ON THE CONVERGENCE OF THE METHOD OF MOMENTS, THE BOUNDARY-RESIDUAL METHOD, AND THE POINT-MATCHING METHOD WITH A RIGOROUSLY CONVERGENT FORMULATION OF THE POINT-MATCHING METHOD

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ABSTRACT

The method of moments reduces to the boundary-residual method or the point-matching method with a suitable weighting function. This paper shows another means by which these three methods can produce equivalent results. Arguments are given as to why point matching can fail to converge, while the other two methods rigorously converge. An example is given to support these arguments.

EQUIVALENCE OF METHODS

The method of moments [1], the boundary-residual method [2, 3], and the point-matching method [4] are three seemingly different methods for field computation. Harrington [1] has shown, however, how the method of moments encompasses the other two methods through the proper selection of weighting functions. Another means exists by which all three methods can become computationally equivalent.

Consider the problem posed from the perspective of the method of moments [1]. A deterministic equation such as

$$L \sum_i \alpha_i f_i(s) = g(s) \tag{1}$$

is to be solved over some range s. The equation, as it applies to electromagnetics, may satisfy the boundary conditions of a particular problem, e.g., the continuity of the tangential fields across the boundary. The summation then represents the field within a region, and the operator L produces the tangential fields at the boundary s. g(s) is the value of the tangential fields from, say, the known incident field. A weighting function W_i can be multiplied on both sides of Eq. 1 and integrated over the boundary s to produce a matrix equation:

$$\bar{\bar{M}}\,\vec{\alpha} = \vec{g} \tag{2}$$

with

$$\bar{\bar{M}}_{ij} = \int_s W_i(s)\, L\, f_j(s)\, ds \tag{3}$$

$$g_i = \int_s W_i(s)\, g(s)\, ds \tag{4}$$

The boundary-residual method can be derived from these equations by setting the weighting functions equal to

$$W_i(s) = \left(L\, f_i(s)\right)^* \tag{5}$$

where * denotes the conjugate operator. The truth of this assertion can be shown by defining the residual along the boundary,

$$R(s) = L \sum_i \alpha_i\, f_i(s) - g(s) \tag{6}$$

and minimizing the integral of the residual magnitude over the boundary in the least-squares sense. The minimization is with respect to each of the unknown coefficients α_i:

$$\begin{aligned}\frac{\partial}{\partial \alpha_i^*} \int_s |R(s)|^2\, ds = 0 &= \frac{\partial}{\partial \alpha_i^*} \Bigg\{ \sum_{i,j} \alpha_i^* \alpha_j \int_s \left(L\, f_i(s)\right)^* L\, f_j(s)\, ds \\ &\quad -2\mathrm{Re} \sum_i \alpha_i^* \int_s \left(Lf_i(s)\right)^* g(s)\, ds + \int_s g^*(s)\, g(s)\, ds \Bigg\} \\ &= 2 \sum_j \alpha_j \int_s \left(L\, f_i(s)\right)^* L\, f_j(s)\, ds \\ &\quad - 2 \int_s \left(L\, f_i(s)\right)^* g(s)\, ds \end{aligned} \tag{7}$$

which implies Eq. 5. For point matching, the weighting function is a delta function given by

$$W_i(s) = \delta\left(s - s_j\right) \tag{8}$$

so that Eq. 3 becomes

$$\bar{\bar{M}}_{ij} = \int_s \delta(s - s_j)\, L\, f_j(s)\, ds$$

$$= L\, f_j(s_i) \tag{9}$$

where s_i is a sample point along the boundary.

Now consider the practical implementation of Eqs. 2-4. The integrals are usually evaluated numerically so that Eqs. 3 and 4 become sums:

$$\bar{\bar{M}}_{ij} = \sum_{p=1}^{m} q_p W_i(s_p)\, L\, f_j(s_p) \tag{10}$$

$$g_i = \sum_{p=1}^{m} q_p\, W_i(s_p)\, g(s_p) \tag{11}$$

where q_p are the weights of a Gaussian quadrature integration method [5]. m is the number of points of the integration method. Assume that the number of functions f_i in Eq. 1 ranges from 1 ... n. The matrix equation to solve becomes

$$\begin{bmatrix} \sum_{p=1}^{m} q_p W_1(s_p) Lf_1(s_p) & \cdots & \sum_{p=1}^{m} q_p W_1(s_p) Lf_n(s_p) \\ \vdots & \vdots & \vdots \\ \sum_{p=1}^{m} q_p W_n(s_p) Lf_1(s_p) & \cdots & \sum_{p=1}^{m} q_p W_n(s_p) Lf_n(s_p) \end{bmatrix} \vec{\alpha} = \begin{bmatrix} \sum_{p=1}^{m} q_p W_1(s_p) g(s_p) \\ \vdots \\ \sum_{p=1}^{m} q_p W_n(s_p) g(s_p) \end{bmatrix} \tag{12}$$

It can be verified through direct matrix multiplication that Eq. 12 is equivalent to

$$\bar{\bar{Q}}^t\, \bar{\bar{P}}\, \vec{\alpha} = \bar{\bar{Q}}^t\, \vec{G} \tag{13}$$

where t denotes the matrix transpose and

$$\bar{\bar{Q}} = \begin{bmatrix} \sqrt{q_1}\, W_1(s_1) & \cdots & \sqrt{q_1}\, W_n(s_1) \\ \vdots & \vdots & \vdots \\ \sqrt{q_m}\, W_1(s_m) & \cdots & \sqrt{q_m}\, W_n(s_m) \end{bmatrix} \tag{14}$$

$$\bar{\bar{P}} = \begin{bmatrix} \sqrt{q_1}\, Lf_1(s_1) & \cdots & \sqrt{q_1}\, Lf_n(s_1) \\ \vdots & \vdots & \vdots \\ \sqrt{q_m}\, Lf_1(s_m) & \cdots & \sqrt{q_m}\, Lf_n(s_m) \end{bmatrix} \tag{15}$$

$$\vec{G} = \begin{bmatrix} \sqrt{q_1}\, g(s_1) \\ \vdots \\ \sqrt{q_m}\, g(s_m) \end{bmatrix} \tag{16}$$

The number of rows of the matrix $\bar{\bar{Q}}$ is m, whereas the number of columns is n. If m is set equal to n, then the matrices in Eq. 13 become square; the problem then becomes equivalent to

$$\bar{\bar{P}}\, \vec{\alpha} = \vec{G} \tag{17}$$

This equation is equivalent to the point-matching method applied to Eq. 1 in which the number of functions f_i equals the number of boundary-sampling points. Because this solution no longer depends on the form of the weighting function, it is also equivalent to the boundary residual solution. Now, the boundary-residual method [2, 3] and the method of moments [1] are rigorously convergent, whereas point matching has been shown to fail to converge to the proper solution [6] in some cases. The discrepancy lies in the discretization inherent in the numerical integration routine used to compute Eqs. 3 and 4. By using too few integration points, to where the number of integration sample points (m) equals the number of fitting functions (n), the method of moments degrades to point matching. This conclusion was also reached by Djordjevic and Sarkar [15] although they do not discuss the failure of point matching as in the next two sections.

THE FAILURE OF POINT MATCHING WITH FUNCTIONS OF UNBOUNDED VARIATION

Why does point matching fail? Return to the first operation imposed by the method of moments on Eq. 1, i.e., integrating with respect to a weighting function:

$$\int_S W_j(s) \sum_i \alpha_i \; L\, f_i(s)\; ds = \int_S W_j(s)\; g(s)\; ds \qquad j = 1, \ldots, n \qquad (18)$$

It is assumed that the integral and the summation may be interchanged in order to create Eq. 2. Titchmarsh [7] proves that an infinite series may be multiplied by a function of bounded variation and integrated term by term. This theorem applies even when the series diverges. Now, the weighting function of the boundary residual method (Eq. 5) is such a function of bounded variation, and so the resulting equations created are valid. The point-matching method uses a delta function as a weighting function (Eq. 8), which is not of bounded variation; bringing the integral inside the summation is not proven to be valid unless the series is uniformly convergent [8], and thus the resulting point-matching equations may or may not be valid. What can be said is that when the series in Eq. 18 satisfies the Rayleigh hypothesis [9], the series converges uniformly [9], and point matching is valid. This view is consistent with Lewin [10].

THE FAILURE OF POINT MATCHING BY REPEATED LIMITS

This paper shows that point matching, and indeed the method of moments and the boundary-residual method, may fail for another reason. Consider the numerical form of Eq. 18:

$$\sum_{p=1}^{m} q_p W_j(s_p) \sum_{i=1}^{n} \alpha_i L f_i(s_p) = \sum_{p=1}^{m} q_p W_j(s_p)\; g(s_p) \qquad j = 1, \ldots, n \qquad (19)$$

The matrix form of Eq. 12 implies that

$$\lim_{m\to\infty} \sum_{p=1}^{m} q_p W_j(s_p) \lim_{n\to\infty} \sum_{i=1}^{n} \alpha_i L f_i(s_p) = \lim_{n\to\infty} \sum_{i=1}^{n} \alpha_i \lim_{m\to\infty} \sum_{p=1}^{m} q_p W_j(s_p) L f_i(s_p) \qquad (20)$$

in order for proper convergence to hold. If the number of integration points (m) is large enough, the series form of Eq. 19 will closely approximate the integral form (Eq. 18), and the interchange of series limits should remain valid. Point-matching forces m = n, and for it to be valid, the simultaneous double limit $(m, n \to \infty)$ must be a valid operation. This validity does not, in general, hold, as shown by a simple example discussed by Carslaw [11]:

$$s_N(x) = \sum_{p=1}^{N} f_p(x) \tag{21}$$

where

$$f_p(x) = \frac{1}{(p - 1)x + 1} - \frac{1}{px + 1} \tag{22}$$

$$s_N(x) = 1 - \frac{1}{Nx + 1} \tag{23}$$

From Eq. 22, at $x = 0$,

$$f_p(0) = 0 \tag{24}$$

Thus,

$$\lim_{N\to\infty} \left(s_N(0)\right) = s_\infty(0) = 0 \tag{25}$$

From Eq. 23, for $x > 0$,

$$\lim_{N\to\infty} \left(s_N(x)\right) = s_\infty(x) = 1 \qquad x > 0 \tag{26}$$

Thus, the infinite series $s_\infty(x)$ has a discontinuity at $x = 0$. It is interesting to note that the partial sum defined by Eq. 21 is a sum of continuous functions, and this is also continuous. The limiting sum s_∞ is not continuous, however, and it is this difference that can cause problems with taking repeated limits.

Consider the limit,

$$\lim_{N\to\infty} \lim_{x\to 0} \left(s_N(x)\right) = A \tag{27}$$

Let

$$x = \frac{c}{N} \tag{28}$$

where c is any positive constant. As N approaches infinity, x will approach zero, and it seems reasonable that

$$A = \lim_{N\to\infty} s_n \frac{c}{N} \tag{29}$$

From Eq. 23,

$$A = 1 - \frac{1}{c + 1} \tag{30}$$

Now, for any $c > 0$, A can be forced to take on any value between 0 and 1 through the proper choice of the constant c. Thus, the substitution given by Eq. 28 is invalid. It is improper to take a repeated limit of a series in this manner.

It is also improper to exchange the order of the limits in Eq. 27 for, in one case, $A = 1$, and in the other, $A = 0$, so that

$$\lim_{N\to\infty} \lim_{x\to 0} \left(s_N(x)\right) \neq \lim_{x\to 0} \lim_{N\to\infty} \left(s_N(x)\right) \tag{31}$$

The failure of Eq. 31 to be valid is due to the nonuniform convergence of the series for $x \geq 0$. That the series defined by Eqs. 21 and 22 is nonuniformly convergent can be seen by considering any x arbitrarily close to zero. For any arbitrarily small positive number ε,

$$\left|s_\infty(0) - s_N(x)\right| = \left|\frac{1}{Nx + 1}\right| \leq \varepsilon \tag{32}$$

it must be true that

$$N > \frac{\frac{1}{\varepsilon} - 1}{x} \tag{33}$$

As x approaches zero, N must become large to satisfy Eq. 32. N must not be dependent on the position within the interval for uniform convergence to hold.

Again, for point matching, the conclusion drawn from this discussion is that point matching is only rigorously valid when the summation in Eq. 1 and Eq. 19 converges uniformly everywhere it is used; satisfaction of the

Rayleigh hypothesis ensures this condition [9]. Unfortunately, determining when a boundary satisfies the Rayleigh hypothesis is not always simple; even a boundary that satisfies the hypothesis can fail through a simple coordinate transformation [10]. Bates [9] suggests using conformal transformations to determine if a boundary satisfies the Rayleigh hypothesis, but this method weakens the main advantage of point matching; i.e., namely simplicity.

Moreover, the implication for the method of moments and the boundary-residual method is that the number of integration points (m) must be much larger than the number of functions $f_i(n)$. Somewhere between this condition (m >> n) and that of point matching (m = n), both of these methods may fail. Indeed, this view is borne out by results found from Ikuno and Yasuura [6] in which their "improved point-matching method" converges for m > 2n, but fails otherwise.

As a final heuristic argument explaining the failure of point matching, consider a "function fitting" view of this method in which a set of functions (Lf_i in Eq. 1) is used to fit a driving function (g in Eq. 1) over an interval (the boundary s):

$$\sum_{i=1}^{n} \alpha_i \, h_i(s) = g(s) \tag{34}$$

where $h_i(s) = L\, f_i(s)$. Point matching forces this equation to be true on a discrete set of n points along s. In between these points, however, the functions h_i are unconstrained and can take on any value. The measure of the residual of the problem (i.e., how well the fitting functions fit the driving function) is over a discrete set of points of g(s), and it is therefore over a set of measure zero on g(s). An infinite number of functions can be found which equal g(s) on a set of measure zero and produce the same point matched solution, even as the number of fitting points (n in Eq. 1) approaches infinity [2]! The method of moments and the boundary-residual method do not fail because the fitting functions are smoothed everywhere along the boundary by the integral in Eqs. 3 and 4. The residual is not over a set of measure zero, and the fitting functions converge in the mean to the proper value [7].

An example will illustrate this view. Consider a set of odd polynomials used to represent sin (2πx) over the interval $0 \leq x \leq 1$:

$$\sum_{n=0}^{N} a_n (2\pi x)^{2n+1} = \sin(2\pi x) \qquad 0 \leq x \leq 1 \tag{35}$$

Figures 1-3 compare the errors of this fit for the case of point matching versus the boundary-residual method. The plots clearly show how the boundary-residual method smooths the error across the entire interval. The error of the point-matching method varies wildly between fitting points, even as the number of fitting functions (N in Eq. 35) increases.

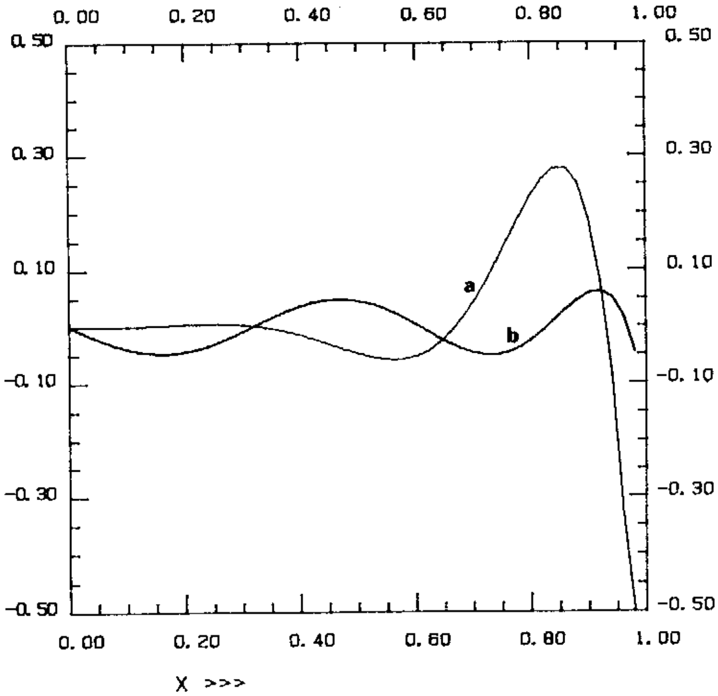

Fig. 1. The errors of Eq. 35 corresponding to the point-matching case (a) and the boundary-residual cases (b) for 4 series terms.

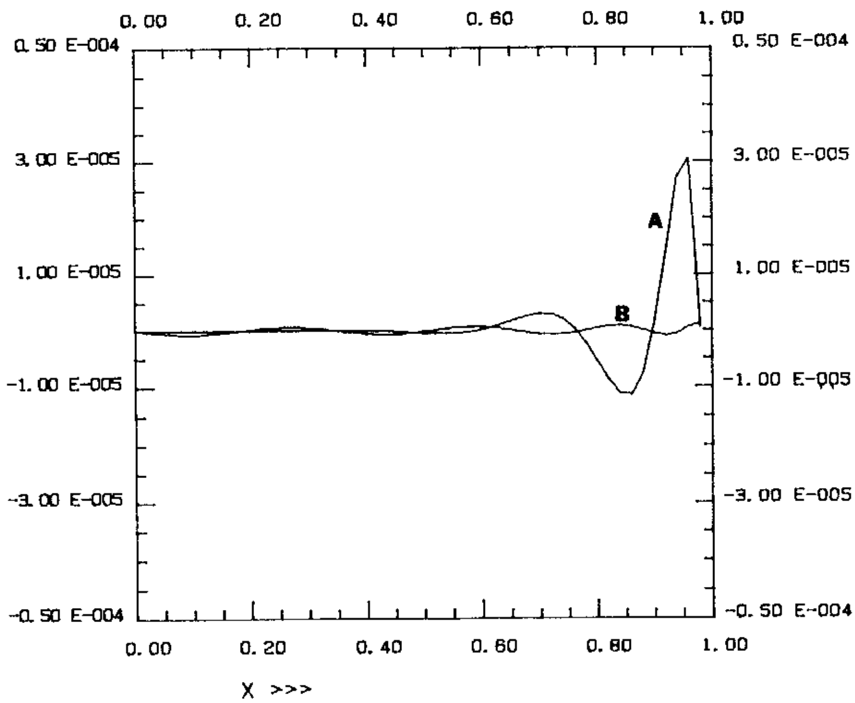

Fig. 2. A comparison of the errors between the point-matched (a) and the boundary-residual solutions (b) for 8 series terms.

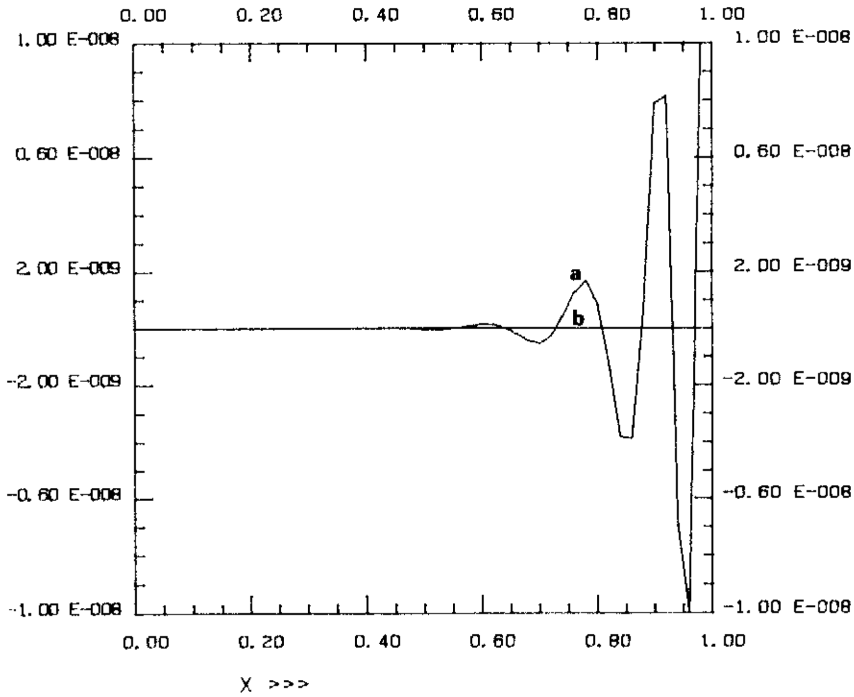

Fig. 3. A comparison of the errors between the point-matched (a) and the boundary-residual solutions (b) for 16 series terms.

A REFORMULATION OF POINT MATCHING

Bunch and Grow [12] have proposed a method of using the boundary-residual method which retains most of the simplicity of point matching, but which is rigorously convergent. Recall that in the boundary-residual case, the weighting functions are given by Eq. 5. Using these in Eq. 13 produces

$$\bar{\bar{P}}^{\dagger} \, \bar{\bar{P}} \, \vec{\alpha} = \bar{\bar{P}}^{\dagger} \, \vec{G} \tag{36}$$

where † denotes the complex conjugate transpose, and $\bar{\bar{P}}$ is given by Eq. 14. Numerically, this equation is equivalent to solving the equation,

$$\bar{\bar{P}} \, \vec{\alpha} \approx \vec{G} \tag{37}$$

in the least-squares sense [12]. Remember, m > n in Eq. 37, and so there are more rows than columns. Rather than calculating the matrix product in Eq. 36, however, Eq. 37 can be solved directly and equivalently using Householder transforms [12, 13] or using a singular value decomposition [12, 13]. This method retains the advantages of point matching in which a wave expansion is set up as in Eq. 1 and forced to satisfy the boundary conditions, yet it retains the convergent properties of the boundary-residual method [2, 3]. The method is rigorously convergent because the boundary-residual method (Eq. 36) is rigorously convergent [2, 3], and it creates an identical numerical solution [12, 13] without having to form the matrix product. It is similar to the method proposed by Ikuno and Yasuura [6], except the connection to the method of moments and the boundary-residual method have been clearly shown.

The direct formulation of Eq. 37 also has the advantages of being more numerically stable and quicker to solve than Eq. 36 [12, 13]. The stability problems occur when one or more eigenvalues of the matrix product $\bar{\bar{P}}^{\dagger} \bar{\bar{P}}$ are close to zero, in which case $\bar{\bar{P}}^{\dagger} \bar{\bar{P}}$ is nearly singular. It is easy to show that the eigenvalues of the product $\bar{\bar{P}}^{\dagger} \bar{\bar{P}}$ are the square of the singular values of the matrix $\bar{\bar{P}}$. The matrix $\bar{\bar{P}}$ can be decomposed into its singular values,

$$\bar{\bar{P}} = \bar{\bar{V}}\,\bar{\bar{\sigma}}\,\bar{\bar{U}} \tag{38}$$

where $\bar{\bar{\sigma}}$ is a diagonal matrix of singular values; $\bar{\bar{V}}$ and $\bar{\bar{U}}$ are orthogonal matrices in which $\bar{\bar{V}}^{\dagger}\bar{\bar{V}} = \bar{\bar{I}}$ $(\bar{\bar{U}}^{\dagger}\bar{\bar{U}} = \bar{\bar{I}})$, where $\bar{\bar{I}}$ is the identity matrix. Forming the product $\bar{\bar{P}}^{\dagger} \bar{\bar{P}}$,

$$\bar{\bar{P}}^{\dagger} \bar{\bar{P}} = (\bar{\bar{V}}\,\bar{\bar{\sigma}}\,\bar{\bar{U}})^{\dagger} (\bar{\bar{V}}\,\bar{\bar{\sigma}}\,\bar{\bar{U}})$$

$$= \bar{\bar{U}}^{\dagger} (\bar{\bar{\sigma}})^2 \,\bar{\bar{U}} \tag{39}$$

Thus, if the matrix $\bar{\bar{P}}$ has a singular value σ close to zero, the matrix product $\bar{\bar{P}}^{\dagger} \bar{\bar{P}}$ will have a corresponding eigenvalue σ^2 even closer to zero. The equation defined by Eq. 36 will thus be more unstable numerically than Eq. 37. Further, solving Eq. 37 directly using a singular value decomposition has the added advantage that the singular values causing numerical instabilities may be discarded in computing the solution [14].

Solving the direct form of the electromagnetic problem may have advantages over using the method of moments. The method of moments creates a matrix problem as in Eq. 12. The matrix consists of a sum for each element due to the numerical integration of the weighting function. The computation of the element sums can be time consuming as the number of sums increases as the square of the matrix size. On the other hand, the direct formulation does not need sums to be computed, but it solves the problem directly. This advantage in speed, however, may be offset by the need for extra storage, as the matrix in the direct formulation is overdetermined (the number of rows is greater than the number of columns). Ikuno and Yasuura [6] have reported good results in a similar formulation when the numbers of rows (corresponding to boundary points) is greater than twice the number of columns (corresponding to wave expansion functions).

Another consideration is that the singular value decomposition is well-behaved and well-suited for solving the direct formulation in a least-squares sense [13]. The singular value decomposition method allows one to have control over the singular values to produce a well-behaved solution even with a nearly singular set of equations [14]. This control is advantageous when the formulation of the problem produces a nearly singular set of equations as when using a large number of wave expansion functions.

The direct method may also be used for solving scattering problems when the induced current on the scattering surface is expanded as a sum of unknown basis functions. Butler and Wilton [16] have investigated the application of the method of moments as applied to thin-wire scatterers with several different basis sets to represent the wire current. They found the convergence of the solution depended strongly on the basis functions used as well as whether the equations solved were cast in Pocklington (electric field) or Hallén (magnetic vector potential) integral form. Their testing functions were delta functions forcing their method of solution to be that of point matching. As stated, point matching may fail to converge to the correct solution; in this case, point matching was satisfactory because the geometry of the scatterer was simple (the Rayleigh hypothesis was satisfied) and no singularities in the fields existed on the scatterer. Using the direct formulation in this case, however, would allow the technique to be extended to scatterers of more complicated geometry. The dependence of convergence on the choice of basis functions used to represent the wire current would still remain, but an advantage of the direct method is that the singular value decomposition would be ideal for the problem of ill-conditioned matrices found in some of their test cases.

A SPHERICAL CAVITY EXAMPLE

To illustrate the use of Eq. 37, we solved the resonances of the spherical cavity using cylindrical wave functions. A scalar expansion for the fields is given by [17]

$$\psi = \sum_n a_n \, J_\ell(\gamma_n \rho) \, e^{j\ell\phi} \, e^{\frac{j2\pi n}{p} z} \tag{40}$$

with

$$\gamma_n = \sqrt{k^2 - \left(\frac{2\pi n}{p}\right)^2} \tag{41}$$

J_ℓ is the cylindrical Bessel function of the first kind of order ℓ [17], k is the wave number (ω/c), and p is the diameter of the cavity.

This wave expansion is used to create the electric field [17] whose tangential value is minimized on the spherical boundary.

Figure 4 shows the minimum singular value over the wave number of the overdetermined matrix using Eq. 40. In this case, we do not have an incident field and so the right-hand side of Eq. 37 is zero. The minimum singular value of the matrix of Eq. 37 gives an indication of how well the wave expansion (Eq. 40) fits the boundary conditions over frequency [18]. The dips in the singular value are the resonances of the cavity, and these gradually approach the exact resonances (shown as dotted lines) as the number of wave functions (n in Eq. 67) increases. As shown, good results are obtained using only a few number of wave functions.

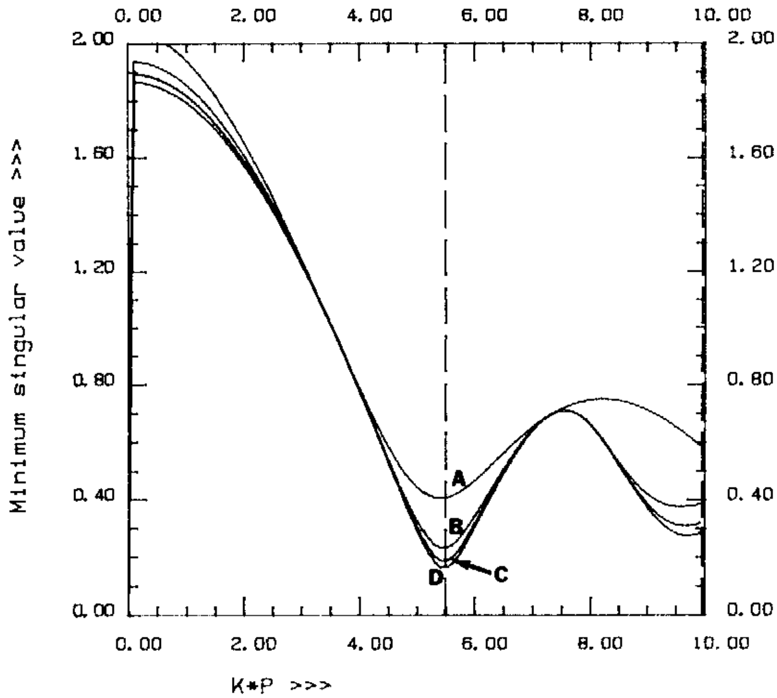

Fig. 4. The minimum singular value over wave number for the spherical cavity with $\ell = 0$. The dotted line shows an exact resonance. The results are shown for n = 0 (a), n = -1, ..., 1 (b), -2, ..., 2 (c), and -3, ..., 3 (d).

CONCLUSIONS

This paper has shown how the method of moments can collapse to the point-matching method and the boundary-residual method; it can do so in two ways. The boundary-residual method has also been shown to revert to point matching in some cases. A large number of sampling points for numerical integration in either the method of moments or the boundary-residual method can prevent this collapse. The point-matching method is unconstrained between data points; an example has shown that the error of the functions between these points can fluctuate wildly. Finally, a formulation has been given which retains the simplicity of point matching while retaining the rigorous convergence properties of the method of moments or the boundary-residual method.

REFERENCES

1. R. Harrington, "Origin and Development of the Method of Moments for Field Computation," IEEE Antennas and Propagation Magazine, Vol. 32, No. 3, June 1990, pp. 31-36.

2. K. J. Bunch, Theoretical and Numerical Foundations of a Boundary-Residual Method for Solving Three-Dimensional Boundary-Value Problems in Electromagnetics, Ph.D. Dissertation, Department of Electrical Engineering, University of Utah, Salt Lake Cityl, Utah, March 1990.

3. J. B. Davies, "A Least-Squares Boundary-Residual Method for the Numerical Solution of Scattering Problems," IEEE Transactions on Microwave Theory and Techniques, Vol. MTT-21, No. 2, February 1973, pp. 99-103.

4. R. H. T. Bates, "The Point-Matching Method for Interior and Exterior Two-Dimensional Boundary-Value Problems," IEEE Transactions on Microwave Theory and Techniques, Vol. MTT-15, March 1967, pp. 185-187.

5. M. Abramowitz and I. A. Stegun, Handbook of Mathematical Functions, Dover Publications, New York, 1972, Chapter 25.

6. H. Ikuno and K. Yasuura, "Improved Point-Matching Method with Application to Scattering from a Periodic Surface," IEEE Transactions on Antennas and Propagation, Vol. AP-21, No. 5, September 1973, pp. 657-662.

7. F. S. Titchmarsh, The Theory of Functions, Second Edition, Oxford University Press, London, England, 1939, pp. 421-422.

8. K. Knopp, Theory of Functions Part I, Dover Publications, New York, 1945, Chapter 6.

9. R. H. T. Bates, "Analytic Constraints on Electromagnetic Field Computations," IEEE Transactions on Microwave Theory and Techniques, Vol. MTT-23, No. 8, August 1975, pp. 605-623.

10. L. Lewin, "On the Restricted Validity of Point-Matching Techniques," IEEE Transactions on Microwave Theory and Techniques, Vol. MTT-18, No. 12, December 1970, pp. 1041-1047.

11. H. S. Carslaw, An Introduction to the Theory of Fourier's Series and Integrals, Third Edition, Dover Publications, New York, 1950, Chapter 5.

12. K. J. Bunch and R. W. Grow, "Numerical Aspects of the Boundary-Residual Method," International Journal of Numerical Modeling, Vol. 3, No. 1, March 1990, pp. 57-71.

13. C. L. Lawson and R. J. Hanson, Solving Least-Squares Problems, Prentice-Hall, Inc., Englewood Cliffs, New Jersey, 1974, Chapters 3 and 4.

14. W. H. Press, B. P. Flannery, S. A. Teukolsky, and W. T. Vetterling, Numerical Recipes, Cambridge University Press, Cambridge, 1986, Section 2.9.

15. A. R. Djordjevic and T. K. Sarkar, "A Theorem on the Moment Methods," IEEE Transactions on Antennas and Propagation, Vol. AP-35, No. 3, March 1987, pp. 353-355.

16. C. M. Butler and D. R. Wilton, "Analysis of Various Numerical Techniques Applied to Thin-Wire Scatterers," IEEE Transactions on Antennas and Propagation, July 1975, pp. 534-540.

17. R. F. Harrington, Time-Harmonic Electromagnetic Fields, McGraw-Hill Book Company, New York, 1961, Chapter 5.

18. K. J. Bunch and R. W. Grow, "Numerical Aspects of the Boundary Residual Method," International Journal of Numerical Modeling, Vol. 3, No. 1, March 1990, pp. 57-71.