3D Diagonalization and Supplementation of Maxwell's Equations in Fully Bi-anisotropic and Inhomogeneous Media Part II: Relative Proof of Consistency

A. R. Baghai-Wadji [1,2]

[1]Electrical Engineering Department
University of Cape Town, Cape Town, 7701, South Africa
alireza.baghai-wadji@uct.ac.za

[2]College of Science
Xi'an University of Science and Technology, Xi'an, Shaanxi, 710054, China

Abstract − Consider fully bi-anisotropic and inhomogeneous media supporting the electromagnetic wave propagation. Assume an $(x, y, z)-$Cartesian coordinate system and a harmonic time-dependence according to $\exp(-j\omega t)$. In the accompanying paper (Part I) it was shown that the Maxwell's equations can be diagonalized with respect to the $z-$axis, resulting in the $\mathcal{D}_c-$form. Furthermore, the existence of the associated supplementary matrix equation, the $\mathcal{S}_c-$form, was demonstrated rigorously. In the present paper "structural," "differential," and "material" matrices have been introduced to explicate the $(\mathcal{D}_a, \mathcal{S}_a)-$, $(\mathcal{D}_b, \mathcal{S}_b)-$, and $(\mathcal{D}_c, \mathcal{S}_c)-$forms, relative to the $x-$, $y-$, and $z-$axes, respectively. As the pinnacle of the theory, it has thoroughly been established that the derived combined $(\mathcal{D}_c, \mathcal{S}_c)-$forms are sharply equivalent with the joint Maxwell's and constitutive equations, and thus internally consistent. The presented proof is relative in the sense that its validity hinges on the consistency of Maxwell's equations and the material realizability conditions.

Index Terms − Bi-anisotropic and inhomogeneous media, diagonalization, Maxwell's equations, supplementation.

I. INTRODUCTION

Natural laws in their originally-conceived manifestations, and more generally, the process of theory construction, occasionally mirror the contents of experiments and their underlying assumptions. The Maxwell's curls equations offer themselves as an archetypical example. They explicate the Faraday's and Ampere's experiments, in an awe-inspiringly elegant manner, and expose the intricately interwoven links between them. On the other hand, not only in logic, but also in mathematics and physics, when constructing theories, one is concerned with the internal consistency of formulations and equations, beside the existence and uniqueness of their solutions. In the 1930's several other epistemologically groundbreaking ideas emerged, e.g., completeness, consistency, provability, and computability, [1], promoting the concepts of, e.g., finitary algorithms, and down the road, complexity and optimality of algorithms for obtaining accurate, robust, and accelerated numerical solutions to engineering problems. In the context of Maxwell's equations, a fundamental question arises as to whether Maxwell's equations' necessarily-heuristic nature renders them, as they stand, optimal for theorizing, algorithms design and computations. Concerning theorizing and algorithm design, the answer depends on the specificities of the theoretical investigations one might be interest in. However, when "taming" infinities and dealing with divergences in computations, the diagonalized $(\mathcal{D}-)$ and supplementary $(\mathcal{S}-)$ forms are considerably more adequate, for reasons substantiated in [2] and [3], and the references therein. This paper completes the exposition in [2] and proves the internal consistency of the $\mathcal{D}-$ and $\mathcal{S}-$forms by showing the sharp equivalence of the $\mathcal{D}-$ and $\mathcal{S}-$forms with the Maxwell's and constitutive equations.

The paper has been organized as follows. Section II starts with introducing three "universal structural" matrices. The attribute "universal" points to the fact that the introduced matrices are independent of the spatial direction along which the diagonalization and the associated supplementation take place: the structural matrices are the same, irrespective of which pairs $(\mathcal{D}_a, \mathcal{S}_a)$, $(\mathcal{D}_b, \mathcal{S}_b)$, or $(\mathcal{D}_c, \mathcal{S}_c)$ are constructed. The attribute "structural" alludes the fact that the entries of

Submitted On: October 16, 2018
Accepted On: October 23, 2018

the matrices, being 0 or 1, merely serve as place holders. Following a discussion of the properties of the structural matrices, a theorem has been stated which formalizes and completes the results obtained in [2]. The expressions for the diagonalized- and supplemented forms with respect to the $x-$axis have been stated, and the counterparts with respect to the $y-$ and $z-$axes have been obtained by cyclic permutations of indices and variables. The theorem comprises Parts I, II, and III, which are dedicated to $(\mathcal{D}_a, \mathcal{S}_a)$, $(\mathcal{D}_b, \mathcal{S}_b)$, and $(\mathcal{D}_c, \mathcal{S}_c)$, respectively. In each part the corresponding matrix differential operators and material matrices have been defined and their properties explained. A reference to the proof in [2] completes this section. Section III proves the internal consistency of the diagonalized- and supplementary equations in terms of the $(\mathcal{D}_c, \mathcal{S}_c)-$forms. Section IV concludes the paper.

II. 3D DIAGONALIZATION AND SUPPLEMENTATION OF MAXWELL'S EQUATIONS

It is assumed that the reader is acquainted with the notation introduced in [2]. Define the following "universal structural" matrices:

$$\mathbf{P}^{4\times4} = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}, \tag{1a}$$

$$\mathbf{P}^{4\times2} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 1 \\ -1 & 0 \end{bmatrix}, \tag{1b}$$

$$\mathbf{P}^{2\times1} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}. \tag{1c}$$

The following relationships do not play any direct role in the diagonalization- and supplementation processes. Nevertheless, their underpinning unified connection to identity matrices deserves to be mentioned,

$$\left(\mathbf{P}^{4\times4}\right)^T\left(\mathbf{P}^{4\times4}\right) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \mathbb{I}^{4\times4}, \tag{2a}$$

$$\left(\mathbf{P}^{4\times2}\right)^T\left(\mathbf{P}^{4\times2}\right) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \mathbb{I}^{2\times2}, \tag{2b}$$

$$\left(\mathbf{P}^{2\times1}\right)^T\left(\mathbf{P}^{2\times1}\right) = \begin{bmatrix} 1 \end{bmatrix} = \mathbb{I}^{1\times1}. \tag{2c}$$

Here, $\mathbb{I}^{N\times N}$ refers to the $N\times N$ identity matrix.

Theorem: Consider the Maxwell's equations in fully bi-anisotropic and inhomogeneous media.

Part A: Define the matrix differential operators:

$$\mathbf{Q}_a^{2\times4} = \begin{bmatrix} 0 & 0 & \partial_{\tilde{z}} & -\partial_{\tilde{y}} \\ -\partial_{\tilde{z}} & \partial_{\tilde{y}} & 0 & 0 \end{bmatrix}, \tag{3a}$$

$$\mathbf{Q}_a^{4\times2} = \begin{bmatrix} \partial_{\tilde{y}} & 0 \\ \partial_{\tilde{z}} & 0 \\ 0 & \partial_{\tilde{y}} \\ 0 & \partial_{\tilde{z}} \end{bmatrix}. \tag{3b}$$

Observe the "intriguing" interplay between the entries of the matrices $\mathbf{Q}_a^{2\times4}$ and $\mathbf{Q}_a^{4\times2}$ leading to,

$$\mathbf{Q}_a^{2\times4}\mathbf{Q}_a^{4\times2} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = \mathbb{O}^{2\times2}. \tag{4}$$

Here $\mathbb{O}^{2\times2}$ refers to the 2×2 null matrix.

Define the following material-specific matrices:

$$\mathbf{M}_a^{4\times4} = \begin{bmatrix} \varepsilon_{22} & \varepsilon_{23} & \xi_{22} & \xi_{23} \\ \varepsilon_{32} & \varepsilon_{33} & \xi_{32} & \xi_{33} \\ \zeta_{22} & \zeta_{23} & \mu_{22} & \mu_{23} \\ \zeta_{32} & \zeta_{33} & \mu_{32} & \mu_{33} \end{bmatrix}, \tag{5a}$$

$$\mathbf{M}_a^{4\times2} = \begin{bmatrix} \varepsilon_{21} & \xi_{21} \\ \varepsilon_{31} & \xi_{31} \\ \zeta_{21} & \mu_{21} \\ \zeta_{31} & \mu_{31} \end{bmatrix}, \tag{5b}$$

$$\mathbf{M}_a^{2\times4} = \begin{bmatrix} \varepsilon_{12} & \varepsilon_{13} & \xi_{12} & \xi_{13} \\ \zeta_{12} & \zeta_{13} & \mu_{12} & \mu_{13} \end{bmatrix}, \tag{5c}$$

$$\mathbf{M}_a^{2\times2} = \begin{bmatrix} \varepsilon_{11} & \xi_{11} \\ \zeta_{11} & \mu_{11} \end{bmatrix}. \tag{5d}$$

Define the essential, $\mathbf{\Psi}_a^{\parallel}$, and its associated nonessential, $\mathbf{\Psi}_a^{\perp}$, field vectors according to,

$$\mathbf{\Psi}_a^{\parallel} = [E_2, E_3, H_2, H_3]^T, \tag{6a}$$

$$\mathbf{\Psi}_a^{\perp} = [E_1, H_1]^T. \tag{6b}$$

Then, the following $\mathcal{D}_a-$ and $\mathcal{S}_a-$forms hold valid.

The $\mathcal{D}_a-$form:

$$\begin{aligned}&\Big\{\mathbf{P}^{4\times4}\mathbf{M}_a^{4\times4}+\left(\mathbf{P}^{4\times4}\mathbf{M}_a^{4\times2}+\mathbf{Q}_a^{4\times2}\right)\\&\times\left[\mathbf{M}_a^{2\times2}\right]^{-1}\left(-\mathbf{M}_a^{2\times4}+\mathbf{Q}_a^{2\times4}\right)\Big\}\mathbf{\Psi}_a^{\parallel}\\&+\left(\mathbf{P}^{4\times4}\mathbf{M}_a^{4\times2}+\mathbf{Q}_a^{4\times2}\right)\left[\mathbf{M}_a^{2\times2}\right]^{-1}\mathbf{P}^{2\times1}\tilde{J}_a^{\perp}\\&+\mathbf{P}^{4\times2}\tilde{\mathbf{J}}_a^{\parallel}=\partial_{\tilde{x}}\mathbf{\Psi}_a^{\parallel}.\end{aligned}\tag{7}$$

The $\mathcal{S}_a-$form:

$$\begin{aligned}\mathbf{\Psi}_a^{\perp}&=\left[\mathbf{M}_a^{2\times2}\right]^{-1}\left(-\mathbf{M}_a^{2\times4}+\mathbf{Q}_a^{2\times4}\right)\mathbf{\Psi}_a^{\parallel}\\&+\left[\mathbf{M}_a^{2\times2}\right]^{-1}\mathbf{P}^{2\times1}\tilde{J}_a^{\perp}.\end{aligned}\tag{8}$$

Part B: Define the matrix differential operators $\mathbf{Q}_b^{2\times4}$ and $\mathbf{Q}_b^{4\times2}$, the material matrices $\mathbf{M}_b^{4\times4}$, $\mathbf{M}_b^{4\times2}$, $\mathbf{M}_b^{2\times4}$, and $\mathbf{M}_b^{2\times2}$, and the field vectors $\mathbf{\Psi}_b^{\parallel}$ and $\mathbf{\Psi}_b^{\perp}$, by performing the cyclic permutations $\tilde{x}\to\tilde{y}$, $\tilde{y}\to\tilde{z}$, $\tilde{z}\to\tilde{x}$, $1\to2$, $2\to3$, $3\to1$, and $a\to b$.

Matrix differential operators $\mathbf{Q}_b^{2\times4}$ and $\mathbf{Q}_b^{4\times2}$:

$$\mathbf{Q}_b^{2\times4}=\begin{bmatrix}0&0&\partial_{\tilde{x}}&-\partial_{\tilde{z}}\\-\partial_{\tilde{x}}&\partial_{\tilde{z}}&0&0\end{bmatrix},\tag{9a}$$

$$\mathbf{Q}_b^{4\times2}=\begin{bmatrix}\partial_{\tilde{z}}&0\\\partial_{\tilde{x}}&0\\0&\partial_{\tilde{z}}\\0&\partial_{\tilde{x}}\end{bmatrix}.\tag{9b}$$

The following relationship holds valid,

$$\mathbf{Q}_b^{2\times4}\mathbf{Q}_b^{4\times2}=\mathbb{O}^{2\times2}.\tag{10}$$

Material matrices $\mathbf{M}_b^{4\times4}$, $\mathbf{M}_b^{4\times2}$, $\mathbf{M}_b^{2\times4}$, and $\mathbf{M}_b^{2\times2}$:

$$\mathbf{M}_b^{4\times4}=\begin{bmatrix}\varepsilon_{33}&\varepsilon_{31}&\xi_{33}&\xi_{31}\\\varepsilon_{13}&\varepsilon_{11}&\xi_{13}&\xi_{11}\\\zeta_{33}&\zeta_{31}&\mu_{33}&\mu_{31}\\\zeta_{13}&\zeta_{11}&\mu_{13}&\mu_{11}\end{bmatrix},\tag{11a}$$

$$\mathbf{M}_b^{4\times2}=\begin{bmatrix}\varepsilon_{32}&\xi_{32}\\\varepsilon_{12}&\xi_{12}\\\zeta_{32}&\mu_{32}\\\zeta_{12}&\mu_{12}\end{bmatrix},\tag{11b}$$

$$\mathbf{M}_b^{2\times4}=\begin{bmatrix}\varepsilon_{23}&\varepsilon_{21}&\xi_{23}&\xi_{21}\\\zeta_{23}&\zeta_{21}&\mu_{23}&\mu_{21}\end{bmatrix},\tag{11c}$$

$$\mathbf{M}_b^{2\times2}=\begin{bmatrix}\varepsilon_{22}&\xi_{22}\\\zeta_{22}&\mu_{22}\end{bmatrix}.\tag{11d}$$

Field vectors $\mathbf{\Psi}_b^{\parallel}$ and $\mathbf{\Psi}_b^{\perp}$:

$$\mathbf{\Psi}_b^{\parallel}=[E_3,E_1,H_3,H_1]^T,\tag{12a}$$

$$\mathbf{\Psi}_b^{\perp}=[E_2,H_2]^T.\tag{12b}$$

Then the following $\mathcal{D}_b-$ and $\mathcal{S}_b-$forms hold valid.

The $\mathcal{D}_b-$form:

$$\begin{aligned}&\Big\{\mathbf{P}^{4\times4}\mathbf{M}_b^{4\times4}+\left(\mathbf{P}^{4\times4}\mathbf{M}_b^{4\times2}+\mathbf{Q}_b^{4\times2}\right)\\&\times\left[\mathbf{M}_b^{2\times2}\right]^{-1}\left(-\mathbf{M}_b^{2\times4}+\mathbf{Q}_b^{2\times4}\right)\Big\}\mathbf{\Psi}_b^{\parallel}\\&+\left(\mathbf{P}^{4\times4}\mathbf{M}_b^{4\times2}+\mathbf{Q}_b^{4\times2}\right)\left[\mathbf{M}_b^{2\times2}\right]^{-1}\mathbf{P}^{2\times1}\tilde{J}_b^{\perp}\\&+\mathbf{P}^{4\times2}\tilde{\mathbf{J}}_b^{\parallel}=\partial_{\tilde{y}}\mathbf{\Psi}_b^{\parallel}.\end{aligned}\tag{13}$$

The $\mathcal{S}_b-$form:

$$\begin{aligned}\mathbf{\Psi}_b^{\parallel}&=\left[\mathbf{M}_b^{2\times2}\right]^{-1}\left(-\mathbf{M}_b^{2\times4}+\mathbf{Q}_b^{2\times4}\right)\mathbf{\Psi}_b^{\parallel}\\&+\left[\mathbf{M}_b^{2\times2}\right]^{-1}\mathbf{P}^{2\times1}\tilde{J}_b^{\perp}.\end{aligned}\tag{14}$$

Part C: Define the matrix differential operators $\mathbf{Q}_c^{2\times4}$ and $\mathbf{Q}_c^{4\times2}$, the material matrices $\mathbf{M}_c^{4\times4}$, $\mathbf{M}_c^{4\times2}$, $\mathbf{M}_c^{2\times4}$, and $\mathbf{M}_c^{2\times2}$, and the field vectors $\mathbf{\Psi}_c^{\parallel}$ and $\mathbf{\Psi}_c^{\perp}$, by performing the cyclic permutations $\tilde{x}\to\tilde{y}$, $\tilde{y}\to\tilde{z}$, $\tilde{z}\to\tilde{x}$, $1\to2$, $2\to3$, $3\to1$, and $b\to c$.

Operator matrices $\mathbf{Q}_c^{2\times4}$ and $\mathbf{Q}_c^{4\times2}$:

$$\mathbf{Q}_c^{2\times4}=\begin{bmatrix}0&0&\partial_{\tilde{y}}&-\partial_{\tilde{x}}\\-\partial_{\tilde{y}}&\partial_{\tilde{x}}&0&0\end{bmatrix},\tag{15a}$$

$$\mathbf{Q}_c^{4\times2}=\begin{bmatrix}\partial_{\tilde{x}}&0\\\partial_{\tilde{y}}&0\\0&\partial_{\tilde{x}}\\0&\partial_{\tilde{y}}\end{bmatrix}.\tag{15b}$$

The following relationship holds valid,

$$\mathbf{Q}_c^{2\times4}\mathbf{Q}_c^{4\times2}=\mathbb{O}^{2\times2}.\tag{16}$$

Material matrices $\mathbf{M}_c^{4\times4}$, $\mathbf{M}_c^{4\times2}$, $\mathbf{M}_c^{2\times4}$, and $\mathbf{M}_c^{2\times2}$:

$$\mathbf{M}_c^{4\times4} = \begin{bmatrix} \varepsilon_{11} & \varepsilon_{12} & \xi_{11} & \xi_{12} \\ \varepsilon_{21} & \varepsilon_{22} & \xi_{21} & \xi_{22} \\ \zeta_{11} & \zeta_{12} & \mu_{11} & \mu_{12} \\ \zeta_{21} & \zeta_{22} & \mu_{21} & \mu_{22} \end{bmatrix}, \tag{17a}$$

$$\mathbf{M}_c^{4\times2} = \begin{bmatrix} \varepsilon_{13} & \xi_{13} \\ \varepsilon_{23} & \xi_{23} \\ \zeta_{13} & \mu_{13} \\ \zeta_{23} & \mu_{23} \end{bmatrix}, \tag{17b}$$

$$\mathbf{M}_c^{2\times4} = \begin{bmatrix} \varepsilon_{31} & \varepsilon_{32} & \xi_{31} & \xi_{32} \\ \zeta_{31} & \zeta_{32} & \mu_{31} & \mu_{32} \end{bmatrix}, \tag{17c}$$

$$\mathbf{M}_c^{2\times2} = \begin{bmatrix} \varepsilon_{33} & \xi_{33} \\ \zeta_{33} & \mu_{33} \end{bmatrix}. \tag{17d}$$

Field vectors $\mathbf{\Psi}_c^{\|}$ and $\mathbf{\Psi}_c^{\perp}$:

$$\mathbf{\Psi}_c^{\|} = [E_1, E_2, H_1, H_2]^T, \tag{18a}$$
$$\mathbf{\Psi}_c^{\perp} = [E_3, H_3]^T. \tag{18b}$$

Then the following $\mathcal{D}_c-$ and $\mathcal{S}_c-$forms hold valid.

The $\mathcal{D}_c-$form:

$$\begin{aligned} &\Big\{\mathbf{P}^{4\times4}\mathbf{M}_c^{4\times4} + \left(\mathbf{P}^{4\times4}\mathbf{M}_c^{4\times2} + \mathbf{Q}_c^{4\times2}\right) \\ &\times \left[\mathbf{M}_c^{2\times2}\right]^{-1}\left(-\mathbf{M}_c^{2\times4} + \mathbf{Q}_c^{2\times4}\right)\Big\}\mathbf{\Psi}_c^{\|} \\ &+ \left(\mathbf{P}^{4\times4}\mathbf{M}_c^{4\times2} + \mathbf{Q}_c^{4\times2}\right)\left[\mathbf{M}_c^{2\times2}\right]^{-1}\mathbf{P}^{2\times1}\tilde{J}_c^{\perp} \\ &+\mathbf{P}^{4\times2}\tilde{\mathbf{J}}_c^{\|} = \partial_{\tilde{z}}\mathbf{\Psi}_c^{\|}. \end{aligned} \tag{19}$$

The $\mathcal{S}_c-$form:

$$\begin{aligned} \mathbf{\Psi}_c^{\perp} &= \left[\mathbf{M}_c^{2\times2}\right]^{-1}\left(-\mathbf{M}_c^{2\times4} + \mathbf{Q}_c^{2\times4}\right)\mathbf{\Psi}_c^{\|} \\ &+ \left[\mathbf{M}_c^{2\times2}\right]^{-1}\mathbf{P}^{2\times1}\tilde{J}_c^{\perp}. \end{aligned} \tag{20}$$

Proof: The constructions of the $\mathcal{D}_c-$ and $\mathcal{S}_c-$forms, expressed in Eqs. (19) and (20), respectively, were performed in exhaustive and painstaking detail in [2], rigorously proving the claims in Part III. The proofs of Parts I and II follow from the proof of Part III by successive cyclic permutations as mentioned above.

III. ON THE CONSISTENCY OF THE $\mathcal{D}_c-$ AND $\mathcal{S}_c$ FORMS

In this section it is rigorously shown that the derived $\mathcal{D}_c-$ and $\mathcal{S}_c-$forms are, taken jointly, sharply equivalent with the originating governing and constitutive equations, and thus internally consistent.

Theorem: The $\mathcal{D}_c-$ and $\mathcal{S}_c-$forms given in Eqs. (19) and (20), respectively, are, taken together, sharply equivalent with Maxwell's equations and constitutive relationships, and thus internally consistent.

Proof: The $\mathcal{D}_c-$ and $\mathcal{S}_c-$forms given in Eqs. (19) and (20), respectively, are the starting point. Writing the $\mathcal{D}_c-$form more explicitly,

$$\begin{aligned} &\mathbf{P}^{4\times4}\mathbf{M}_c^{4\times4}\mathbf{\Psi}_c^{\|} + \left(\mathbf{P}^{4\times4}\mathbf{M}_c^{4\times2} + \mathbf{Q}_c^{4\times2}\right) \\ &\times \left[\mathbf{M}_c^{2\times2}\right]^{-1}\left(-\mathbf{M}_c^{2\times4} + \mathbf{Q}_c^{2\times4}\right)\mathbf{\Psi}_c^{\|} \\ &+ \left(\mathbf{P}^{4\times4}\mathbf{M}_c^{4\times2} + \mathbf{Q}_c^{4\times2}\right)\left[\mathbf{M}_c^{2\times2}\right]^{-1}\mathbf{P}^{2\times1}\tilde{J}_c^{\perp} \\ &+\mathbf{P}^{4\times2}\tilde{\mathbf{J}}_c^{\|} = \partial_{\tilde{z}}\mathbf{\Psi}_c^{\|}. \end{aligned} \tag{21}$$

Rewriting the $\mathcal{S}_c-$form,

$$\begin{aligned} &\left[\mathbf{M}_c^{2\times2}\right]^{-1}\left(-\mathbf{M}_c^{2\times4} + \mathbf{Q}_c^{2\times4}\right)\mathbf{\Psi}_c^{\|} \\ &\quad = \mathbf{\Psi}_c^{\perp} - \left[\mathbf{M}_c^{2\times2}\right]^{-1}\mathbf{P}^{2\times1}\tilde{J}_c^{\perp}. \end{aligned} \tag{22}$$

Using (22) for the term in the second line in (21),

$$\begin{aligned} &\mathbf{P}^{4\times4}\mathbf{M}_c^{4\times4}\mathbf{\Psi}_c^{\|} + \left(\mathbf{P}^{4\times4}\mathbf{M}_c^{4\times2} + \mathbf{Q}_c^{4\times2}\right) \\ &\times \left(\mathbf{\Psi}_c^{\perp} - \left[\mathbf{M}_c^{2\times2}\right]^{-1}\mathbf{P}^{2\times1}\tilde{J}_c^{\perp}\right) \\ &+ \left(\mathbf{P}^{4\times4}\mathbf{M}_c^{4\times2} + \mathbf{Q}_c^{4\times2}\right)\left[\mathbf{M}_c^{2\times2}\right]^{-1}\mathbf{P}^{2\times1}\tilde{J}_c^{\perp} \\ &+\mathbf{P}^{4\times2}\tilde{\mathbf{J}}_c^{\|} = \partial_{\tilde{z}}\mathbf{\Psi}_c^{\|}. \end{aligned} \tag{23}$$

Terms associated with $\tilde{J}_c^{\perp}$ drop off,

$$\begin{aligned} &\mathbf{P}^{4\times4}\mathbf{M}_c^{4\times4}\mathbf{\Psi}_c^{\|} + \left(\mathbf{P}^{4\times4}\mathbf{M}_c^{4\times2} + \mathbf{Q}_c^{4\times2}\right)\mathbf{\Psi}_c^{\perp} \\ &+\mathbf{P}^{4\times2}\tilde{\mathbf{J}}_c^{\|} = \partial_{\tilde{z}}\mathbf{\Psi}_c^{\|}. \end{aligned} \tag{24}$$

Rewriting and factoring out $\mathbf{P}^{4\times4}$,

$$\begin{aligned} &\mathbf{P}^{4\times4}\underbrace{\left(\mathbf{M}_c^{4\times4}\mathbf{\Psi}_c^{\|} + \mathbf{M}_c^{4\times2}\mathbf{\Psi}_c^{\perp}\right)}_{=\mathbf{\Phi}_c^{\|}} + \mathbf{Q}_c^{4\times2}\mathbf{\Psi}_c^{\perp} \\ &+\mathbf{P}^{4\times2}\tilde{\mathbf{J}}_c^{\|} = \partial_{\tilde{z}}\mathbf{\Psi}_c^{\|}. \end{aligned} \tag{25}$$

Recognizing the indicated term in (25) to be equal to $\mathbf{\Phi}_c^{\|}$ (defined in [2]),

$$\mathbf{P}^{4\times4}\mathbf{\Phi}_c^{\parallel} + \mathbf{Q}_c^{4\times2}\mathbf{\Psi}_c^{\perp} + \mathbf{P}^{4\times2}\tilde{\mathbf{J}}_c^{\parallel} = \partial_{\tilde{z}}\mathbf{\Psi}_c^{\parallel}. \quad (26)$$

Multiplying (20) from the L.H.S. by $\mathbf{M}_c^{2\times2}$,

$$\mathbf{M}_c^{2\times2}\mathbf{\Psi}_c^{\perp} = \left(-\mathbf{M}_c^{2\times4} + \mathbf{Q}_c^{2\times4}\right)\mathbf{\Psi}_c^{\parallel} + \mathbf{P}^{2\times1}\tilde{J}_c^{\perp}. \quad (27)$$

Rearranging (27),

$$\underbrace{\mathbf{M}_c^{2\times4}\mathbf{\Psi}_c^{\parallel} + \mathbf{M}_c^{2\times2}\mathbf{\Psi}_c^{\perp}}_{=\mathbf{\Phi}_c^{\perp}} = \mathbf{Q}_c^{2\times4}\mathbf{\Psi}_c^{\parallel} + \mathbf{P}^{2\times1}\tilde{J}_c^{\perp}. \quad (28)$$

Recognizing the indicated term in (28) as $\mathbf{\Phi}_c^{\perp}$ (defined in [2]),

$$\mathbf{\Phi}_c^{\perp} = \mathbf{Q}_c^{2\times4}\mathbf{\Psi}_c^{\parallel} + \mathbf{P}^{2\times1}\tilde{J}_c^{\perp}. \quad (29)$$

Taking the derivative with respect to $\tilde{z}$ of both sides, unpacking $\mathbf{\Phi}_c^{\perp}$ and $\mathbf{\Psi}_c^{\parallel}$, and noting $\tilde{J}_c^{\perp} = \tilde{J}_3$,

$$\partial_{\tilde{z}}\begin{bmatrix} D_3 \\ B_3 \end{bmatrix} = \partial_{\tilde{z}}\mathbf{Q}_c^{2\times4}\begin{bmatrix} E_1 \\ E_2 \\ H_1 \\ H_2 \end{bmatrix} + \partial_{\tilde{z}}\mathbf{P}^{2\times1}\tilde{J}_3. \quad (30)$$

Using the commutativity relationships,

$$\partial_{\tilde{z}}\mathbf{Q}_c^{2\times4} = \mathbf{Q}_c^{2\times4}\partial_{\tilde{z}}, \quad (31a)$$
$$\partial_{\tilde{z}}\mathbf{P}^{2\times1} = \mathbf{P}^{2\times1}\partial_{\tilde{z}}, \quad (31b)$$

equation (30) transforms into,

$$\partial_{\tilde{z}}\begin{bmatrix} D_3 \\ B_3 \end{bmatrix} = \mathbf{Q}_c^{2\times4}\partial_{\tilde{z}}\begin{bmatrix} E_1 \\ E_2 \\ H_1 \\ H_2 \end{bmatrix} + \mathbf{P}^{2\times1}\partial_{\tilde{z}}\tilde{J}_3. \quad (32)$$

Remembering the definition $\mathbf{\Psi}_c^{\parallel} = [E_1, E_2, H_1, H_2]^T$, [2], and considering (26), Eq. (32) reads,

$$\begin{aligned}\partial_{\tilde{z}}\begin{bmatrix} D_3 \\ B_3 \end{bmatrix} &= \mathbf{Q}_c^{2\times4}\left\{\mathbf{P}^{4\times4}\begin{bmatrix} D_1 \\ D_2 \\ B_1 \\ B_2 \end{bmatrix} + \mathbf{Q}_c^{4\times2}\begin{bmatrix} E_3 \\ H_3 \end{bmatrix}\right. \\ &+ \left.\mathbf{P}^{4\times2}\begin{bmatrix} \tilde{J}_1 \\ \tilde{J}_2 \end{bmatrix}\right\} + \mathbf{P}^{2\times1}\partial_{\tilde{z}}\tilde{J}_3. \quad (33)\end{aligned}$$

Here, $\mathbf{\Phi}_c^{\parallel}$, $\mathbf{\Psi}_c^{\perp}$, and $\tilde{\mathbf{J}}_c^{\parallel}$ have been unpacked for greater clarity. Written more explicitly,

$$\begin{aligned}\partial_{\tilde{z}}\begin{bmatrix} D_3 \\ B_3 \end{bmatrix} &= \mathbf{Q}_c^{2\times4}\mathbf{P}^{4\times4}\begin{bmatrix} D_1 \\ D_2 \\ B_1 \\ B_2 \end{bmatrix} \\ &+\mathbf{Q}_c^{2\times4}\mathbf{Q}_c^{4\times2}\begin{bmatrix} E_3 \\ H_3 \end{bmatrix} + \mathbf{Q}_c^{2\times4}\mathbf{P}^{4\times2}\begin{bmatrix} \tilde{J}_1 \\ \tilde{J}_2 \end{bmatrix} \\ &+\mathbf{P}^{2\times1}\partial_{\tilde{z}}\tilde{J}_3. \quad (34)\end{aligned}$$

In (16) it was established that $\mathbf{Q}_c^{2\times4}\mathbf{Q}_c^{4\times2} = \mathbb{O}^{2\times2}$, with $\mathbb{O}^{2\times2}$ being the 2×2 null matrix. Thus (34) reads,

$$\begin{aligned}\partial_{\tilde{z}}\begin{bmatrix} D_3 \\ B_3 \end{bmatrix} &= \mathbf{Q}_c^{2\times4}\mathbf{P}^{4\times4}\begin{bmatrix} D_1 \\ D_2 \\ B_1 \\ B_2 \end{bmatrix} \\ &+\mathbf{Q}_c^{2\times4}\mathbf{P}^{4\times2}\begin{bmatrix} \tilde{J}_1 \\ \tilde{J}_2 \end{bmatrix} + \mathbf{P}^{2\times1}\partial_{\tilde{z}}\tilde{J}_3. \quad (35)\end{aligned}$$

Considering,

$$\begin{aligned}&\mathbf{Q}_c^{2\times4}\mathbf{P}_c^{4\times4} \\ &= \begin{bmatrix} 0 & 0 & \partial_{\tilde{y}} & -\partial_{\tilde{x}} \\ -\partial_{\tilde{y}} & \partial_{\tilde{x}} & 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}, \quad (36a) \\ &= \begin{bmatrix} -\partial_{\tilde{x}} & -\partial_{\tilde{y}} & 0 & 0 \\ 0 & 0 & -\partial_{\tilde{x}} & -\partial_{\tilde{y}} \end{bmatrix}, \quad (36b)\end{aligned}$$

and

$$\begin{aligned}&\mathbf{Q}_c^{2\times4}\mathbf{P}_c^{4\times4} \\ &= \begin{bmatrix} 0 & 0 & \partial_{\tilde{y}} & -\partial_{\tilde{x}} \\ -\partial_{\tilde{y}} & \partial_{\tilde{x}} & 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 1 \\ -1 & 0 \end{bmatrix}, \quad (37a) \\ &= \begin{bmatrix} \partial_{\tilde{x}} & \partial_{\tilde{y}} \\ 0 & 0 \end{bmatrix}, \quad (37b)\end{aligned}$$

and using the definition $\mathbf{P}^{2\times1} = [1, 0]^T$, (35) reads,

$$\begin{aligned}\partial_{\tilde{z}}\begin{bmatrix} D_3 \\ B_3 \end{bmatrix} &= \begin{bmatrix} -\partial_{\tilde{x}} & -\partial_{\tilde{y}} & 0 & 0 \\ 0 & 0 & -\partial_{\tilde{x}} & -\partial_{\tilde{y}} \end{bmatrix}\begin{bmatrix} D_1 \\ D_2 \\ B_1 \\ B_2 \end{bmatrix} \\ &+ \begin{bmatrix} \partial_{\tilde{x}} & \partial_{\tilde{y}} \\ 0 & 0 \end{bmatrix}\begin{bmatrix} \tilde{J}_1 \\ \tilde{J}_2 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix}\partial_{\tilde{z}}\tilde{J}_3. \quad (38)\end{aligned}$$

Multiplying out,

$$\begin{bmatrix} \partial_{\tilde z} D_3 \\ \partial_{\tilde z} B_3 \end{bmatrix} = \begin{bmatrix} -\partial_{\tilde x} D_1 - \partial_{\tilde y} D_2 \\ -\partial_{\tilde x} B_1 - \partial_{\tilde y} B_2 \end{bmatrix} + \begin{bmatrix} \partial_{\tilde x} \tilde J_1 + \partial_{\tilde y} \tilde J_2 \\ 0 \end{bmatrix} + \begin{bmatrix} \partial_{\tilde z} \tilde J_3 \\ 0 \end{bmatrix}. \quad (39)$$

Writing component-wise,

$$\begin{aligned} \partial_{\tilde z} D_3 &= -\partial_{\tilde x} D_1 - \partial_{\tilde y} D_2 \\ &\quad + \partial_{\tilde x} \tilde J_1 + \partial_{\tilde y} \tilde J_2 + \partial_{\tilde z} \tilde J_3, && (40a) \\ \partial_{\tilde z} B_3 &= -\partial_{\tilde x} B_1 - \partial_{\tilde y} B_2. && (40b) \end{aligned}$$

Rearranging, remembering the definitions $\partial_{\tilde x} = \partial_x / j\omega$, $\partial_{\tilde y} = \partial_y / j\omega$, and $\partial_{\tilde z} = \partial_z / j\omega$, introduced in [2], assuming $\omega \neq 0$, and multiplying by $j\omega$,

$$\begin{aligned} \text{div}\mathbf{D} &= \text{div}\tilde{\mathbf{J}}, && (41a) \\ \text{div}\mathbf{B} &= 0. && (41b) \end{aligned}$$

Remembering the definition of $\tilde{\mathbf{J}} = \mathbf{J}/(j\omega)$, [2],

$$\text{div}\mathbf{D} = \text{div}\frac{1}{j\omega}\mathbf{J}. \quad (42)$$

Multiplying by $j\omega(\neq 0)$,

$$j\omega\text{div}\mathbf{D} = \text{div}\mathbf{J}. \quad (43)$$

Considering the electric charge conservation law,

$$\text{div}\mathbf{J} + \frac{\partial \rho}{\partial t} = 0, \quad (44)$$

and recalling the $\exp(-j\omega t)$ time-harmonic assumption, Eq. (44) can equivalently be written in the form,

$$\text{div}\mathbf{J} = j\omega\rho. \quad (45)$$

Thus, Eq. (43) results in,

$$j\omega\text{div}\mathbf{D} = j\omega\rho. \quad (46)$$

Dividing by $j\omega(\neq 0)$,

$$\text{div}\mathbf{D} = \rho. \quad (47)$$

Summarizing our results, it can be stated that by assuming the charge conservation law, Eq. (44), and using the derived $\mathcal{D}_c-$ and $\mathcal{S}_c-$forms, Maxwell's divergence equations have been established,

$$\begin{aligned} \text{div}\mathbf{D} &= \rho, && (48a) \\ \text{div}\mathbf{B} &= 0. && (48b) \end{aligned}$$

This completes the relative proof of the internal consistency of the drived $\mathcal{D}_c-$ and the associated $\mathcal{S}_c-$forms, in virtue of their sharp equivalence with the originating Maxwell's equations and the constitutive equations, characterizing fully bi-anisotropic and inhomogeneous media.

IV. CONCLUSION

In the accompanying paper, [2], it was shown that the Maxwell's equations in fully bi-anisotropic and inhomogeneous media can be diagonalized resulting in the $\mathcal{D}-$form. In addition, the existence of an associated supplementary equation, the $\mathcal{S}-$form, was demonstrated. In this work it was stringently proved that the derived $(\mathcal{D}, \mathcal{S})-$forms are internally consistent, based on the fact that they are sharply equivalent with the originating Maxwell's equations and constitutive relationships. A relative consistency proof was presented which proceeded along the following line of argument: (i) It is a known fact that the Maxwell's curl equations together with the electric charge conservation law imply the Maxwell's divergence equations. (ii) Within the framework of the classical electrodynamics it is assumed that the Maxwell's equations are self-consistent (internal consistency). (iii) Employing the $\mathcal{D}-$ and the $\mathcal{S}-$forms constructed in [2], and utilizing the charge conservation law, this paper established the Maxwell's divergence equations. (iv) The sharp equivalence of the $\mathcal{D}-$ and the $\mathcal{S}-$forms with Maxwell's curl equations, and the consistency of Maxwell's curl equations, imply the consistency of the $\mathcal{D}-$ and $\mathcal{S}-$forms. The discussion in [3] provides a glimpse on possible wide-ranging implications of the proposed theoretical framework.

REFERENCES

[1] B. J. Copeland, C. J. Posy, and O. Shagrir, Eds., ‘‘Computability - Turing, Gödel, Church, and Beyond,’’ The MIT Press, Cambridge, Massachusetts, London, England, 2015.

[2] A. R. Baghai-Wadji, ‘‘3D Diagonalization and supplementation of Maxwell's equations in fully bi-anisotropic and inhomogeneous media, Part I: Proof of existence by construction,’’ ACES Journal, this issue.

[3] A. R. Baghai-Wadji, ‘‘The Path from Monadic to Tetradic Green's Functions,’’ Proc. International Conference on Electromagnetics in Advanced Applications (ICEAA), 2018, pp. 1-4.

Alireza Baghai-Wadji: For a biography, please refer to the accompanying paper, [2], in this issue.