The 3D Fractional Modeling of Electromagnetic Sub-Diffusion Based on FDTD

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Abstract **─** The anomalous diffusion has been discovered in many natural motions, it is defined as a phenomenon that does not conform to FICK's diffusion law. One of the anomalous diffusions is the electromagnetic sub-diffusion, which indicated the power law decay rate is slower than normal -2/5. In this paper, we modeled electromagnetic sub-diffusion based on 3D finite-different time-domain (FDTD) method. Through the introduction of roughness parameter in the definition of conductivity and the discretization of fractional integrations, the electromagnetic sub-diffusion can be efficiently modeled. The improved method is verified by homogeneous half-space models and anomalous models with 3D bodies, the results show that it can model 3D electromagnetic sub-diffusion with high precisions and has a good performance in the recognitions of anomalous bodies.

Index Terms **─** Electromagnetic sub-diffusion, finite-different time-domain method, fractional calculus.

I. INTRODUCTION

In the modeling of electromagnetic propagation, the electrical conductivity of the ground is usually imagined to be uniformly and constant [1-4]. However, the ground conductivity usually presents heterogeneity and nonlinearity which results in anomalous diffusion occurred in the measured data [5-7]. One of the anomalous diffusions is called sub-diffusion [8]. It manifests as the measured data decays slower especially in late time. In this case, the measured data can't be explained accurately based on the classical electromagnetic theory, which has hindered the application of electromagnetic method in the mineral resource's exploration and other fields in a way. The previous researches have indicated that the fractional diffusion equation can provide the theoretical basis for the electromagnetic sub-diffusion which only need to introduce roughness parameter in the expression of electrical conductivity [9-10]. Accordingly, the fractional calculous should be solved in time domain which makes the discretization of the electromagnetic fields difficult. With the development of fractional derivative calculation in mathematics, lots of fractional order finite difference algorithms are developed [11-13], which provides a possibility for the electromagnetic sub-diffusion modeling in time domain.

In this paper, we introduce roughness parameter in the expression of electrical conductivity in frequency domain, and discrete fractional items after the frequency-time transformation. Accordingly, the iterative equations of electromagnetic fields are derived based on a FDTD method. At last the improved method is verified by different models, the results indicated that it can model electromagnetic sub-diffusion well and provide basis for a future study on the electromagnetic anomalous induction in time domain.

II. METHOD

After introduced the roughness parameter β ($0 < \beta < 1$), the electoral conductivity σ_β can be expressed as [9, 10]:

$$\sigma_\beta(\omega) = \sigma_0 + k\sigma_0 (i\omega)^{-\beta}. \tag{1}$$

The Ampere's law without sources in frequency domain:

$$\nabla \times H(\omega) = \sigma_\beta(\omega) E(\omega) + j\omega\varepsilon E(\omega), \tag{2}$$

can be transformed into time domain:

$$\nabla \times H(t) = \sigma_\beta(t) * E(t) + \varepsilon \frac{\partial E(t)}{\partial t}. \tag{3}$$

The convolution item in equation (3) can be expressed as:

$$\begin{aligned} \sigma_\beta(t) * E(t) &= \sigma_0 E(t) + k\sigma_0 \, {}_0D_t^{-\beta} E(t) \\ &= \sigma_0 E(t) + \frac{k\sigma_0}{\Gamma(\beta)} \int_0^t (t-\tau)^{\beta-1} E(\tau) d\tau \end{aligned}, \tag{4}$$

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where $\Gamma(\beta)$ indicates the Gamma function:

$$\Gamma(z)=\int_0^{\infty} e^{-t}t^{z-1}dt, \qquad \mathrm{Re}(z)>0. \tag{5}$$

Discretized (4) can get:

$$\begin{aligned}
&\int_0^{t_n}(t_n-\tau)^{\beta-1}E(\tau)d\tau=\int_0^{\Delta t_1}(t_n-\tau)^{\beta-1}E(\tau)d\tau+\\
&\int_{\Delta t_1}^{\Delta t_1+\Delta t_2}(t_n-\tau)^{\beta-1}E(\tau)d\tau+\int_{\Delta t_1+\Delta t_2}^{\Delta t_1+t\Delta t_2+\Delta t_3}(t_n-\tau)^{\beta-1}E(\tau)d\tau+\\
&\ldots\ldots+\int_{t_n-\Delta t_n}^{t_n}(t_n-\tau)^{\beta-1}E(\tau)d\tau \quad .\\
&=\frac{1}{\beta}\left(\begin{array}{l}\frac{E_0+E_1}{2}\left((t_n)^{\beta}-(t_n-t_1)^{\beta}\right)+\frac{E_1+E_2}{2}\left((t_n-t_1)^{\beta}-(t_n-t_2)^{\beta}\right)\\ +\ldots\ldots+\frac{E_{n-1}+E_n}{2}\left((t_n-t_{n-1})^{\beta}-(t_n-t_n)^{\beta}\right)\end{array}\right)
\end{aligned} \tag{6}$$

Inserting (6) into (3), and after discretion based on FDTD [14-17], we can get the iterative formulation of electric field:

$$\begin{aligned}
&E_x^{n+1}(i+\tfrac{1}{2},j,k)=\frac{-\sigma_0\Delta t_n+2\varepsilon}{\sigma_0\Delta t_n+2\varepsilon}E_x^{n+1}(i+\tfrac{1}{2},j,k)+\\
&\frac{2\Delta t_n}{(\sigma_0\Delta t_n+2\varepsilon)\Delta y(i,j,k)}(H_z^{n+1/2}(i+\tfrac{1}{2},j+\tfrac{1}{2},k)-H_z^{n+1/2}(i+\tfrac{1}{2},j-\tfrac{1}{2},k))-\\
&\frac{2\Delta t_n}{(\sigma_0\Delta t_n+2\varepsilon)\Delta z(i,j,k)}(H_y^{n+1/2}(i+\tfrac{1}{2},j,k+\tfrac{1}{2})-H_y^{n+1/2}(i+\tfrac{1}{2},j,k-\tfrac{1}{2}))-\\
&\frac{2k\sigma_0\Delta t}{(\sigma_0\Delta t_n+2\varepsilon)\beta\Gamma(\beta)\Delta z(i,j,k)}B_x^{n+1/2}(i+\tfrac{1}{2},j,k)
\end{aligned}, \tag{7}$$

where

$$\begin{aligned}
&B_x^{n+1/2}(i+\tfrac{1}{2},j,k)=\\
&\frac{E_0+E_1}{2}\left((t_{n+1/2}\text{-}t_0)^{\beta}-(t_{n+1/2}-t_1)^{\beta}\right)+\frac{E_1+E_2}{2}\left((t_{n+1/2}-t_1)^{\beta}-(t_{n+1/2}-t_2)^{\beta}\right)+\ldots\ldots\\
&+\frac{E_{n-1}+E_n}{2}\left((t_{n+1/2}-t_{n-1})^{\beta}-(t_{n+1/2}-t_n)^{\beta}\right)+E_n\left((t_{n+1/2}-t_n)^{\beta}-(t_{n+1/2}-t_{n+1/2})^{\beta}\right)
\end{aligned}. \tag{8}$$

For the magnetic fields, we chose equation (9) as the control equation of H_x and H_y and equation (10) as the control equation of H_z,

$$\nabla\times E=-\mu\frac{\partial H}{\partial t}, \tag{9}$$

$$\nabla\cdot H=0. \tag{10}$$

Equation (9) and (10) can be expressed in components as equation (11-13):

$$\frac{\partial E_z}{\partial y}-\frac{\partial E_y}{\partial z}=\mu\frac{\partial H_x}{\partial t}, \tag{11}$$

$$\frac{\partial E_x}{\partial z}-\frac{\partial E_z}{\partial x}=\mu\frac{\partial H_y}{\partial t}, \tag{12}$$

$$\frac{\partial H_z}{\partial z}=-\frac{\partial H_x}{\partial x}-\frac{\partial H_y}{\partial y}. \tag{13}$$

The discrete form of equation (11, 13) is performed by a FDTD method [9-11] as:

$$\begin{aligned}
&H_x^{n+1/2}(i+1,j+\frac{1}{2},k\frac{1}{2})=H_x^{n-1/2}(i+1,j+\frac{1}{2},k\frac{1}{2})+\\
&\frac{\Delta t_{n-1}+\Delta t_n}{2\mu}\left[\frac{E_y^n\left(i,j+\frac{1}{2},k+1\right)-E_y^n\left(i,j+\frac{1}{2},k\right)}{\Delta z(i,j,\mathrm{k})}-\frac{E_z^n\left(i,j+1,k+\frac{1}{2}\right)-E_z^n\left(i,j,k+\frac{1}{2}\right)}{\Delta y(i,j,\mathrm{k})}\right]
\end{aligned}, \tag{14}$$

$$\begin{aligned}
&H_z^{n+1/2}(i+\frac{1}{2},j+\frac{1}{2},k)=H_z^{n+\frac{1}{2}}(i+\frac{1}{2},j+\frac{1}{2},k+1)+\Delta z(i,j,1)\times\\
&\left[\frac{H_x^{n+1/2}\left(i+1,j+\frac{1}{2},k+\frac{1}{2}\right)-H_x^{n+1/2}\left(i,j+\frac{1}{2},k+\frac{1}{2}\right)}{\Delta x(i,j,1)}+\frac{H_y^{n+1/2}\left(i+1,j+\frac{1}{2},k+\frac{1}{2}\right)-H_y^{n+1/2}\left(i+\frac{1}{2},j,k+\frac{1}{2}\right)}{\Delta y(i,j,1)}\right]
\end{aligned}. \tag{15}$$

III. EXAMPLES

To test the effectiveness of the improved method, homogeneous half-space models and anomalous models with 3D bodies are designed. All models have 117×117×58 grids. The grid is non-uniform with a smallest spacing of 10 m and a largest spacing of 120 m. The transmitting coil is located at the center of the model with a 120 m height, the radius is 7.5 m. The transmitting current is 30 A. The receiving coil is 130 m away from the transmitting coil with a height of 60 m. The electrical conductivity is set as 10 S/m and k=1. In Fig. 1, the responses with different roughness parameters are compared.

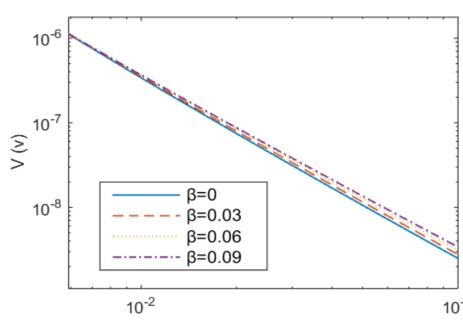
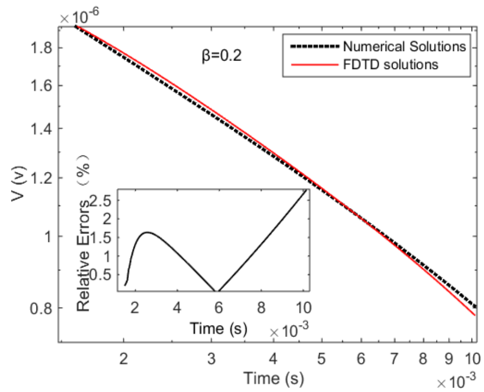
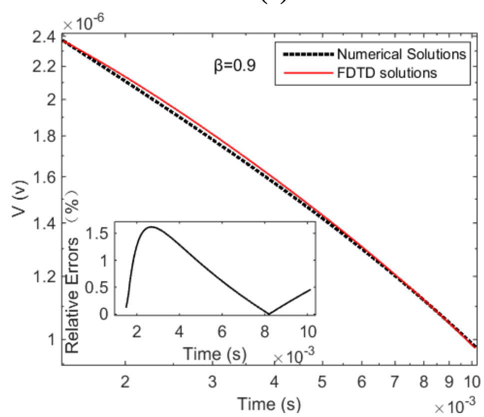


Fig. 1. The induced voltage with different roughness parameters.

The roughness parameter is set as 0, 0.03, 0.06 and 0.09. From Fig. 1 we can find that the induced voltage decay slowly as the increase of β which have indicated the improved method can model electromagnetic sub-diffusion efficiently. To verify the precision of the FDTD method, the FDTD solutions are compared with the numerical solutions calculated by integral method [18] in Fig. 2. The roughness parameters are chosen as 0.2 and 0.9. Figure 2 (a) shows the comparison of the two solutions and the relative errors responsibly when β=0.2. We can find the two solutions coincide well with a max relative error of 2.6%. Figure 2 (b) shows the comparison and the relative errors responsibly when β=0.9. The relative errors are less than 1.6% in 10 ms. The electromagnetic responses in the air with different roughness parameters are shown in Fig. 3.



(a)

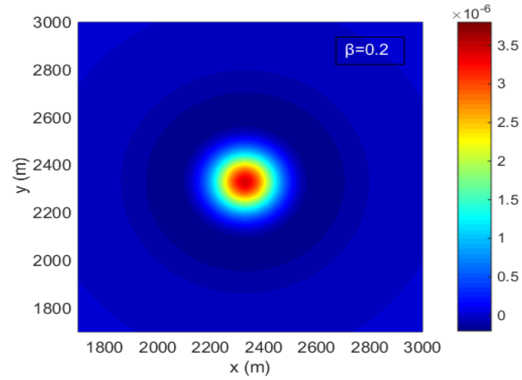


(b)

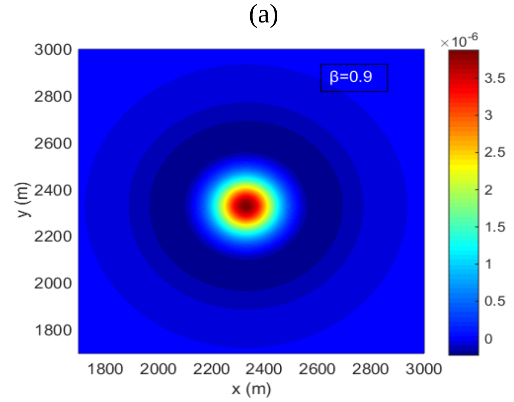
Fig. 2. The comparison of FDTD solutions and numerical solutions and the relative errors when: (a) β=0.2 and (b) β=0.9.

According to the definition of the generalized electrical conductivity, the conductivity varies with time, so the roughness parameter doesn't affect the diffusion pattern of electromagnetic wave. Accordingly, the responses decay slowly with the increase of roughness parameter.

The anomalous model is designed as Fig. 4. The roughness parameter is 0.7. The depth of the 3D body is 100 m, the size of the body is 410 m×410 m×450 m and is set in the center of x-y plane. The conductivity of the body is 100 S/m and the conductivity of the background is 5 S/m. The slices of electromagnetic responses of 1.5 ms and 5 ms are shown in Fig. 5. From these slices we can find that the responses can reflect the information of the 3D body well, which has verified the effectiveness of the improved method well again.



(a)



(b)

Fig. 3. The electromagnetic responses in the air with different roughness parameters: (a) β=0.2 and (b) β=0.9.

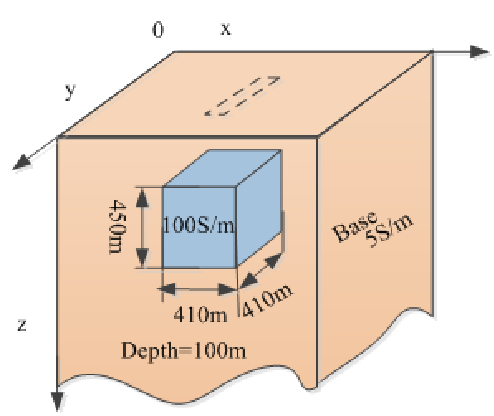
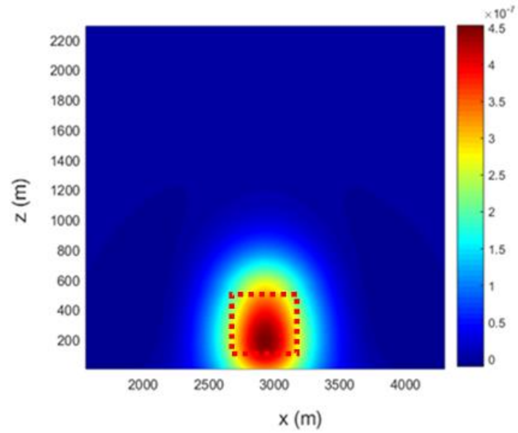
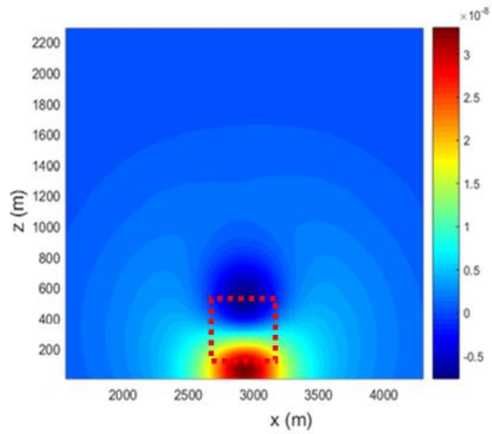


Fig. 4. Anomalous model with single 3D body.



(a)



(b)

Fig. 5. The slices of electromagnetic responses at the time of: (a) 1.5 ms and (b) 5 ms.

IV. CONCLUSIONS

We have introduced roughness parameter in the expression of electrical conductivity. After the discretization of the fractional item we got the iterative formulation of electric field based on FDTD. The modeling results validated the effectiveness of the improved method in the modeling of electromagnetic sub-diffusion. As the discretization of the fractional item involved electric fields of every time-step, large memory needed consequently. For high-resistance models, as more time-steps divided, the method may be limited by the computer's storage. How to reduce the memory consumption is the focus of our following research.

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