New Stability Criterion for Unstructured Mesh Upwinding FVTD Schemes for Maxwell's Equations

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Abstract – A new stability criterion applicable to explicit upwind FVTD schemes for solving Maxwell's equations on unstructured meshes is derived. This criterion is based on L_2-norm estimates of specially constructed matrices G_i for each finite volume i. Each such matrix is constructed using the scalar product of the eigenvectors corresponding to the unity eigenvalues of the fluxsplitting operators associated with the facets of volume i. The new stability criterion is obtained numerically once the grid is constructed using these matrices over the mesh and is therefore mesh dependent. The new criterion gives a time-step that is larger than the time-step calculated using previously published stability criteria. On structured meshes the new criterion gives the same time-step limit as the von Neumann analysis. The method incurs a small computational expense at the beginning of each run of the algorithm. The method is generalizable but the extent to which it can be generalized to other time-evolving physical phenomenon is not considered in this paper.

Keywords: Finite-volume time-domain, unstructured mesh, Maxwell's equations, and stability criterion.

I. INTRODUCTION

One of the classical drawbacks of using explicit time-stepping numerical schemes is that a stringent timestep limit must be adhered to for stability. For structured meshes this time-step limit can usually be obtained using von Neumann analysis, but it is not possible to use von Neumann analysis with unstructured meshes. Thus, several authors have used analyses based on the energ-ynorm in the mesh to obtain estimates for the time-step bound. Previously published maximum time-step bounds have been sufficient to ensure stability but not tight bounds: they generally restrict the time-step to a value that is smaller than necessary. Obviously, using a smaller than necessary time-step increases execution time for any particular problem, but in addition, a timestep that is too small compared to the necessary limit may also result in poorer solution accuracy. A sufficient maximum time-step criterion for FVTD upwind schemes for Maxwell's equations was presented in [1, 2]. The time-step limit given in [2] is,

$$\Delta t = min_i \frac{V_i}{cA_i} \tag{1}$$

for an unstructured mesh, where c_i is the speed of light in element i, $1 \leq i \leq N$ is a number identifying the elements in the unstructured mesh (with N elements), V_i and A_i are the volume and total facet area for the i-th element, respectively. On the other hand, in [1] the timestep limit for an unstructured mesh is given as,

$$\Delta t = min_i \frac{2V_i}{cA_i} \tag{2}$$

twice that of equation (1) reported in [2]. Unfortunately, as was stated in [1], the bound given by equation (1) is merely a sufficient condition and not necessary: a larger time-step is possible.

For a structured cubical mesh with the edge-size of the elements h and the same speed of light c on all elements with von Newman analysis in [2] was derived,

$$\Delta t = \frac{h}{2c} \tag{3}$$

which is necessary condition and 1.5 times large then criterion equation (2) applied to the cubical elements on structured mesh. Note that a structured cubical mesh is a special case of an unstructured mesh in which all elements have the same cubical shape. But, criterion equation (2) does not reduce to equation (3) when applied to a structured cubical mesh and therefore equation (2) cannot be the necessary stability condition.

In this paper we derive the necessary stability criterion for the first-order Euler explicit scheme which can then be easily extended for higher-order time integration schemes [3]. As in [1], the derivation is based on the natural physical constraint that in a mesh that is free of sources of energy the total energy in the mesh should not increase with time. The electromagnetic energy in a particular region, $\Omega \subset \mathbb{R}^3$, gives rise to the mathematical concept of an energy-norm which can be calculated as,

$$\|\boldsymbol{u}\| = \sqrt{\frac{1}{2}\sum_{i=1}^{N} V_i(\boldsymbol{\epsilon}_i \boldsymbol{E}_i \cdot \boldsymbol{E}_i + \boldsymbol{\mu}_i \boldsymbol{H}_i \cdot \boldsymbol{H}_i)} \ . \tag{4}$$

The key difference in the derivation that allows us to obtain the necessary criterion is that we express the summation of fluxes over facets as a single matrix operator for which the norm can be determined numerically. Therefore, once an arbitrary mesh is generated, the necessary time-step limit is obtained by computing a simple formula over each element in the mesh at the beginning of each FVTD run.

II. THE TIME-STEP CRITERION IN TERMS OF ENERGY

Suppose we have a domain $\Omega \subset \mathbb{R}^3$ upon which is specified an unstructured mesh $\omega = \bigcup_{i=1}^{N} \Omega_i$, where Ω_i are the elements of the mesh each having a volume V_i. Here we suppose that the electromagnetic material parameters ϵ_i and μ_i are constants on each element Ω_i. The finite-volume time-domain method is formulated in terms of a generalized solution vector containing the electric and magnetic field vectors

$$\boldsymbol{u}(\boldsymbol{x}) = [\boldsymbol{E}^T(\boldsymbol{x})\boldsymbol{H}^T(\boldsymbol{x})]^T,$$

and solved for the averaged values

$$\boldsymbol{u}_i = \frac{1}{V_i}\int_{\Omega_i} \boldsymbol{u}(\boldsymbol{x})dv,$$

on each element. An equivalent discrete energy-norm over the domain Ω can be written as,

$$\|\boldsymbol{u}\| = \sqrt{\frac{1}{2}\sum_{i=1}^{N} V_i(\boldsymbol{\epsilon}_i\boldsymbol{E}_i \cdot \boldsymbol{E}_i + \boldsymbol{\mu}_i\boldsymbol{H}_i \cdot \boldsymbol{H}_i)} \qquad (5)$$

where $\boldsymbol{E}_i$ represents the averaged value of the electric field over the element i, and similarly for the magnetic field vector $\boldsymbol{H}_i$.

Consider now the Euler approximation for the time-dependent Maxwell's equations cast as a conservation law (see [4]). It can be written concisely as,

$$\boldsymbol{u}^{n+1} = \boldsymbol{u}^n - \Delta t L\boldsymbol{u}^n \qquad (6)$$

where L represents the discretization of the spatial derivatives. More specifically, for the case of the FVTD method, L represents the integration of the fluxes over the facets of each element. We are interested in the maximum value of δt that keeps the scheme stable.

A numerical scheme is L^2-stable if the energy doesn't grow in time; that is, if the following is true,

$$\|\boldsymbol{u}^{n+1}\|^2 \leq \|\boldsymbol{u}^n\|^2. \qquad (7)$$

We can define an inner-product in the mesh for Maxwell's equations as $(\boldsymbol{u}, \boldsymbol{w}) = \sum_{i=1}^{N} V_i \boldsymbol{u}_i^T \alpha_i \boldsymbol{w_i}$, where

$$\alpha_i = \begin{pmatrix} \boldsymbol{\epsilon}_i & 0 \\ 0 & \boldsymbol{\mu}_i \end{pmatrix},$$

and $\boldsymbol{\epsilon}_i$, $\boldsymbol{\mu}_i$ are permittivity and permeability matrices for volume i.

The energy-norm is obtained as $\|u\| = \sqrt{(\boldsymbol{u}, \boldsymbol{u})}$. It can be easily verified, that $(\boldsymbol{u}, \boldsymbol{w})$ satisfies the mathematical properties of an inner product.

Taking the inner product of equation (6) with $\boldsymbol{u}^n$ we get $(\boldsymbol{u}^{n+1}, \boldsymbol{u}^n) - (\boldsymbol{u}^n, \boldsymbol{u}^n) = -\Delta t(L\boldsymbol{u}^n, \boldsymbol{u}^n)$, and using the property that,

$$\begin{aligned}(\boldsymbol{u}^{n+1}, \boldsymbol{u}^n) = \tfrac{1}{2}(\boldsymbol{u}^{n+1}, \boldsymbol{u}^{n+1}) + \tfrac{1}{2}(\boldsymbol{u}^n, \boldsymbol{u}^n) - \\ \tfrac{1}{2}(\boldsymbol{u}^{n+1} - \boldsymbol{u}^n, \boldsymbol{u}^{n+1} - \boldsymbol{u}^n)\end{aligned} \qquad (8)$$

we can rewrite this as

$$\begin{aligned}&(\boldsymbol{u}^{n+1}, \boldsymbol{u}^{n+1}) + (\boldsymbol{u}^n, \boldsymbol{u}^n) - \\ &\quad(\boldsymbol{u}^{n+1} - \boldsymbol{u}^n, \boldsymbol{u}^{n+1} - \boldsymbol{u}^n) = -2\Delta t(L\boldsymbol{u}^n, \boldsymbol{u}^n).\end{aligned}$$

This last equation together with,

$$(\boldsymbol{u}^{n+1} - \boldsymbol{u}^n, \boldsymbol{u}^{n+1} - \boldsymbol{u}^n) = (\Delta t L\boldsymbol{u}^n, \Delta t L\boldsymbol{u}^n)$$

and the energy constraint of equation (7) gives

$$\begin{aligned}&(\boldsymbol{u}^{n+1}, \boldsymbol{u}^{n+1}) - (\boldsymbol{u}^n, \boldsymbol{u}^n) = \\ &\quad\Delta t^2(L\boldsymbol{u}^n, L\boldsymbol{u}^n) - 2\Delta t(L\boldsymbol{u}^n, \boldsymbol{u}^n).\end{aligned}$$

This gives us a condition for the maximum time-step, based on the non-increasing energy stability criterion for the Euler scheme, that depends on any spatial discretization L,

$$\Delta t(L\boldsymbol{u}^n, L\boldsymbol{u}^n) \leq 2(L\boldsymbol{u}^n, \boldsymbol{u}^n). \qquad (9)$$

III. FVTD SOLUTION OF MAXWELL'S EQUATIONS

A. The FVTD scheme for Maxwell's equations

We can write the FVTD scheme for Maxwell's equations with Euler explicit time integration and first order spatial upwinding, as [1, 2],

$$\begin{aligned}&\boldsymbol{u}_i^{n+1} = \boldsymbol{u}_i^n - \Delta t \tfrac{1}{V_i}\sum_{j=1}^{m_i} A_i(j)\times \\ &\left(T_i^+(j)B_i^+(j)\boldsymbol{u}_i^n + T_i^-(j)B_i^-(j)\boldsymbol{u}_{i_j}^n\right)\end{aligned} \qquad (10)$$

where m_i is the number of facets defining finite-volume Ω_i, and $A_i(j)$ is the area of the j^{th} facet of the i^{th} volume. The subscript i_j denotes the element neighboring facet j. The transmission operators are given as

$$T_i^{\pm}(j) = \alpha_i^{-1}\begin{pmatrix} \frac{2Y_i^{\mp}(j)}{Y_i^+(j)+Y_i^-(j)}I & 0 \\ 0 & \frac{2Z_i^{\mp}(j)}{Z_i^+(j)+Z_i^-(j)}I \end{pmatrix},$$

for facets between dielectrics, where

$$\begin{aligned}Y_i^+(j) &= \tfrac{1}{Z_i^+(j)} = \sqrt{\tfrac{\boldsymbol{\epsilon}_i}{\boldsymbol{\mu}_i}}, \\ Y_i^-(j) &= \tfrac{1}{Z_i^-(j)} = \sqrt{\tfrac{\boldsymbol{\epsilon}_{i_j}}{\boldsymbol{\mu}_{i_j}}}.\end{aligned}$$

For facets located on a perfect electric conductor (PEC) these become

$$T_i^+(j) = \alpha_i^{-1}\begin{pmatrix} 2I & 0 \\ 0 & 0 \end{pmatrix}, T_i^-(j) = 0,$$

or equivalently, using the image principle [4], we have the same $T_i^{\pm}(j)$ and $B_i^{\pm}(j)$ operators as for facets between dielectrics, but $\boldsymbol{u}_i^n = [(\boldsymbol{E}_i^n)^T(\boldsymbol{H}_i^n)^T]^T$ and $\boldsymbol{u}_{i_j}^n = [-(\boldsymbol{E}_i^n)^T(\boldsymbol{H}_i^n)^T]^T$. This equivalent formulation allows us to extend the stability criterion results for meshes with PEC facets more easily. For a facet at the external boundary of the mesh, we consider the first-order absorbing boundary condition (ABC)

$$T_i^+(j) = \alpha_i^{-1}\begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix}, T_i^-(j) = 0.$$

The flux splitting operators are given as,

$$\begin{cases} B_i(j)^+ = \frac{1}{2}\begin{pmatrix} -S_i^2(j) & -S_i(j) \\ S_i(j) & -S_i^2(j) \end{pmatrix} \\ B_i(j)^- = \frac{1}{2}\begin{pmatrix} S_i^2(j) & -S_i(j) \\ S_i(j) & S_i^2(j) \end{pmatrix} \end{cases} \tag{11}$$

where the matrix operator $S_i(j)$ applied to an arbitrary vector $\boldsymbol{a}$ produces the cross-product of the outward normal $\hat{\boldsymbol{n}}_i(j)$, the normal to the j-th facet of element i, with $\boldsymbol{a}$, that is, $S_i(j)\boldsymbol{a} = \hat{\boldsymbol{n}}_i(j) \times \boldsymbol{a}$.

B. Expressing $(L\boldsymbol{u}, \boldsymbol{u})$ and $(L\boldsymbol{u}, L\boldsymbol{u})$ for FVTD

For simplicity, we now consider only the case when we have the same ϵ and μ for all elements. For the case when we have non-uniform ϵ and μ the time-step will be changed in the same way as in [1] according to the speed of light in the element ($c_i = (\epsilon_i\mu_i)^{-1/2}$). We write,

$$\begin{aligned}(L\boldsymbol{u}, \boldsymbol{u}) = \tfrac{1}{2}\sum_{i=1}^{N}\sum_{k=1}^{m_i} A_i(k)\times \\ \left[T_i^+(k)B_i^+(k)\boldsymbol{u}_i + T_i^-(k)B_i^-(k)\boldsymbol{u}_{i_k}\right]\cdot\boldsymbol{u}_i\end{aligned} \tag{12}$$

$$\begin{aligned}(L\boldsymbol{u}, L\boldsymbol{u}) = \tfrac{1}{2}\sum_{i=1}^{N}\tfrac{c}{V_i}\cdot \\ \left\{\sum_{k=1}^{m_i} A_i(k)\left(T_i^+(k)B_i^+(k)\boldsymbol{u}_i + \right.\right. \\ \left. T_i^-(k)B_i^-(k)\boldsymbol{u}_{i_k}\right)\cdot \\ \left.\sum_{j=1}^{m_i} A_i(j)\left(T_i^+(j)B_i^+(j)\boldsymbol{u}_i + \right.\right. \\ \left.\left. T_i^-(j)B_i^-(j)\boldsymbol{u}_{i_j}\right)\right\}.\end{aligned} \tag{13}$$

The flux-splitting operators, $B_i^+(k)$ and $B_i^-(k)$, when applied to the field value at the center of an element give the flux at facet k which when summed over all facets of the element give zero [2]. Thus, we have

$$\sum_{k=1}^{m_i} A_i(k)T_i^+(k)B_i^+(k)\boldsymbol{u}_i + A_i(k)T_i^-(k)B_i^-(k)\boldsymbol{u}_i = 0,$$

which can be written as,

$$\sum_{k=1}^{m_i} A_i(k)B_i^+(k)\boldsymbol{u}_i = -\sum_{k=1}^{m_i} A_i(k)B_i^-(k)\boldsymbol{u}_i \tag{14}$$

Combining equation (13) with equation (14) allows us to write

$$\begin{aligned}(L\boldsymbol{u}, L\boldsymbol{u}) = \tfrac{1}{2}\sum_{i=1}^{N}\tfrac{c}{V_i} \\ \left[\sum_{k=1}^{m_i} A_i(k)B_i^-(k)(\boldsymbol{u}_{i_k} - \boldsymbol{u}_i)\right]\cdot \\ \left[\sum_{j=1}^{m_i} A_i(j)B_i^-(j)(\boldsymbol{u}_{i_j} - \boldsymbol{u}_i)\right],\end{aligned}$$

whereas combining equation (12) with equation (14) gives

$$(L\boldsymbol{u}, \boldsymbol{u}) = \frac{1}{2}\sum_{i=1}^{N}\boldsymbol{u}_i\cdot\sum_{k=1}^{m_i} A_i(k)\left(B_i^-(k)(\boldsymbol{u}_{i_k} - \boldsymbol{u}_i)\right).$$

For our case, because $S = -S^T$, we have $B = B^T$ we can write

$$B\boldsymbol{a}\cdot\boldsymbol{b} = \frac{1}{2}B\boldsymbol{b}\cdot\boldsymbol{b} + \frac{1}{2}B\boldsymbol{a}\cdot\boldsymbol{a} - \frac{1}{2}B(\boldsymbol{b}-\boldsymbol{a})\cdot(\boldsymbol{b}-\boldsymbol{a}),$$

and therefore

$$\begin{aligned}\boldsymbol{u}_i\cdot\sum_{k=1}^{m_i} A_i(k)\left(B_i^-(k)(\boldsymbol{u}_{i_k} - \boldsymbol{u}_i)\right) = \\ \tfrac{1}{2}\sum_{k=1}^{m_i} A_i(k)\left[\boldsymbol{u}_{i_k}\cdot B_i^-(k)\boldsymbol{u}_{i_k} - \boldsymbol{u}_i\cdot B_i^-(k)\boldsymbol{u}_i - \right. \\ \left.(\boldsymbol{u}_{i_k} - \boldsymbol{u}_i)\cdot B_i^-(k)(\boldsymbol{u}_{i_k} - \boldsymbol{u}_i)\right].\end{aligned}$$

C. Expressions for the maximum energy criterion

Introducing the expressions for $(L\boldsymbol{u}, \boldsymbol{u})$ and $(L\boldsymbol{u}, L\boldsymbol{u})$ in to formula (9) we get,

$$\begin{aligned}\Delta t\sum_{i=1}^{N}\tfrac{c}{V_i}\left[\sum_{k=1}^{m_i} A_i(k)B_i^-(k)(\boldsymbol{u}_{i_k} - \boldsymbol{u}_i)\right]\cdot \\ \left[\sum_{j=1}^{m_i} A_i(j)B_i^-(j)(\boldsymbol{u}_{i_j} - \boldsymbol{u}_i)\right] \le \\ \sum_{i=1}^{N}\sum_{k=1}^{m_i} A_i(k)\left[\boldsymbol{u}_{i_k}\cdot B_i^-(k)\boldsymbol{u}_{i_k} - \right. \\ \boldsymbol{u}_i\cdot B_i^-(k)\boldsymbol{u}_i - (\boldsymbol{u}_{i_k} - \boldsymbol{u}_i)\cdot \\ \left. B_i^-(k)(\boldsymbol{u}_{i_k} - \boldsymbol{u}_i)\right].\end{aligned} \tag{15}$$

It is also easy to check that the following is true,

$$\begin{aligned}\sum_{i=1}^{N}\sum_{k=1}^{m_i} A_i(k)\left[\boldsymbol{u}_{i_k}\cdot B_i^-(k)\boldsymbol{u}_{i_k} - \right. \\ \left.\boldsymbol{u}_i\cdot B_i^-(k)\boldsymbol{u}_i\right] = \\ -\sum_{i=1}^{N_b}\sum_{k=1}^{m_i^s} A_i(k)\boldsymbol{u}_i\cdot B_i^-(k)\boldsymbol{u}_i \ge 0\end{aligned} \tag{16}$$

where N_b is the number of elements with facets on the domain boundary, and m_i^s is the number of facets of i-th element on that boundary. In the case if k-th facet of i-th element is PEC facet, we can write the terms in the sum on the left side of formula (16) as,

$$\begin{aligned}\boldsymbol{u}_{i_k}\cdot B_i^-(k)\boldsymbol{u}_{i_k} - \boldsymbol{u}_i\cdot B_i^-(k)\boldsymbol{u}_i = \\ = \begin{bmatrix} \boldsymbol{E}_i^n \\ \boldsymbol{H}_i^n \end{bmatrix}\tfrac{1}{2}\begin{pmatrix} S_i^2(j) & -S_i(j) \\ S_i(j) & S_i^2(j) \end{pmatrix}\begin{bmatrix} \boldsymbol{E}_i^n \\ \boldsymbol{H}_i^n \end{bmatrix} - \\ \begin{bmatrix} -\boldsymbol{E}_i^n \\ \boldsymbol{H}_i^n \end{bmatrix}\tfrac{1}{2}\begin{pmatrix} S_i^2(j) & -S_i(j) \\ S_i(j) & S_i^2(j) \end{pmatrix}\begin{bmatrix} -\boldsymbol{E}_i^n \\ \boldsymbol{H}_i^n \end{bmatrix} = 0.\end{aligned} \tag{17}$$

Therefore, in equation (16) the PEC facets interior to the mesh do not contribute to the inequality and the inequality remains the same. Thus, using equation (16), we can rewrite equation (15) as a non-tight bound,

$$\begin{aligned}&\sum_{i=1}^{N}\tfrac{c\Delta t}{V_i}\left[\sum_{k=1}^{m_i}A_i(k)B_i^-(k)(\boldsymbol{u}_{i_k}-\boldsymbol{u}_i)\right]\cdot\\&\left[\sum_{j=1}^{m_i}A_i(j)B_i^-(j)(\boldsymbol{u}_{i_j}-\boldsymbol{u}_i)\right]\leq\\&-\sum_{i=1}^{N}\sum_{k=1}^{m_i}A_i(k)(\boldsymbol{u}_{i_k}-\boldsymbol{u}_i)\cdot\\&B_i^-(k)(\boldsymbol{u}_{i_k}-\boldsymbol{u}_i)\leq 2(L\boldsymbol{u},\boldsymbol{u}).\end{aligned}\tag{18}$$

This is the fundamental global inequality that imposes the stability constraint on Δt. If not for the ABC facets this would be a tight bound which implies that the inequality (18) defines the necessary condition on Δt for the FVTD algorithm on an infinite domain or inside a PEC enclosure.

It is not a simple task to derive a global constraint on Δt based on this formula. Therefore, we have to make due with imposing this inequality locally on a finite-volume by finite-volume basis. This removes the summation over all finite-volumes, but leaves the inner summations. Note that on a uniform mesh dealing only with the inner summation keeps the inequality (18) exact.

In order to get a manageable constraint for Δt, even limiting ourselves to a local constraint, requires that we somehow remove the inner summations over facets while keeping the formula exact. We proceed by first constructing a block-diagonal matrix $Z_i = diag\{-B_i^-(k)\}_{k=1}^{m_i}$ as well as a block-row vector of m_i identity matrices $W = \{I, ..., I\}$, where the dimension of W is $6 \times 6m_i$ and I is the 6×6 identity matrix. We also construct a column vector made up of the solution vector differences across each facet,

$$\boldsymbol{x} = vector\{\boldsymbol{u}_{i_k} - \boldsymbol{u}_i\}_{k=1}^{m_i}$$

which is a vector of length $6m_i$. Hence a new local inequality, based on inequality (18), can be written concisely using these constructions as,

$$\frac{c\Delta t}{V_i}(WZ_i\boldsymbol{x}, WZ_i\boldsymbol{x}) \leq (Z_i\boldsymbol{x}, \boldsymbol{x}), \quad \forall i. \tag{19}$$

D. Efficient computation of stability criterion

Using the property $B_i^-(k)^T = B_i^-(k)$, we have also $Z_i = Z_i^T$. Hence the square-root of Z_i can be expressed as

$$Z_i = Q^T\Lambda_i Q = Q^T\sqrt{\Lambda_i}Q\ Q^T\sqrt{\Lambda_i}Q = \sqrt{Z_i}\sqrt{Z_i},$$

where $\Lambda_i = diag\{\lambda_j\}_{j=1}^{6m_i}$ is the diagonal matrix of eigenvalues of Z_i, and

$$\sqrt{\Lambda_i} = diag\{\sqrt{\lambda_j}\}_{j=1}^{6m_i}.$$

Now defining the new variable $\boldsymbol{y} = \sqrt{Z_i}\boldsymbol{x}$, which means that $\sqrt{Z_i}\boldsymbol{y} = \boldsymbol{x}$ and therefore $\boldsymbol{y} \notin Ker(\sqrt{Z_i})$, we have equation (19) rewritten as

$$\frac{c\Delta t}{V_i}(W\sqrt{Z_i}\boldsymbol{y}, W\sqrt{Z_i}\boldsymbol{y}) \leq (\boldsymbol{y}, \boldsymbol{y}),$$

or

$$\frac{V_i}{c\Delta t} \geq \frac{(W\sqrt{Z_i}\boldsymbol{y}, W\sqrt{Z_i}\boldsymbol{y})}{(\boldsymbol{y}, \boldsymbol{y})}, \ \forall i.$$

Finally, this last inequality can be written in terms of the original summations over the facets as, $\forall i$

$$\begin{aligned}\tfrac{V_i}{c\Delta t} \geq &\left(\sum_{k=1}^{m_i}(\boldsymbol{y}_k, \boldsymbol{y}_k)\right)^{-1}\times\\&\left[\sum_{k=1}^{m_i}\sqrt{A_i(k)}\sqrt{-B_i^-(k)}\boldsymbol{y}_k\right]\cdot\\&\left[\sum_{j=1}^{m_i}\sqrt{A_i(j)}\sqrt{-B_i^-(j)}\boldsymbol{y}_j\right].\end{aligned}\tag{20}$$

From the above equation we can evaluate the maximum Δt for stability by numerically evaluating the right hand side over all elements. This calculation can be simplified considerably if we use the property that $S_i^3(k) = -S_i(k)$ and notice that

$$\begin{aligned}&[-B_i^-(k)]^2 =\\&\tfrac{1}{4}\begin{pmatrix}S_i^2(k) & -S_i(k)\\ S_i(k) & S_i^2(k)\end{pmatrix}\times\begin{pmatrix}S_i^2(k) & -S_i(k)\\ S_i(k) & S_i^2(k)\end{pmatrix} =\\&\tfrac{1}{2}\begin{pmatrix}-S_i^2(k) & S_i(k)\\ -S_i(k) & -S_i^2(k)\end{pmatrix} = -B_i^-(k),\end{aligned}$$

hence $\sqrt{-B_i^-(k)} = -B_i^-(k)$. This means that the eigenvalues of the operator $-B_i^-(k)$ are only 1, with multiplicity 2, and 0, with multiplicity 4. Now the eigenvector $\hat{\boldsymbol{e}}_k$ of $-B_i^-(k)$ which corresponds to the eigenvalue 1 can be written (without facet index k) as

$$\begin{pmatrix}\hat{\boldsymbol{a}}\\ \hat{\boldsymbol{b}}\end{pmatrix} = \frac{1}{2}\begin{pmatrix}-S^2 & S\\ -S & -S^2\end{pmatrix}\begin{pmatrix}\hat{\boldsymbol{a}}\\ \hat{\boldsymbol{b}}\end{pmatrix}, \quad \begin{aligned}2\hat{\boldsymbol{a}} &= -S^2\hat{\boldsymbol{a}} + S\hat{\boldsymbol{b}}\\ S\hat{\boldsymbol{a}} &= -2\hat{\boldsymbol{b}} - S^2\hat{\boldsymbol{b}}\end{aligned}.$$

This is satisfied for any $\hat{\boldsymbol{a}} = S\hat{\boldsymbol{b}} = \hat{\boldsymbol{n}} \times \hat{\boldsymbol{b}}$. Normalizing the eigenvector for any facet k we have $\hat{\boldsymbol{e}}_k = \sqrt{2}^{-1}\left(\hat{\boldsymbol{n}}_k \times \hat{\boldsymbol{b}}_k, \hat{\boldsymbol{b}}_k\right)^T$, where we choose $\hat{\boldsymbol{b}}_k$ as an arbitrary vector in the plane of the k-th facet (i.e., $\hat{\boldsymbol{n}}_k \cdot \hat{\boldsymbol{b}}_k = 0$). If we choose two orthogonal vectors $\hat{\boldsymbol{b}}_k^1$ and $\hat{\boldsymbol{b}}_k^2$ on the facet using the formula $\hat{\boldsymbol{b}}_k^2 = \hat{\boldsymbol{n}}_k \times \hat{\boldsymbol{b}}_k^1$ we can write two eigenvectors for the facet as

$$\hat{\boldsymbol{e}}_k^1 = \frac{1}{\sqrt{2}}\left(\hat{\boldsymbol{n}}_k \times \hat{\boldsymbol{b}}_k, \hat{\boldsymbol{b}}_k\right)^T,$$

$$\hat{\boldsymbol{e}}_k^2 = \frac{1}{\sqrt{2}}\left(-\hat{\boldsymbol{b}}_k, \hat{\boldsymbol{n}}_k \times \hat{\boldsymbol{b}}_k\right)^T.$$

The four eigenvectors corresponding to the zero eigenvalue of $-B_i^-(k)$ can be written as

$$\hat{\boldsymbol{e}}_k^{3,4} = \frac{1}{\sqrt{2}}\left(\hat{\boldsymbol{n}}_k, \pm\hat{\boldsymbol{n}}_k\right)^T,$$

$$\hat{\boldsymbol{e}}_k^{5,6} = \frac{1}{\sqrt{2}}\left(-\hat{\boldsymbol{n}}_k \times \hat{\boldsymbol{b}}_k^{1,2}, \hat{\boldsymbol{b}}_k^{1,2}\right)^T.$$

Thus, with the single vector $\hat{\boldsymbol{b}}_k$ in the plane of the k^{th} facet we can define all eigenvectors of the operator $\sqrt{-B_i^-(k)}$.

To efficiently evaluate formula (20) as in [2] we can decompose each $\boldsymbol{y}_k$ as a sum of eigenvectors, with the only ones taking part being the ones corresponding to non-zero eigenvalues $\sqrt{\lambda_i(k)} = 1$ of $\sqrt{-B_i^-(k)}$

$$\boldsymbol{y}_k = \sum_{s=1}^{2} w_k^s \hat{\boldsymbol{e}}_k^s.$$

The value of $\hat{\boldsymbol{b}}_k^1$ can be chosen arbitrarily, for example the edge of the facet. Substituting this decomposition into the inequality (20) we get the formula

$$\frac{V_i}{c\Delta t} \geq \left(\sum_{k=1}^{m_i} \sum_{s=1}^{2} (w_k^s)^2 \right)^{-1} \times \left[\sum_{k=1}^{m_i} \sqrt{A_i(k)} \sum_{s=1}^{2} w_k^s \hat{\boldsymbol{e}}_k^s \right] \cdot \left[\sum_{j=1}^{m_i} \sqrt{A_i(j)} \sum_{s=1}^{2} w_j^s \hat{\boldsymbol{e}}_j^s \right],$$

simplifying we get the formula which is used to obtain the time-step limit

$$\frac{V_i}{c\Delta t} \geq \frac{\sum_{k=1}^{m_i} \sum_{j=1}^{m_i} \sqrt{A_i(k) A_i(j)} \sum_{s_1=1}^{2} \sum_{s_2=1}^{2} \hat{\boldsymbol{e}}_j^{s_1} \cdot \hat{\boldsymbol{e}}_j^{s_2} w_j^{s_1} w_j^{s_2}}{\sum_{k=1}^{m_i} \sum_{s=1}^{2} (w_k^s)^2}.$$

This can be written concisely as,

$$\frac{V_i}{c\Delta t} \geq \frac{(G_i \boldsymbol{w}, G_i \boldsymbol{w})}{(\boldsymbol{w}, \boldsymbol{w})} \tag{21}$$

where the elements of the matrix $G_i \in \mathbb{R}^{2m_i \times 2m_i}$ are written as

$$G_i = \left[g_j^i g_k^i \hat{\boldsymbol{e}}^j \cdot \hat{\boldsymbol{e}}^k \right]_{j,k=1}^{2m_i}, \; g_{2k-1}^i = g_{2k}^i = \sqrt{A_i(k)},$$

$$\hat{\boldsymbol{e}}^{2k} = \hat{\boldsymbol{e}}_k^2, \; \hat{\boldsymbol{e}}^{2k-1} = \hat{\boldsymbol{e}}_k^1; \; k = 1..m_i.$$

Using this notation the stability criterion becomes

$$\Delta t \leq \min_i \frac{V_i}{c\|G_i\|}. \tag{22}$$

As in [1] this can be generalized concisely for elements with an individual ϵ_i and μ_i as

$$\Delta t \leq \min_i \frac{V_i}{c_i\|G_i\|}, \tag{23}$$

where $c_i = (\epsilon_i \mu_i)^{=0.5}$.

The norm of matrix G_i can be computed relatively quickly because $4 \leq m_i \leq 6$ for a cell-centered FVTD mesh which contains tetrahedrons, prisms, pyramids and hexahedrons.

IV. NUMERICAL EXPERIMENTS

To test the increase in the allowed time-step due to our new limit we conducted a wide set of numerical experiments on our cell-centered FVTD code [4]. These were conducted for both unstructured as well as structured meshes. For unstructured tetrahedral mesh we had a 5-15% increase in the allowable time-step over the time-step limit given by formula (2) from [1]. For a structured cubical mesh of equation (23) gives the same result as the von Neumann method applied to the FVTD approximation of Maxwell's equations [2],

$$\Delta t \leq \frac{h}{2c}.$$

This is a 1.5 times larger time-step that allowed by (2) when it is applied to a structured cubical mesh (h taken as the cubical element edge size).

V. SCATTERING FROM A PEC SPHERE

We present the FVTD results for scattering from a perfectly electrical conducting (PEC) sphere as an exact series solution is available in the frequency domain [5], and a time domain solution may be easily obtained using the inverse Fourier transform. The geometry of the problem is shown in Fig. 1. The radius of sphere is 3 meters. This problem was selected as a benchmark for the FVTD engine as the irregular surface of the sphere coincides with one of the primary reasons for developing finite-volume methods on irregular grids: eliminating the need for stair-stepping at arbitrarily shaped boundaries. The average edge-length of the tetrahedrons for the results shown in the figure set to 0.75 m.

An x-polarized electric-field plane-wave transient pulse $\boldsymbol{E} = g(t)\hat{\boldsymbol{x}}$ incident in the z-direction and varying as the derivative of a Gaussian was selected where, for $t \geq 0$,

$$g(t) = -2A(t - t_0)b^{-2} exp(-b^{-2}(t - t_0)^2) \tag{24}$$

with $A = 1$, $b = 1.14 \times 10^{-8}$s, $t_0 = 4.0 \times 10^{-8}$. These give a shortest free-space wavelength of about 3 meters resulting in significant energy in the resonance region of the sphere.

The results in the right side of the figure were computed using the second-order MUSCL-type scheme for two scattering locations: side-scatter and back-scatter (see Fig. 1). These are compared to the analytic solution for these same two points. The FVTD results shown were computed for the time-step given by equation (2), but almost identical results were obtained when the time-step was set to that of equations (1) and (23). Note that all the time-step criteria discussed in this paper are for the firstorder upwind schemes, but we have found that using any of these criteria for the higher-order schemes gives stable results. This was also reported by Piperno [1]. Table 1. shows the relative error of computational results for

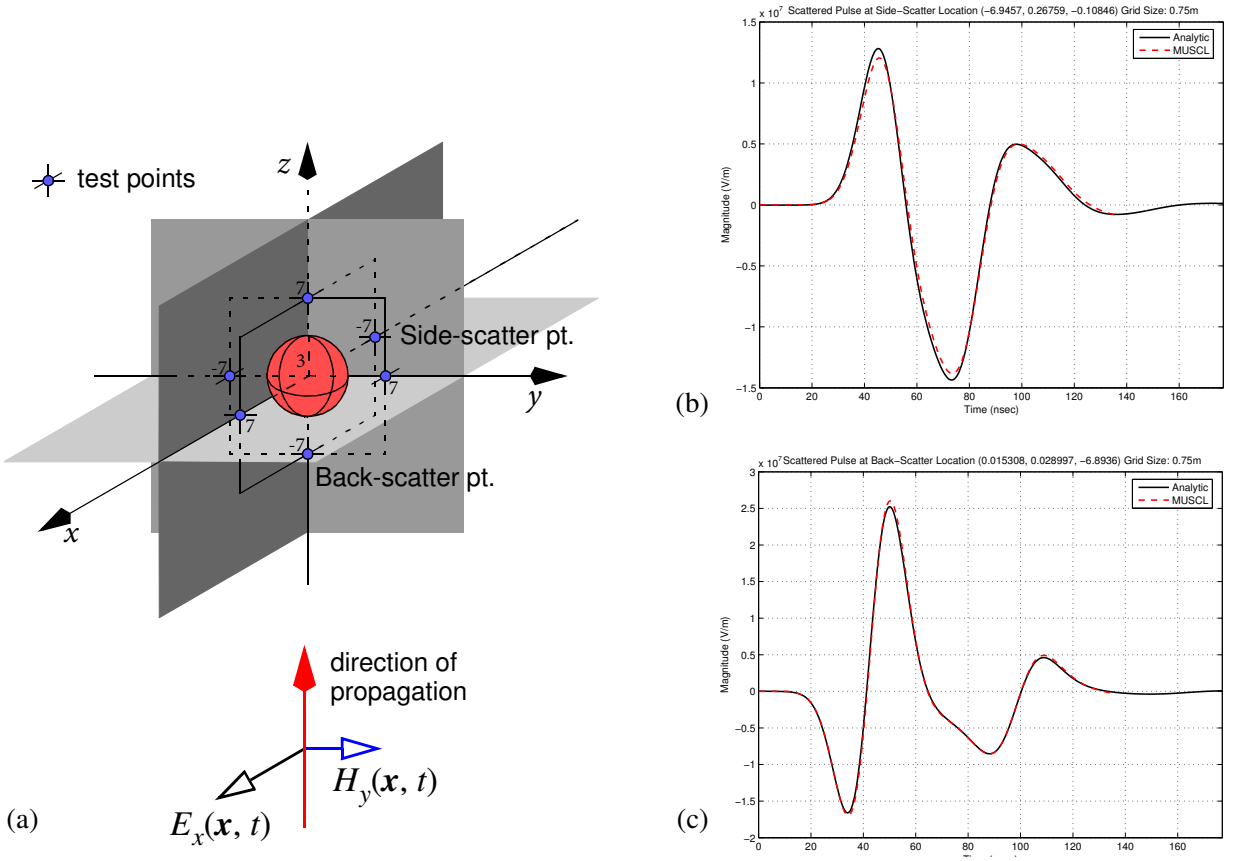

Fig. 1. Scattering from a sphere. (a) Geometry of the problem, (b) Side-scattered electric field E_x, and (c) back-scattered electric field E_x. Time-step used was that of equation (23). Analytic results are also shown.

Table 1. Comparison of PEC sphere back-scattering and side-scattering results on time interval $(0,1.4 \times 10^7)$.

		Side Scatter		Back Scatter	
Time step algorithm	# of time steps (acceleration factor)	L_2 Error of E_x	L_∞ Error of E_x	L_2 Error of E_x	L_∞ Error of E_x
Eq. (1)	1497 (1.0)	4.15%	4.49%	4.37%	4.21%
Eq. (2)	749 (2.0)	4.52%	4.19%	4.23%	3.99%
Eq. (21)	704 (2.13)	4.63%	4.47%	4.25%	4.50%

the x-component of the scattered field at the back-scatter location (0, 0, -7), as well as at the side-scatter location (-7, 0, 0) for the three different time-step criteria: equations (1), (2) and (23). For both test locations, the analytic solution is compared to solutions obtained using the MUSCL finite-volume methods computed on a mesh with an average cell edge length of 0.75 m. All results shown use the second-order predictor-corrector time-integration scheme.

VI. CONCLUSION

The derivation we have provided gives a new time-step limit for the explicit upwinding finite-volume time-domain approximation scheme on arbitrary unstructured meshes for Maxwell's equations. On uniform meshes the new criterion is necessary for stability. In fact, the time-step limit provided by the formula given herein gives the same time-step limit on structured meshes as the standard von Neumann analysis. On mixed structured and unstructured meshes, the new criterion provides the maximum time-step allowable for retaining stability. The derivation can be easily extended to other FVTD approximations of partial differential equations on unstructured meshes; which is a subject of future work.

ACKNOWLEDGMENTS

This work was supported by the Department of National Defence and the Natural Sciences and Engineering Research Council.

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