. Introduction

In this paper hybrid formulations for solving nonlinear 3D magnetostatic and eddy current problems in terms of magnetic field H are presented. Using the proposed formulations problems with conducting and magnetic subregions can be solved without discretizing air regions.

The approaches to form and solve a system of equations providing a solution for H within a bounded region Ω (and its boundary Γ) can be characterized as follows. A partial differential equation (PDE) governs the magnetic field within Ω. In addition, an integral operator $\mathcal{B}$ or $\mathcal{H}$ yields the normal component of flux density $B \cdot n$ or the tangential component of magnetic field $H \times n$ on Γ as a function of magnetization $M = \chi H$ and currents $J = \operatorname{curl} H$. Thus, the proposed formulations share the advantages of both differential and integral operators, such that computationally efficient differential operators are employed in interior region, and integral operators are used to provide boundary conditions on Γ in order to avoid meshing air regions.

II. Methods

A. Assumptions and Definitions

It is assumed that region Ω and its boundary Γ are simply connected, even though the formulations can be extended also in multiply connected regions [1],[2]. It is also assumed that conductivity σ is piecewise constant and strictly positive within Ω, and that permeability μ is bounded and positive. Susceptibility χ is defined by $\chi = \mu/\mu_0 - 1$. It is also assumed that $\sigma = 0$ in $\mathbb{R}^3 - \Omega$, and in addition, that there is no current flow across Γ (i.e. $J \cdot n = 0$). The magnetic field strength, the magnetic flux density, the electric field strength, the electric current density, the magnetization, and the magnetic vector potential are denoted by H, B, E, J, M, and A, respectively. When source currents are denoted by J^s, the constitutive laws can be given as $J = \sigma E + J^s$, $B = \mu H$, and $M = \chi H$. For simplicity, it is also assumed that there are no source currents J^s in Ω.

All formulations presented here are based on finding a solution for H. As the magnetic energy is always finite, H evidently belongs to the space of square integrable vector fields $\mathbb{L}^2(\Omega)$. The space $\mathbb{L}^2(\Omega)$ can be split into complementary gradient and curl parts such that $\mathbb{L}^2(\Omega) = G \oplus C^0$, where $G = \{ H' \mid \operatorname{curl} H' = 0\}$ and $C^0 = \{ J' \mid \operatorname{div} J' = 0,\ J' \cdot n = 0 \text{ on } \Gamma \}$, where n is the normal pointing outward from Ω [3]. The space of square integrable scalar fields is denoted by $L^2(\Omega)$. The space of scalar fields belonging to $L^2(\Omega)$ whose gradients belong to $\mathbb{L}^2(\Omega)$ is abbreviated as $L^2_{\text{grad}}(\Omega)$ and the space of vector fields belonging to $\mathbb{L}^2(\Omega)$ whose curls belong to $\mathbb{L}^2(\Omega)$ as $\mathbb{L}^2_{\text{curl}}(\Omega)$.

B. $B \cdot n$-hybrid Formulations

In order to derive the hybrid formulations to find a solution for H, the Gauss' law for magnetics is multiplied by a test function $\varphi' \in L^2(\Omega)$, and then integration by parts yields

$$\int_\Omega H' \cdot \mu H = \int_\Gamma \varphi' \, B \cdot n \qquad \forall \{\varphi', H'\} \in \mathcal{G}, \tag{1}$$

where $\mathcal{G}$ is the set of all pairs $\{\varphi', H'\}$ for which $\varphi' \in L^2_{\text{grad}}(\Omega)$ and $H' \in G$, such that $\operatorname{grad} \varphi' = H'$. Knowing $B \cdot n$ on Γ, (1) can be used to set up a system of equations yielding a PDE solution for H in the magnetostatic case.

However, $B \cdot n$ is usually not known on the boundary of the magnetic and/or conducting material. The first option to circumvent this problem is to decompose $B \cdot n$ term in (1) into two parts: $B^s \cdot n$ due to the known source currents J^s and $\mathcal{B}(\operatorname{curl} H, \chi H) \cdot n$ due to induced currents $J = \operatorname{curl} H$ and magnetization $M = \chi H$. The integral operator $\mathcal{B}$ yielding B due to currents and magnetization in Ω is defined such that

$$(\mathcal{B}(J,M))(r) = \frac{\mu_0}{4\pi} \int_\Omega \frac{J(r') \times (r - r')}{|r - r'|^3} dr' + \mu_0 M(r) + \frac{\mu_0}{4\pi} \int_\Omega \frac{-M(r')}{|r - r'|^3} + \frac{3[M(r') \cdot (r - r')](r - r')}{|r - r'|^5} \, dr'. \tag{2}$$

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The magnetic flux density B^s due to currents external to Ω is given by the Biot-Savart's law

$$B^s(r) = \frac{\mu_0}{4\pi} \int\limits_{\mathbb{R}^3-\Omega} \frac{J^s(r') \times (r-r')}{|r-r'|^3} dr' . \qquad (3)$$

Now substitution of $B = \mathcal{B}(J,M) + B^s$ into (1) yields

$$\int\limits_\Omega H' \cdot \mu H - \int\limits_\Gamma \varphi' \mathcal{B}(\mathrm{curl}\, H, \chi H) \cdot n = \int\limits_\Gamma \varphi' B^s \cdot n \quad (4)$$

for all $\{\varphi', H'\} \in \mathcal{G}$ [4],[2]. In (4) only $B^s \cdot n$ is required to be known on Γ, and the unknown part of "the boundary condition" due to $M = \chi H$ is solved within the system of equations.

As it is assumed that $J^s \equiv 0$ in Ω, in magnetostatic case H is a gradient field. Thus *the magnetostatic $B \cdot n$-hybrid formulation* can be stated as follows: Find $H \in G$ such that (4), where $\mathrm{curl}\, H = 0$, holds for all $\{\varphi', H'\} \in \mathcal{G}$.

In eddy current problems the induced currents are taken into account by using also Ampère's and Faraday's law such that the solution H also satisfies

$$\int\limits_\Omega J' \cdot \frac{1}{\sigma} \mathrm{curl}\, H + \int\limits_\Omega W' \cdot \frac{\partial \mu H}{\partial t} = \int\limits_\Omega J' \cdot \frac{1}{\sigma} J^s \quad (5)$$

for all pairs $\{W', J'\} \in \mathcal{C}$, where $\mathcal{C}$ is the set of all pairs $\{W', J'\}$ for which $W' \in \mathbb{L}^2_{\mathrm{curl}}(\Omega)$ and $J' \in C^0$ and $\mathrm{curl}\; W' = J'$ [4],[2]. Now *the $B \cdot n$-hybrid eddy current formulation* is given: Find $H \in \mathbb{L}^2(\Omega)$ such that (4) holds for all $\{\varphi', H'\} \in \mathcal{G}$ and (5) holds for all $\{W', J'\} \in \mathcal{C}$.

C. $H \times n$-hybrid Formulations

There is also another possibility to avoid meshing air such that the total field on Γ need not be known. Instead of exploiting directly the theory of the orthogonal subspaces of $\mathbb{L}^2(\Omega)$ [3], these methods are presented here using a similar approach as in the derivation of the standard PDE methods (FEM).

First it is provisionally assumed that $H \times n$ is known on Γ (i.e. $H \times n = h$ on Γ, where h is a known function). Then the solution for H is searched from an affine subspace of $\mathbb{L}^2(\Omega)$ yielding the known boundary conditions h on Γ. When knowing h in (1), it may be chosen that φ' is constant and $H' \times n = 0$ on Γ. Thus (1) is replaced by the given boundary condition and

$$\int\limits_\Omega H' \cdot \mu H \; = \; 0 \qquad \forall \; H' \in G^0 , \qquad (6)$$

where $G^0 = \{ H' \mid \mathrm{curl}\, H' = 0,\; H' \times n = 0 \text{ on } \Gamma \}$. By imposing $H \times n = h$ explicitly on Γ with *additional equations* an equivalent formulation to the standard FEM approach for magnetostatics can be stated as: Find $H \in G$ such that (6) hold together with $H \times n = h$ on Γ.

In practice h is usually not known on the boundary of the magnetic and/or conducting material. Thus equation $H \times n = h$ is modified such that only the part of $H \times n$ on Γ due to known source currents J^s need to be known a priori, and the boundary condition due to magnetization and eddy currents is solved at the same time as the solution inside Ω. In other words H is decomposed into two parts similarly as B above, such that $H = \mathcal{H}(J,M) + H^s$, where $H^s = B^s/\mu_0$ and the integral operator $\mathcal{H}(J,M)$ yielding H due to M and J in Ω is defined as:

$$(\mathcal{H}(J,M))(r) = \frac{1}{4\pi} \int\limits_\Omega \frac{J(r') \times (r-r')}{|r-r'|^3} dr' +$$
$$\frac{1}{4\pi} \int\limits_\Omega \frac{-M(r')}{|r-r'|^3} + \frac{3[M(r') \cdot (r-r')](r-r')}{|r-r'|^5} dr'. \; (7)$$

Now the boundary values can be solved from equation

$$H \times n - \mathcal{H}(\mathrm{curl}\; H, \chi H) \times n = H^s \times n \quad \text{on } \Gamma. \qquad (8)$$

Thus *the magnetostatic $H \times n$-hybrid formulation* can be stated as: Find $H \in G$ such that (6) and (8) hold. Notice that in magnetostatic case $\mathrm{curl}\, H = 0$ in (8). Similarly *the $H \times n$-hybrid eddy current formulation* is given: Find $H \in \mathbb{L}^2(\Omega)$ such that (6) and (8) hold together with (5) holding for all $\{W', J'\} \in \mathcal{C}$.

D. Discretization

Consistency between the continuous and the discrete form of the proposed hybrid formulations is retained by employing Whitney edge elements in a tetrahedral mesh. Thus H is approximated as

$$H = \sum_{e \in \mathcal{E}} h_e \, w_e \, , \qquad (9)$$

where h_e represents the degree of freedom (DoF) related to edge e (i.e. the circulation of H along e), w_e is the basis function of edge e and $\mathcal{E}$ is the set of edges.

The discrete analogies of G, G^0 and C^0 can be created using the spanning tree technique [5],[6]. Since any gradient field can be presented using the tree edges $\mathcal{E}^T$, the basis functions of G are related to the tree edges and they are linear combinations of the basis functions w_e [5]. In order to form the discrete analogy of G^0, the tree must be created first on Γ ($\mathcal{E}^T_\Gamma$) and after that inside Ω ($\mathcal{E}^T_\Omega$). Now the independent basis of G^0 is related to the interior tree edges $\mathcal{E}^T_\Omega$ in the same way as G to all tree edges $\mathcal{E}^T$. The "boundary equations" (8) of the $H \times n$-hybrid approaches are discretized by forcing the solution of H to yield correct circulations along the boundary tree edges $\mathcal{E}^T_\Gamma$, such that (8) is replaced by

$$\int\limits_e H \cdot t - \int\limits_e \mathcal{H}(\mathrm{curl}\, H, \chi H) \cdot t = \int\limits_e H^s \cdot t \quad \forall e \in \mathcal{E}^T_\Gamma, \; (10)$$

where t is a unit tangent vector of a curve.

Also for the discrete C^0, the tree must be created first on Γ. Then the basis functions w_e of the co-tree (i.e. complement of tree) edges interior to Ω form the independent basis of W (i.e. the space where all W' belong) and the curl of them the basis functions of C^0 [2]. Thus, no basis functions of G or C^0 are related to co-tree edges on Γ, but correspondingly also circulations h_e along co-tree edges on Γ are not DoFs, since they can be expressed as a linear combinations of DoFs related to boundary tree edges $\mathcal{E}_\Gamma^T$ due to the assumption $J \cdot n = 0$ on Γ [2]. Thus in all the proposed H-orientated approaches the number of DoFs and the number of equations are equal: number of tree edges in magnetostatic case and number of tree edges and interior co-tree edges for eddy current problems.

In this paper transient eddy current problems are solved using the backward difference implicit time-marching method.

In the proposed hybrid formulations the integral operators are related only to the boundary tree edges. Thus the system matrices is sparse (as the matrix in FEM) except the rows related to the boundary tree edges which are fully populated. The relative size of this dense block depends highly on the fraction of the edges on Γ. Even though the system matrix is only partly dense, so far the systems of equations are solved using a standard LU-solver for dense matrices. However, it has been shown that iterative solvers can be much more efficient in solving dense systems than LU-solver [7]. Moreover, according to some preliminary tests, systems arising from this kind of hybrid methods could be solved tens or even hundreds times faster with iterative methods than with LU-decomposition.

III. Results

A. $B \cdot n$- versus $H \times n$-hybrid methods

When solving problems with the $B \cdot n$-hybrid method, it was noticed that in order to get satisfactory results the boundary integrals should be computed with very high accuracy or a very dense mesh should be used if the boundary Γ is on the surface of the ferromagnetic region. The problem can be circumvented by adding one or two element layers of air on the top of the iron parts, but this kind of trick contradicts the aim to avoid meshing air. On the contrary, the $H \times n$-hybrid formulations provide accurate results with moderate accuracy of the numerical integration of the line integrals along boundary tree edges and also with coarse meshes. Therefore the $H \times n$-hybrid approaches have been selected to closer examination, and all the hybrid results presented in this paper are obtained using the $H \times n$-hybrid formulations.

B. TEAM problem 13

The first test problem is TEAM problem 13, which is a nonlinear magnetostatic problem consisting of thin steel plates, which are excited below the saturation level [8]. The model is symmetrical such that fields in one fourth of the model need to be solved. One of the main difficulties in this problem is the narrow air gap between the central steel plates and the U-shaped parts and the sharp corner in the U-shaped plates. According to definition of TEAM problem 13 [9], the corner in the plate is sharp. However, it has been found out recently that the measured reference results are obtained by using a device where the outer radius of the rounded corner is about 5 mm [10]. Since the average flux density is measured and calculated near the corner the effect of this inaccuracy is remarkable. Because of this, the computational results for problem 13 presented in this paper are obtained using models with a rounded corner.

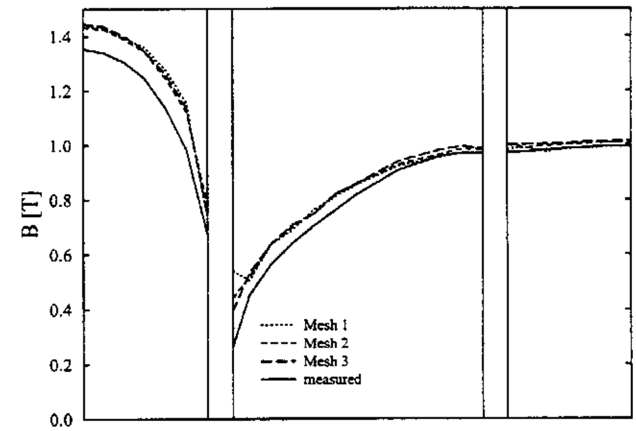

Fig. 1. Average flux density in the steel plates of TEAM problem 13 obtained using the $H \times n$-hybrid method.

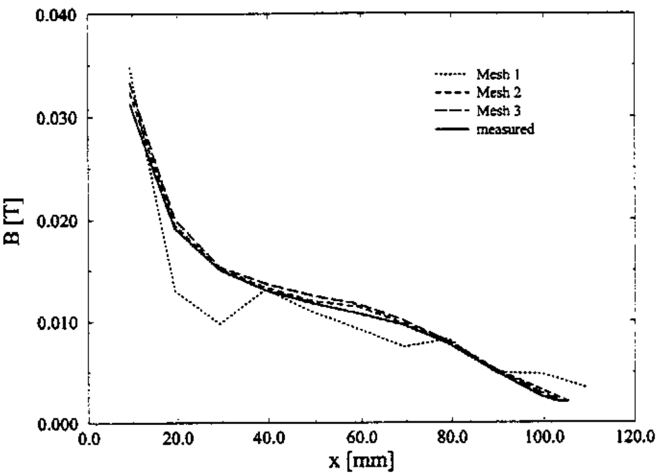

Fig. 2. Magnetic field in air in TEAM problem 13 obtained using the $H \times n$-hybrid method.

Results – requested in the problem definition [8] – for TEAM problem 13 at the lower excitation current level (1000 A) with three meshes solved using the $H \times n$-hybrid method are shown in Figs. 1 and 2. Mesh 1 consists of 480 nodes and 1615 tetrahedra, Mesh 2 of 1350 nodes and 4780 tetrahedra and Mesh 3 of 3601 nodes and 14134 tetrahedra. The resulting linear systems have 462 equations for Mesh 1, 1320 equations for Mesh 2 and 3551 equations for Mesh 3. Both meshes are refined approximately equally near the air gap and near the corner, and the Mesh 2 is shown in Fig. 3.

Computed and measured values of both the average flux densities in the plates and the magnetic field in air are in fairly good agreement with the measurements. The computed average flux densities in the plates are quite independent of the mesh except near the air gap, and they converge to a slightly larger values as obtained from the measurements. However, only one independent measure-

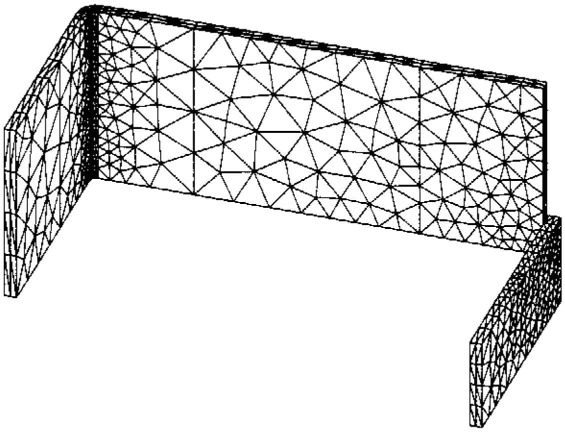

Fig. 3. Mesh 2 used in solving TEAM problem 13. There are 1350 nodes and 4780 tetrahedra in the mesh.

ments have been done for TEAM problem 13, there is always the possibility that the measured curves does not correspond exactly the model definition (e.g. due to hysteresis of steel). In addition, now when the corner of the computational model is rounded, the computed and measured average flux densities are similar also in the corner region. In air only the results of the coarsest mesh (Mesh 1) differ remarkably from the measurements near the region where there are largest elements.

C. TEAM problem 4

In the FELIX brick problem a rectangular aluminum brick with a rectangular hole through it is placed in a exponentially decreasing uniform external magnetic field [11]. Results for the FELIX brick using the $H \times n$-hybrid formulation and two meshes are presented in Figs. 4 – 6. The coarse mesh consists only 16 nodes, 26 tetrahedra and 27 DoFs, whereas the dense mesh has 459 nodes, 1945 tetrahedra and 2140 DoFs. The solution the coarse mesh for 40 time steps on a DEC AlphaStation 600 5/333 workstation took only couple of CPU-seconds. The elapsed CPU-time to compute the hybrid solution for the dense mesh was 1260 seconds, and 92% of the total time took the 40 dense LU-solutions. Thus, at least in this case an iterative solver fully utilizing the sparse matrix hybrid method could reduce the computing time significantly. The computed eddy current distribution in the dense mesh is shown in Fig. 4.

In Fig. 5 the total eddy current circulating in the brick are shown. Global quantities (e.g. total power and total current) are close to those obtained using other methods [6],[12] also with very coarse meshes. The induced magnetic field (i.e. the external field is subtracted from the total field) at the center of the brick are presented In Fig. 6. Also this integrated field outside the conducting region are in good agreement with other methods [6],[12].

D. Conducting Iron Ring

The other test problem is a magnetic and conducting ring, which is rotationally symmetric and thus reliable comparisons can be made to results obtained with 2D

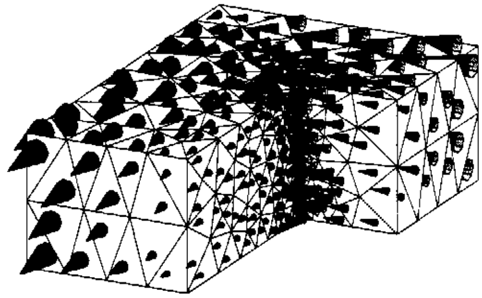

Fig. 4. Eddy current distribution in the dense mesh of the FELIX brick at 10 ms.

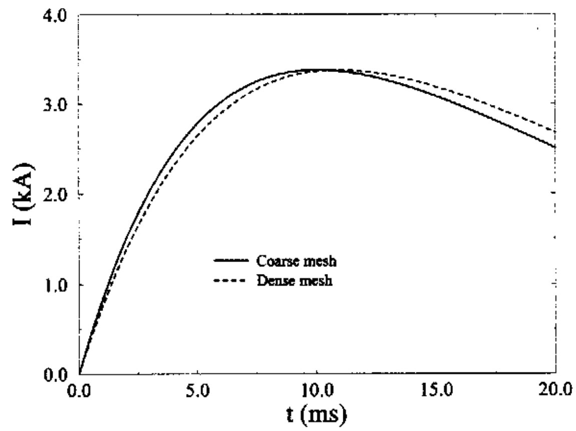

Fig. 5. The total circulating eddy current in the FELIX brick.

methods. The symmetry axis of the ring is the z-axis. The cross-section of the ring is rectangular such that the inner radius of the ring is 30 mm, the outer radius is 60 mm, and thickness is 20 mm. The relative permeability of the material is 1000 and the conductivity is 10^6 S/m. The ring is placed in spatially uniform external magnetic field B_z, which is in the z-direction. For $t < 0$, $B_z = 0$, and for $t \geq 0$, $B_z = 0.1\ \mathrm{T}(1 - e^{-t/1\ \mathrm{s}})$.

For testing the 3D method one eighth of the ring is modeled, even though the problem is in fact a 2D problem. The results of the developed methods are compared to 2D FEM results (Opera-2d [13]) computed using a high number of DoFs. Results for the conducting iron ring problem using two meshes are presented in Figs. 7 – 9. The coarse mesh consists of 122 nodes and 361 tetrahedra, whereas the dense mesh includes 991 nodes and 3703 tetrahedra. The resulting number of DoFs are 358 and 3976, respectively.

Also in this case the global quantities (e.g. total ohmic power and total current) can be computed accurately with quite coarse meshes as shown in Fig. 8. In Fig. 9 the local behavior of the current density is shown along the surface of the ring ($z = 10$ mm) as a function of the distance from th z-axes. As the use of Whitney elements imply that the current density is constant in each element, the agreement with the 2D results cannot be very good, since there are only small fraction of elements in the 2D cross section of the 3D model compared to the 2D model.

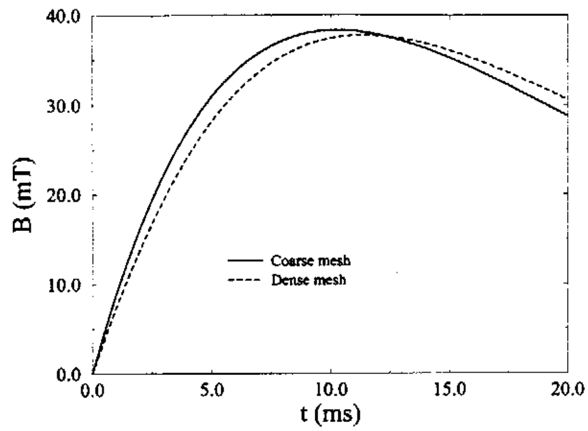

Fig. 6. The induced magnetic flux density B_z at the center of the FELIX brick (i.e. in the center of the hole).

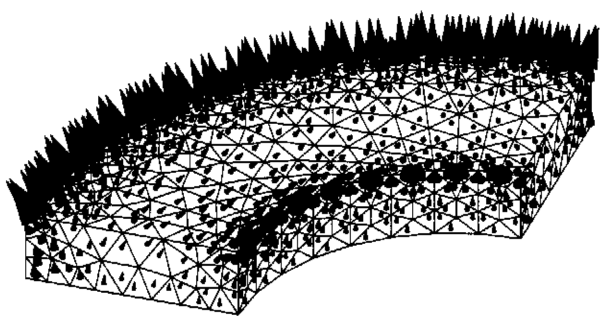

Fig. 7. Magnetic flux density distribution in the dense mesh of the conducting iron ring at 0.2 s.

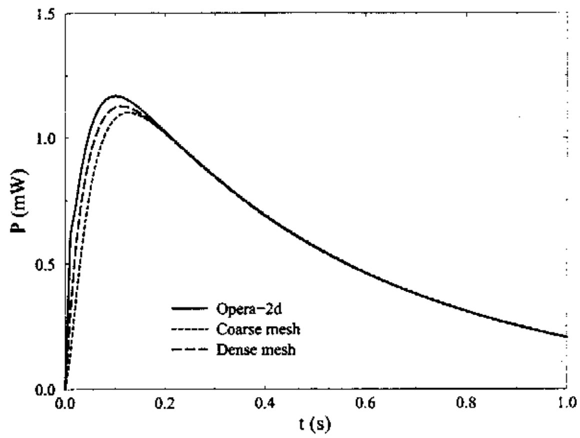

Fig. 8. Ohmic power in the conducing iron ring.

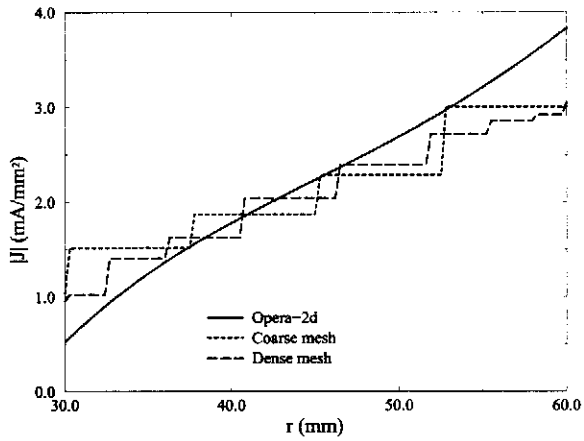

Fig. 9. Eddy currents on the top surface (z = 10mm) of the conducting iron ring.

IV. Conclusion

In this paper new types of hybrid methods to combine differential and integral operators in formulating 3D magnetostatic and eddy current problems are presented. By using the proposed hybrid methods problems can be solved without discretizing air regions, which is from the end user's point of you a remarkable practical advantage compared to standard FEM formulation. The results presented for the test problems validate the methods and results with moderate accuracy can be obtained using only a very small number of elements. However, in order to achieve a good performance with these hybrid formulations it is essential to have a solver which efficiently utilizes the sparsity of the system matrix.

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