# A New Iterative Moment Method Solution for Conducting Bodies of Revolution

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***Abstract*-An iterative solution of the *combined field integral equation* (CFIE) has been obtained for perfectly conducting bodies. The proposed technique avoids the inversion of the *moment method* (MoM) matrix by using an iterative algorithm which shows fast convergence properties, mainly for large bodies. The algorithm is applied to several 3D problems involving *bodies of revolution* (BOR). Accurate currents are computed with important speedups over direct MoM solutions.**

## I. Introduction

The scattering by large metallic bodies has been treated extensively in the literature using high frequency methods [1]. This choice was mainly motivated by the excessive computer time and storage requirements of more rigorous low frequency methods based on integral equation formulations.

But this boundary between high and low frequency problems is changing due to the developments in hardware technology with faster processors and the ability to handle large amounts of data. However, these improvements, on their own, are not able to extend the range of applicability of low frequency methods enough.

To overcome these problems, several algorithms have been developed. These considerably reduce the amount of data and computer time with respect to the usual moment method solution. It is worth mentioning, among others, the impedance matrix localization (IML) technique [2], the application of wavelet expansions [3], the fast multipole method (FMM) [4], the spatial decomposition technique (SDT) [5] and the multilevel matrix decomposition algorithm (MLMDA) [6].

Other methods exploit the fact that the *magnetic field integral equation* (MFIE) is a Fredholm equation of second type, which can be solved iteratively. In [7], this approach is used to obtain the scattered fields by two-dimensional perfect electric conducting (PEC) bodies. In [8], a similar algorithm was successfully combined with aperture integration [9] to obtain the radar cross section (RCS) of large open-ended cavities.

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The outline of this paper is as follows. In section II we present a modified version of the MFIE which shows better convergence properties than the original equation. Next, in section III, another iterative method based on the *combined field integral equation* (CFIE) [10] is formulated, showing its numerical implementation and discussing its relation with respect to conventional iterative algorithms. In section IV, several results are presented to show the efficiency and accuracy of the proposed approach. Finally, section V summarizes the main conclusions of the paper.

## II. Magnetic Field Iterative Algorithm

### A. Original Approach

The MFIE is obtained by imposing the boundary condition of the magnetic field over a perfectly conducting surface $S$. This can be expressed as

$$\mathbf{J}(\mathbf{r}) = \hat{n}\times\left(\mathbf{H}^i(\mathbf{r})+\mathbf{H}^s(\mathbf{r})\right), \quad \mathbf{r}\in S \tag{1}$$

where $\hat{n}$ is the unit outward vector to $S$, $\mathbf{H}^s$ the scattered magnetic field due to the electric surface currents on $S$ and $\mathbf{H}^i$ is the incident magnetic field. For smooth surfaces $\mathbf{H}^s$ is given by

$$\mathbf{H}^s(\mathbf{r}) = \mathbf{J}(\mathbf{r})\times\frac{\hat{n}}{2} + L_H(\mathbf{J}) \tag{2}$$

with

$$L_H(\mathbf{J}) = PV\iint_{S'}\nabla\times\left(\mathbf{J}(\mathbf{r}')G(R)\right)dS' \tag{3}$$

where $PV$ stands for the *principal value* of the integral, and $G(R)=\exp(-jkR)/4\pi R$ is the *free space Green's function*, where $R=|\mathbf{r}-\mathbf{r}'|$ is the distance between the test and source points. Therefore, the expression of the MFIE is

$$\mathbf{J}(\mathbf{r}) = \frac{\mathbf{J}(\mathbf{r})}{2} + \hat{n}\times\left(\mathbf{H}^i(\mathbf{r}) + L_H(\mathbf{J})\right) \tag{4}$$

which can be rewritten in a more usual form as

$$\mathbf{J}(\mathbf{r}) = 2\hat{n}\times\left(\mathbf{H}^i(\mathbf{r}) + L_H(\mathbf{J})\right) \tag{5}$$

Even though this equation applies to the actual value of the current, an iterative solution can be obtained by assuming that the currents **J** in the right-hand side (RHS) and left-hand side (LHS) correspond to the solution of the stages $k$ and $(k+1)$, respectively. Nevertheless, this algorithm may diverge, and otherwise the rate of convergence usually is too slow due to the properties of the corresponding iteration matrix.

### *B. Relaxed Iteration*

To overcome this problem, this equation can be rearranged by adding the term $(1-\gamma)\mathbf{J}/\gamma$ to both sides of the equation, where $\gamma$ is a positive real number. After some manipulations, this leads to the following equation

$$\mathbf{J}_{k+1}(\mathbf{r}) = 2\gamma\hat{n}\times\left(\mathbf{H}^i(\mathbf{r}) + L_H(\mathbf{J}_k)\right) + (1-\gamma)\mathbf{J}_k \tag{6}$$

It is worth noting that this iteration corresponds to (5) when $\gamma = 1$, and, as mentioned before, the convergence is not guaranteed in this case. On the other hand, as the value of $\gamma$ approaches zero, the rate of convergence decreases. Numerical tests have shown that a value of $\gamma = 1/2$ usually gives the best results. In this case, the algorithm given by (6) is equivalent to the iteration of the original expression of the MFIE given by (4).

Nevertheless, even for $\gamma = 1/2$ the convergence is still slow, so the application of this method is impractical in most cases. This fact motivated the search for a new iterative method, that gave rise to the algorithm described in the following section.

## III. Combined Field Iterative Algorithm

### *A. Formulation*

The CFIE is obtained as a linear combination of the MFIE and the *electric field integral equation* (EFIE) [10]. This second equation is obtained by imposing the boundary condition of the electric field given by

$$\mathbf{E}^s_{tan}(r) = -\mathbf{E}^i_{tan}(r), \quad \mathbf{r}\in S \tag{7}$$

where $\mathbf{E}^i_{tan}$ and $\mathbf{E}^s_{tan}$ are the tangential components on $S$ of the incident and scattered electric fields, respectively. For smooth surfaces, $\mathbf{E}^s_{tan}$ can be expressed by

$$E^s_{tan}(r) = L_E(\mathbf{J}) = \left(-j\omega\mathbf{A} - \frac{j}{\beta\eta}\nabla(\nabla\cdot\mathbf{A})\right)_{tan} \tag{8}$$

where $\beta$ and $\eta$ are the propagation constant and characteristic impedance of free space, respectively, and **A** is the vector potential, given by

$$\mathbf{A}(\mathbf{r}) = \mu\iint_{S'} \mathbf{J}(\mathbf{r'})G(R)dS' \tag{9}$$

Therefore the expression of the CFIE is

$$\mathbf{J}(\mathbf{r}) = 2\hat{n}\times\left(\mathbf{H}^i(\mathbf{r}) + L_H(\mathbf{J})\right) + \frac{\alpha}{\eta}\left(\mathbf{E}^i_{tan}(\mathbf{r}) + L_E(\mathbf{J})\right) \tag{10}$$

where the factor $\alpha$ is an arbitrary constant. It can be shown that the solution of (10) is unique whenever $\alpha$ is a positive real number [10].

Applying the relaxation parameter described in the previous section for the MFIE, one can find a relaxed version of the CFIE which shows much better behavior for our purposes

$$\mathbf{J}_{k+1}(\mathbf{r}) = 2\gamma\hat{n}\times\left(\mathbf{H}^i(\mathbf{r}) + L_H(\mathbf{J}_k)\right) + (1-\gamma)\mathbf{J}_k + \frac{\alpha}{\eta}\left(\mathbf{E}^i_{tan}(\mathbf{r}) + L_E(\mathbf{J}_k)\right) \tag{11}$$

### *B. Matrix Solution*

To obtain a numerical solution of eq. (11), the current distribution must be approximated by using an interpolation method in order to evaluate the RHS of (11). This step corresponds to the *basis functions* selection process of the conventional MoM [11]. Finally, the matrix equation is obtained applying the dot product with a set of *testing functions*. In our approach *delta* functions are used, which leads to the conventional *point matching* [11]. Therefore, the equation is imposed at a discrete set of points, which in this case are selected to be the midpoints of each basis domain. This matrix equation can be written as

$$J_{k+1} = \left(2\gamma\left[H_{MoM}\right] + \alpha/\eta\left[E_{MoM}\right] + (1-\gamma)\left[I\right]\right) J_k + \left(2\gamma H^{inc} + \alpha/\eta\, E^{inc}\right) \tag{12}$$

It can be seen that the part of (12) which is related to the MFIE corresponds to the *simultaneous overrelaxation method* (JOR method) [12], and the other, which comes from the EFIE corresponds to the *stationary Richardson* (RF) method [12], so the whole iteration is a linear combination of both of them.

As in any other iterative method, the behavior of the algorithm is mainly given by the spectral radius $\rho$ of the iteration matrix [12], which depends on the value of the weighting parameter $\alpha$ and the relaxation parameter $\gamma$. It has to be pointed that the election of the optimum values of $\alpha$ and $\gamma$ implies an eigenvalue problem which is as expensive as the direct MoM solution. Therefore, approximate empirical rules have to be used to find them.

The relaxation parameter $\gamma$ can be chosen following the reasoning used for the magnetic field iteration. In relation to the weighting parameter, it has to be noticed that $\alpha = 0$ turns (11) into (6), so it does not make any sense. On the other hand, numerical tests have shown that when $\alpha$ approaches one, the method becomes unstable, thus an intermediate value of $\alpha = 1/2$ was selected for the numerical examples shown in this paper.

*D. Computational Cost*

The computational cost can be divided into two parts: 1) the time spent for filling-in the impedance matrix, and 2) the time required to obtain the current distribution for a given excitation. The first task involves the same number of operations for both direct MoM and the iterative method. The numerical advantage is achieved in the second step, because, while the inversion of the impedance matrix is an $O(N^3)$ problem, where $N$ is the number of unknowns, the iterative approach is just $O(N_{\text{iter}} \cdot N^2)$. As a consequence, this algorithm is computationally efficient if the number of required iterations is lower than the number of unknowns. In practice, convergence is achieved in much less iterations, as will be shown in the following section.

## IV. Numerical Results

A computer program has been written which implements the algorithm described below for a body with rotational symmetry illuminated by an incident plane wave (Fig. 1). Several results for a test geometry are shown to verify the efficiency and accuracy of the proposed approach.

For numerical purposes, a coordinate system is defined $(n, \phi, t)$ depicted in Fig. 1. In order to apply the integral equation, the surface currents **J** and the electric and magnetic fields (**E**, **H**) on surface $S$ are expressed as a superposition of two orthogonal components, $t$ and $\phi$. To take advantage of the rotational symmetry of the problem, the currents, the fields and scalar Green's functions are expanded in Fourier series in coordinate $\phi$. For example, the current distribution **J** can be expressed as a sum of azimuthal modes as

$$\mathbf{J} = \sum_{m=0}^{\infty} J_t^m(t) \begin{matrix}\cos\\ \sin\end{matrix}(m\phi)\,\hat{t} + \sum_{m=0}^{\infty} J_\phi^m(t) \begin{matrix}\sin\\ -\cos\end{matrix}(m\phi)\,\hat{\phi} \qquad (13)$$

where the upper and lower symbols apply for even and odd modes, respectively. While the mode expansion is an infinite series, the series can be truncated in practice [13]. Each term $J_t^m(t)$ and $J_\phi^m(t)$, only show variations with coordinate $t$, thus the original 3D problem is reduced to a series of 2D problems [10,13–15].

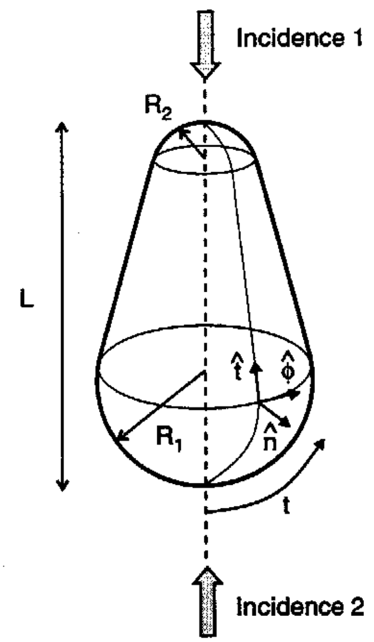



Fig. 1 Cone with spherical caps used as test geometry, showing the coordinate system. $R_1 = 0.5$ m, $R_2 = 0.2$ m, $L = 1.5$ m.

For the numerical examples shown in this paper, axial incidence is assumed, thus only the mode $m = 1$ has to be considered.

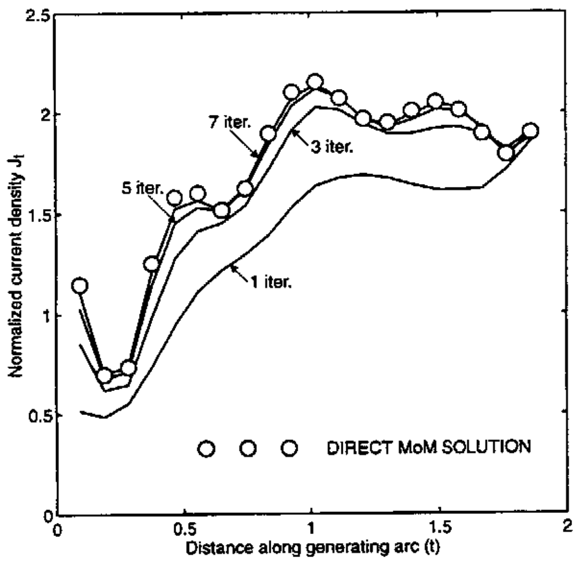
To check the accuracy of the results, the mean relative error in the current distribution with respect to a reference solution obtained by direct inversion of the MoM matrix (CFIE) is defined as

$$\xi = \frac{\text{mean}\left|\mathbf{J} - \mathbf{J}_{ref}\right|}{\max\left|2\,\mathbf{H}^i\right|} \times 100\ (\%) \qquad (14)$$

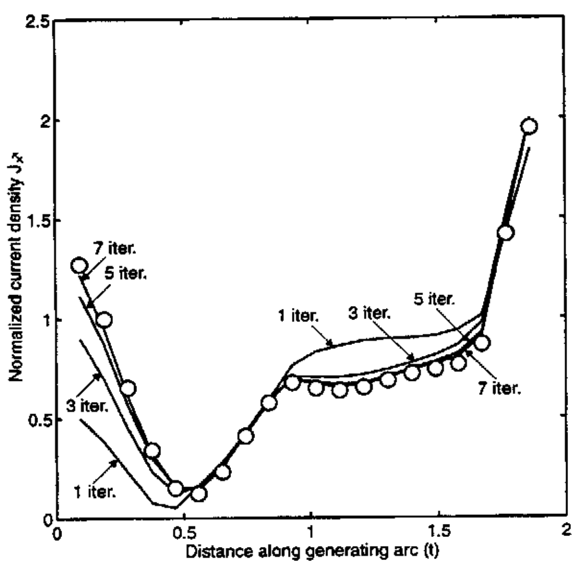
Fig. 2.a and 2.b show the normalized current densities $J_t(t)/H^i$ and $J_\phi(t)/H^i$ for direction of incidence 1 (Fig. 1). Fig. 3.a and 3.b show the same result, but for direction of incidence 2. The convergence in terms of the number of iterations is shown, demostrating that 7 iterations give an excellent accuracy, in both cases. A discretization of 20 triangular basis functions along the generating arc was employed, which average 10 basis per linear wavelength. This grid spacing is used for the following results.

Fig. 4 shows the mean relative error defined in (14) versus the number of iterations, comparing the combined field with the magnetic field iterative algorithms. It can be seen that the convergence is much faster for the combined field algorithm than for the previous magnetic field approach.

Fig. 5.a and 5.b show the error defined in (14) for the two components of the current distribution in terms of the number of iterations. Two different frequencies have been considered: 0.3 GHz and 1.8 GHz, maintaining the same grid spacing of 10 basis per linear wavelength. It can be seen that the convergence is faster for the higher frequency, which makes the algorithm suitable for analyzing electrically large bodies.

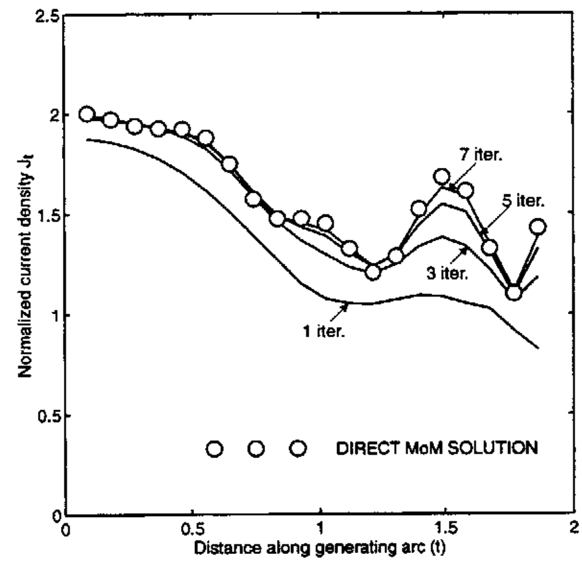



(a)

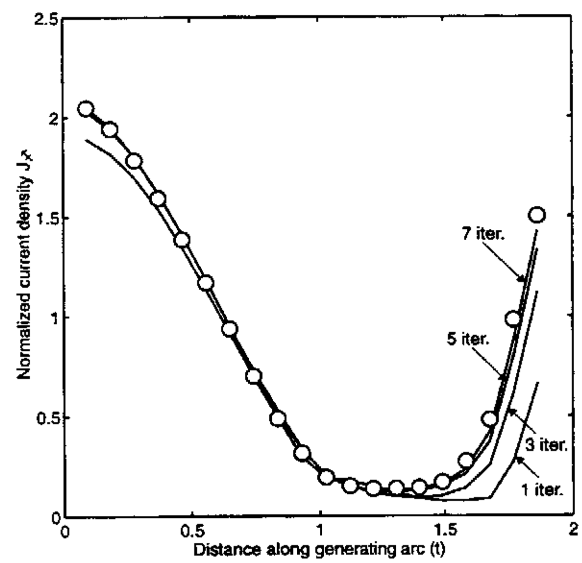



(b)

Fig. 2. Current distribution induced on cone-sphere for direction of incidence 1 in Fig. 1. The frequency is 0.3 GHz.




(a)




(b)

Fig. 3. Current distribution induced on cone-sphere for direction of incidence 2 in Fig. 1. The frequency is 0.3 GHz.

Table I shows the relative computational cost with respect to the direct MoM solution to obtain a normalized mean error in the current of 2 %. These data were obtained for direction of incidence 1. Similar results were obtained for the other incidence.

TABLE I
RELATIVE COMPUTATIONAL COST WITH RESPECT TO MOM SOLUTION

| Frequency (GHz) | CPU time (%) |
|---|---|
| 0.3 | 40 |
| 1.8 | 5 |

## V. CONCLUSIONS

This study of new iterative algorithms has so far led to several conclusions:

1) The unrelaxed magnetic field iterative algorithm does not always give a correct result for the induced currents.
2) The application of a relaxation parameter improves the stability, but convergence is too slow.
3) The relaxed combined field iterative algorithm produces accurate results in a small number of iterations.
4) Furthermore, the convergence rate increases with body size, making this algorithm suitable for high frequency scattering problems.

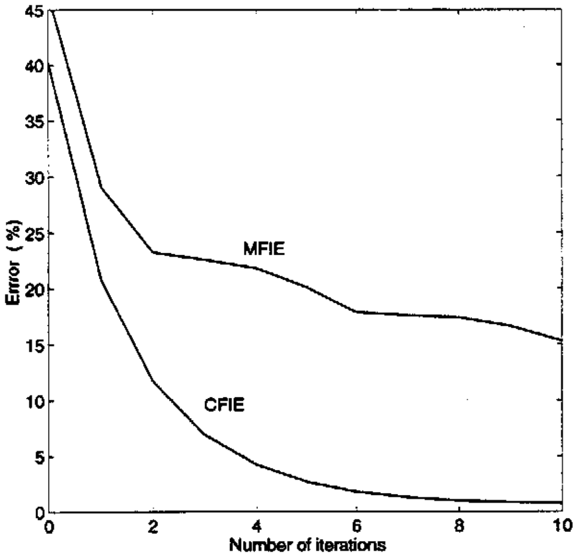



Fig. 4. Comparison between combined field and magnetic field iterative algorithms for direction of incidence 1. The frequency is 0.3 GHz.

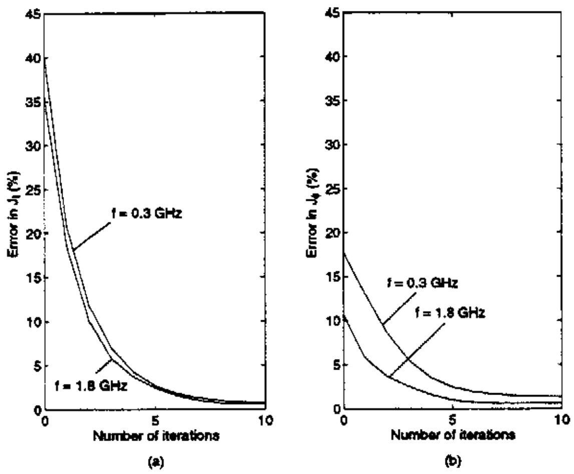



Fig. 5. Mean relative error for two diferent frequencies: 0.3 and 1.8 GHz (direction of incidence 1).

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