A DC to HF Volume PEEC Formulation Based on Hertz Potentials and the Cell Method

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***Abstract*—A new Partial Element Equivalent Circuit formulation based on Hertz potentials and the Cell Method is presented. Conductive, dielectric, and magnetic linear-homogeneous media are considered by means of conduction, polarization, and magnetization current densities. The use of edge unknowns leads to reduce system size with respect to typical face-based approaches.**

I. Introduction

A large body of literature shows that integral methods are particularly suited to the solution of high frequency electromagnetic problems (EM) involving large domains with the characteristics of vacuum. In particular, the Partial Element Equivalent Circuit (PEEC) method has been shown to be well suited for the analysis of electromagnetic devices, such as printed circuit boards and antennas. The aim of this work is to present a novel 3-D Hertz-PEEC formulation, including conductive, dielectric, and magnetic media. This approach is based on the theory of Hertz potentials [1] and magnetization currents for the magnetic media [2], [3], while in literature Amperian currents [4] or the magnetization [5] are applied. The discretization of the formulation is obtained by means of the Cell Method (CM) [6] and the current density vector, which is the only unknown, is expanded by Whitney face functions [7]. The proposed approach leads to a unified treatment of magnetic and dielectric media which can be handled with a single set of robust and efficient semi-analytical integration routines. The approach is thus capable of solving EM problems over a larger frequency range compared to extant PEEC approaches for magnetic media [5]. Furthermore, the formulation uses edge unknowns, leading to the reduction of the linear system size with respect to other approaches.

II. Integral Formulation

Conductive, dielectric, and magnetic domains Ω_c, Ω_d, and Ω_m (with boundaries Γ_c, Γ_d, and Γ_m, respectively) are considered in the formulation. The domains have no intersection and their disjoint union is $\Omega = \Omega_c \sqcup \Omega_d \sqcup \Omega_m$.

When conductive, dielectric, and magnetic media are considered, the following constitutive relations can be introduced:

$$\mathbf{J}_f = \sigma\mathbf{E} \; in \; \Omega_c, \qquad \mathbf{D} = \varepsilon_0\mathbf{E} + \mathbf{P} \; in \; \Omega_d, \qquad \mathbf{B} = \mu_0\mathbf{H} + \mu_0\mathbf{M} \; in \; \Omega_m, \tag{1}$$

where $\mathbf{J}_f$ is the conduction current density, $\mathbf{E}$ and $\mathbf{H}$ are the electric and the magnetic field, $\mathbf{D}$ is the electric displacement and $\mathbf{B}$ is the magnetic flux density, σ is the conductivity, ε_0 and μ_0 are the vacuum permittivity and permeability. The polarization $\mathbf{P}$ and the magnetization $\mathbf{M}$ are introduced as equivalent sources in vacuum, so the effects of dielectric and magnetic media are taken into account. With the introduction of (1), following the approach and notation of [8] (chap. 5, 7), Maxwell's equations in frequency domain can be re-written in a more symmetric form, avoiding the use of $\mathbf{D}$ and $\mathbf{B}$:

$$\begin{aligned} \nabla\cdot\mathbf{E} &= {\varepsilon_0}^{-1}(\varrho_f + \varrho_p), & -\nabla\times\mathbf{E} &= \mathbf{J}_{m^*} + i\omega\mu_0\mathbf{H}, \\ \nabla\cdot\mathbf{H} &= {\mu_0}^{-1}\varrho_{m^*}, & \nabla\times\mathbf{H} &= \mathbf{J}_f + \mathbf{J}_p + i\omega\varepsilon_0\mathbf{E}, \end{aligned} \tag{2}$$

where ϱ_f is the free electric charge density, $\mathbf{J}_p = i\omega\mathbf{P}$ and $\varrho_p = -\nabla\cdot\mathbf{P}$ are the polarization current and bound dielectric charge densities, $\mathbf{J}_{m^*} = i\omega\mu_0\mathbf{M}$ is the magnetization current density and $\varrho_{m^*} = -\nabla\cdot\mu_0\mathbf{M}$ is the bound magnetic charge density. As shown in [1] and [9], it is possible to write the electric field $\mathbf{E}$ and the magnetic field $\mathbf{H}$ as:

$$\begin{aligned} \mathbf{E} &= -(i\omega)^2\mathbf{\Pi}_e - \nabla\varphi_e - {\varepsilon_0}^{-1}\nabla\times(i\omega\mathbf{\Pi}_m) + \mathbf{E}_0, \\ \mathbf{H} &= -(i\omega)^2\mathbf{\Pi}_m - \nabla\varphi_m + {\mu_0}^{-1}\nabla\times(i\omega\mathbf{\Pi}_e) + \mathbf{H}_0, \end{aligned} \tag{3}$$

where $\mathbf{\Pi}_e$ and $\mathbf{\Pi}_m$ are the Hertz electric and magnetic vector potentials, φ_e and φ_m are electric and magnetic scalar potentials and $\mathbf{E}_0$ and $\mathbf{H}_0$ are imposed sources.

By defining $\mathbf{A}_e = i\omega\mathbf{\Pi}_e$, and $\mathbf{A}_m = i\omega\mathbf{\Pi}_m$ as "new" electric and magnetic vector potentials, applying Lorenz gauge, $\nabla\cdot\mathbf{A}_e = -i\omega\varepsilon_0\mu_0\varphi_e$, $\nabla\cdot\mathbf{A}_m = -i\omega\varepsilon_0\mu_0\varphi_m$, and letting (3) into (2), four partial differential equations can be obtained:

$$\begin{aligned} \Box\mathbf{A}_e &= \mu_0\mathbf{J}_e, & \Box\varphi_e &= {\varepsilon_0}^{-1}(\varrho_f + \varrho_p), \\ \Box\mathbf{A}_m &= \varepsilon_0\mathbf{J}_{m^*}, & \Box\varphi_m &= {\mu_0}^{-1}\varrho_{m^*}, \end{aligned} \tag{4}$$

where $\mathbf{J}_e = \mathbf{J}_f$ in Ω_c and $\mathbf{J}_e = \mathbf{J}_p$ in Ω_d, $\Box = (i\omega)^2\frac{1}{c^2} - \nabla^2$ is the d'Alembert operator and c is the speed of light. The solution of the first equation of (4) is:

$$\mathbf{A}_e(\mathbf{r}_y) = \mu_0\int_\Omega \mathbf{J}_e(\mathbf{r}_x)g(\mathbf{r}_x,\mathbf{r}_y)dr_x, \tag{5}$$

where $\mathbf{r}_y$ is the field point, $\mathbf{r}_x$ is the integration point and $g(\mathbf{r}_x,\mathbf{r}_y) = \frac{e^{-i\omega c^{-1}|\mathbf{r}_y-\mathbf{r}_x|}}{|\mathbf{r}_y-\mathbf{r}_x|}$ is the scalar retarded free space Green function. The solution of the other three equations in (4) is obtained likewise (5).

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III. Cell Method Discretization

The electric domains Ω_c and Ω_d are discretized into primal tetrahedral grids $\mathcal{G}_{\Omega_c}$ and $\mathcal{G}_{\Omega_d}$, consisting of n_c, n_d nodes, e_c, e_d edges, f_c, f_d faces, and v_c, v_d volumes. Then, dual grids $\tilde{\mathcal{G}}_{\Omega_c}$ and $\tilde{\mathcal{G}}_{\Omega_d}$ can be obtained by taking the barycentric subdivision of $\mathcal{G}_{\Omega_c}$ and $\mathcal{G}_{\Omega_d}$ [10]. The same approach can be applied to the magnetic domain Ω_m, but in this case $\tilde{\mathcal{G}}_{\Omega_m}$ is chosen to be made up of tetrahedral elements, while $\mathcal{G}_{\Omega_m}$ is obtained by its barycentric subdivision. The following incidence matrices can be obtained: $\mathbf{G}_{\Omega_\alpha}$ (edges to nodes), $\mathbf{C}_{\Omega_\alpha}$ (faces to edges), and $\mathbf{D}_{\Omega_\alpha}$ (volumes to faces), on $\mathcal{G}_{\Omega_\alpha}$, where $\alpha = c, d, m$ indicates the domain. Dual matrices can be obtained for $\tilde{\mathcal{G}}^a_{\Omega_\alpha}$, i.e., $\tilde{\mathbf{G}}^a_{\Omega_\alpha}$, $\tilde{\mathbf{C}}^a_{\Omega_\alpha}$, $\tilde{\mathbf{D}}^a_{\Omega_\alpha}$, where the superscript a indicates that the matrix is *augmented* [10]. To build the CM formulation, following arrays of degrees of freedom are introduced:

- $\mathbf{j_e} = (j_{e_i})$ on faces $f_i \in \mathcal{G}_{\Omega_c}$, $j_{e_i} = \int_{f_i} \mathbf{J}_e \cdot d\mathbf{S}$,
- $\tilde{\mathbf{j}}_{\mathbf{m}^*} = (\tilde{j}_{m_i^*})$ on faces $\tilde{f}_i \in \tilde{\mathcal{G}}_{\Omega_m}$, $\tilde{j}_{m_i^*} = \int_{\tilde{f}_i} \mathbf{J}_{m^*} \cdot d\mathbf{S}$,
- $\mathbf{h} = (h_i)$ on edges $e_i \in \mathcal{G}_{\Omega_m}$, $h_i = \int_{e_i} \mathbf{H} \cdot d\mathbf{l}$,
- $\tilde{\mathbf{e}} = (\tilde{e}_i)$ on edges $\tilde{e}_i \in \tilde{\mathcal{G}}_{\Omega_c \sqcup \Omega_d}$, $\tilde{e}_i = \int_{\tilde{e}_i} \mathbf{E} \cdot d\mathbf{l}$,
- $\mathbf{a}_m = (a_{m_i})$ on edges $e_i \in \mathcal{G}_{\Omega_m}$, $a_{m_i} = \int_{e_i} \mathbf{A}_m \cdot d\mathbf{l}$,
- $\tilde{\mathbf{a}}_e = (\tilde{a}_{e_i})$ on edges $\tilde{e}_i \in \tilde{\mathcal{G}}_{\Omega_c \sqcup \Omega_d}$, $\tilde{e}_i = \int_{\tilde{e}_i} \mathbf{A}_e \cdot d\mathbf{l}$,
- $\tilde{\boldsymbol{\phi}}_e = (\tilde{\phi}_{e_i})$ on nodes $\tilde{n}_i \in \tilde{\mathcal{G}}_{\Omega_c \sqcup \Omega_d}$, $\tilde{\phi}_{e_i} = \varphi_e(\mathbf{r}_{n_i})$,
- $\boldsymbol{\phi}_m = (\phi_{m_i})$ on nodes $n_i \in \mathcal{G}_{\Omega_m}$, $\phi_{m_i} = \varphi_m(\mathbf{r}_{n_i})$.

The coupling between the domains is enforced by weakly imposing (1):

$$\begin{aligned} \int_{\Omega_c \sqcup \Omega_d} \mathbf{w}_i^f \cdot \big(\rho_e \mathbf{J}_e(\mathbf{r}) - \mathbf{E}(\mathbf{r})\big) d\mathbf{r} = 0, \\ \int_{\Omega_m} \mathbf{w}_i^f \cdot \big(\rho_{m^*} \mathbf{J}_{m^*}(\mathbf{r}) - \mathbf{H}(\mathbf{r})\big) d\mathbf{r} = 0, \end{aligned} \tag{6}$$

where $\rho_e = (\sigma_c)^{-1}$ in Ω_c and $\rho_e = (i\omega\varepsilon_0(\varepsilon_r - 1))^{-1}$ in Ω_d, $\rho_{m^*} = (i\omega\mu_0(\mu_r - 1))^{-1}$, $\mathbf{w}_i^f$ is the Whitney face basis function. By expanding $\mathbf{J}_e$ and $\mathbf{J}_{m^*}$ with $\mathbf{w}_i^f$ and letting (3) into (6), the following system is obtained:

$$\begin{bmatrix} \mathbf{Z}_{11} & \mathbf{Z}_{12} \\ \mathbf{Z}_{21} & \mathbf{Z}_{22} \end{bmatrix} \begin{bmatrix} \mathbf{j}_e \\ \tilde{\mathbf{j}}_{m^*} \end{bmatrix} = \begin{bmatrix} \tilde{\mathbf{e}}_0 \\ \mathbf{h}_0 \end{bmatrix}, \tag{7}$$

where:

- $\mathbf{Z}_{11} = \mathbf{R}_e + i\omega\mathbf{L}_e - \frac{1}{i\omega}\tilde{\mathbf{G}}^a_{\Omega_c \sqcup \Omega_d} \mathbf{P}_e \mathbf{D}^a_{\Omega_c \sqcup \Omega_d}$,
- $\mathbf{Z}_{12} = \mathbf{M}_{1/\varepsilon_0} \mathbf{C}_{\Omega_c \sqcup \Omega_d} \mathbf{L_{em}}$,
- $\mathbf{Z}_{21} = -\mathbf{M}_{1/\mu_0} \tilde{\mathbf{C}}_{\Omega_m} \mathbf{L_{me}}$,
- $\mathbf{Z}_{22} = \mathbf{R}_m + i\omega\mathbf{L}_m - \frac{1}{i\omega}\tilde{\mathbf{G}}^a_{\Omega_m} \mathbf{P}_m \mathbf{D}^a_{\Omega_m}$.

$\mathbf{R}$, $\mathbf{L}$, and $\mathbf{P}$ are the "traditional" PEEC resistance, inductance and potential matrices, respectively, $\mathbf{L_{em}}$ and $\mathbf{L_{me}}$ are "inductance" matrices representing the coupling between electric and magnetic domains, whereas $\mathbf{M}_{1/\varepsilon_0}$ and $\mathbf{M}_{1/\mu_0}$ are mass matrices. Then, the system can be solved by applying a change of variables (from faces to edges of the mesh) and a projection into a reduced set of equations [4].

IV. Numerical Results

The 3-D Hertz-PEEC code has been developed with MATLAB® for the system assembly and data handling, while MEX-FORTRAN functions combined with OpenMP libraries have been adopted for the computation of the matrix coefficients and post-processing.

The code has been validated on several benchmarks, including the case, shown here, of two spheres with 1m radius, a dielectric one ($\varepsilon_r = 2$) and a magnetic one ($\mu_r = 10$) placed 3m apart on the y-axis and excited by a linearly polarized plane wave $\mathbf{E_0} = e^{-ik_0x}\mathbf{u}_z$. Where $k_0 = 2\pi f\sqrt{\varepsilon_0\mu_0}$, $f = 30$MHz. The magnitude of the real and imaginary part of the scattered electric field has been compared with the Radio-Frequency module of COMSOL® with good agreement (Fig. 1). Small discrepancies are due to the sphere meshes required by PEEC and FEM, the intrinsic differences of the two approaches, and the numerical post processing adopted for PEEC.

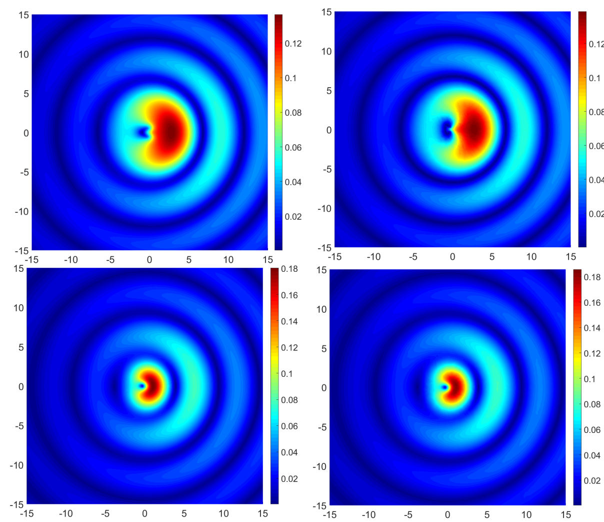

Fig. 1. Scattered Electric Field magnitude V/m, xz-plane, dimensions in m. Left: 3-D PEEC-Hertz. Right: COMSOL®. Top: $\Re$ part. Bottom: $\Im$ part.

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