A Practical Fourth Order Finite-Difference Time-Domain Algorithm for the Solution of Maxwell's Equations

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***Abstract*—Implementing a practical fourth order accurate in time and second order accurate in space finite difference time domain simulation using MATLAB is the goal of this paper. The formulation presented for the fourth order approximation is simple to integrate into an existing second order accurate in time and second order accurate in space formulation and well-established code. The fourth order formulation has been verified and simulation accuracy is confirmed through the application of radiation from a single and an array of dipole antennas.**

Keywords*—*FDTD, finite difference time domain, fourth order approximation, higher order.

I. INTRODUCTION

The finite difference time domain (FDTD) method is a highly effective method of numerically solving Maxwell's equations in the time domain [1]. The standard derivative approximation is a second order accurate central differencing scheme for all derivatives in Maxwell's equations (second order) [1]. This paper will present the implementation for a practical FDTD scheme using fourth order accurate central differencing derivative approximations in space and second order accurate central differencing derivative approximations in time (fourth order). Fourth order FDTD simulations allow the cell size of the simulation to grow while maintaining necessary solution accuracy. Larger cell sizes are imperative when geometries become electrically large and the computational memory becomes too excessive. Many other papers have studied the benefits of fourth order FDTD, but none have simulated practical antenna problems with a simple formulation [2], [3], [4], [5]. The goal of this paper is to present a fourth order formulation that is straightforward and at the same time can simulate practical problems such as antenna arrays.

II. FOURTH ORDER FDTD FORMULATION

Fourth order FDTD can be simply derived by using fourth order accurate derivative approximations (equation 2) for spatial derivatives and using the second order accurate derivative approximation for the time derivative in Maxwell's equations. Consider, for example, the E_x component of Ampere's Maxwell equation as presented in [1]:

$$\frac{\partial E_x}{\partial t} = \frac{1}{\epsilon_x}\left(\frac{\partial H_z}{\partial y} - \frac{\partial H_y}{\partial z} - \sigma_x^e E_x - J_{ix}\right). \quad (1)$$

Upon using the fourth order accurate approximation for the spatial derivatives, that is:

$$f'\left(x - \frac{\Delta x}{2}\right) \approx \frac{-f\left(x+\Delta x\right) + 27 f\left(x\right) - 27 f\left(x - \Delta x\right) + f\left(x - 2\Delta x\right)}{24\Delta x}, \quad (2)$$

the updating equation of the E_x component will take the following form:

$$E_x^{n+1}(i,j,k) = \begin{aligned}&+\frac{2\epsilon_x(i,j,k) - \Delta t\sigma_x^e(i,j,k)}{2\epsilon_x(i,j,k) + \Delta t\sigma_x^e(i,j,k)} E_x^n(i,j,k) - \frac{2\Delta t}{2\epsilon_x(i,j,k) + \Delta t\sigma_x^e(i,j,k)}\beta \\ &+\frac{2\Delta t}{2\epsilon_x(i,j,k) + \Delta t\sigma_x^e(i,j,k)}\alpha - \frac{2\Delta t}{2\epsilon_x(i,j,k) + \Delta t\sigma_x^e(i,j,k)} J_x^{n+\frac{1}{2}}(i,j,k)\end{aligned}, \quad (3)$$

where,

$$\alpha = \frac{-H_z^{n+\frac{1}{2}}(i,j+1,k) + 27H_z^{n+\frac{1}{2}}(i,j,k) - 27H_z^{n+\frac{1}{2}}(i,j-1,k) + H_z^{n+\frac{1}{2}}(i,j-2,k)}{24\Delta y},$$

and

$$\beta = \frac{-H_y^{n+\frac{1}{2}}(i,j,k+1) + 27H_y^{n+\frac{1}{2}}(i,j,k) - 27H_y^{n+\frac{1}{2}}(i,j,k-1) + H_y^{n+\frac{1}{2}}(i,j,k-2)}{24\Delta z}.$$

Note that based on the above representation, if the second order derivative approximation is anticipated, making this second order FDTD, all what is needed is to change the definitions of α and β. The other coefficients in equation (3) are the same for the second order and fourth order formulations. This makes the integration of this fourth order formulation into an existing second order code [1] rather simple. The updating equations for voltage and current sources as well as resistors and convolutional perfectly matched layer (CPML) can all be easily derived and integrated.

III. 1D VERIFICATION EXAMPLE

A simple 1D Gaussian wave propagating in free space was simulated with second order and fourth order formulations. The 1D domain is 1 meter long discretized with 300 cells of air. The source is an infinite sheet of J_z producing a Gaussian pulse with 5 cells per λ_{min}. For second order, 20 cells per λ_{min} is recommended [1] meaning this is a course grid.

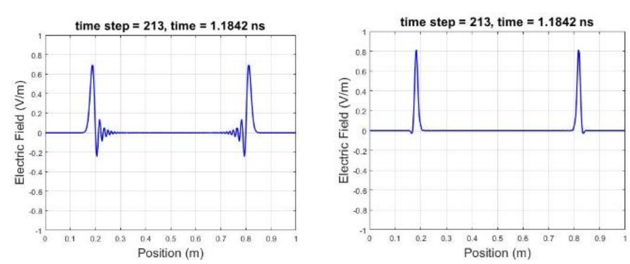

Fig. 1. The second order (right) and fourth order (left) 1D results.

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As Fig. 1 shows the fourth order results have less dispersion than the second order results, showing the fourth order simulation is more accurate than the second order simulation.

IV. Practical Problem: Thin Wire Dipole

Although many papers have presented the formulation for fourth order FDTD, none have simulated practical problems such as antenna problems complete with CPML boundaries, and sources and loads in terms of voltage sources, resistors, and other circuit elements [2], [3], [4], [5]. A thin wire dipole antenna simulation using the developed fourth order FDTD is presented here. The improved thin wire formulation from [1, Ch. 10] is used with the necessary modifications to implement fourth order FDTD. The length of the dipole is 20mm oriented in the Z-direction, with a center gap of 1 cell length for the voltage source. The cell size is defined by dx=dy=dz=0.25mm. The voltage source supports a Gaussian waveform with a max frequency when 20 cells are used per λ_{min} and an internal impedance of 50Ω. A theoretical dipole transmits best when acting as a half wavelength dipole antenna [6]. Based on the length of this dipole, the minimum S_{11} reflection coefficient is predicted to be at 7.5 GHz.

The minimum reflection coefficient in Fig. 2 (left) occurs at 7.52 GHz which shows a difference of 0.27% when compared to the analytical expected value of 7.5 GHz. Additionally, the directivity in the x-y plane matches the expected isotropic radiation pattern for a half wavelength dipole antenna.

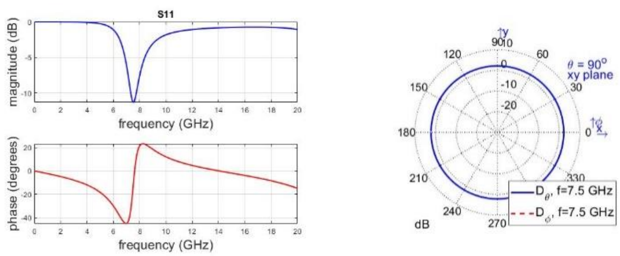

Fig. 2. The simulation results for a dipole antenna, (left) is the amplitude and phase of the reflection coefficient for a wide frequency band, (right) is the directivity pattern in the x-y plane.

V. Practical Problem: Thin Wire Dipole Array

The real benefit of using fourth order FDTD is evident when simulating electrically large problems, such as antenna arrays. For simplicity and ease of comparison to analytical solutions a simple two element array of thin wire dipole antennas is simulated. The array is in the x-direction with a spacing of $\lambda/2$ = 20mm. The problem space setup and each antenna geometry match that of the antenna simulated in Fig. 2.

Note that Fig. 2 (left) and Fig. 3 (left) do not match exactly because FDTD is a full wave solution technique and takes into account the coupling between the two antennas.

As this is a simple two element uniform linear array, the normalized array factor without coupling can be written as follows [6]:

$$(AF)_N = \cos\left(\frac{1}{2}(kd\cos\phi + \beta)\right). \tag{5}$$

The polar plot of the array factor has maximums at $\phi = 90^o$ and $\phi = 270^o$ and minimums at $\phi = 0$ and at $\phi = 180^o$.

Since the single element dipole directivity is isotropic in the x-y plane, the total directivity pattern of the array should match the shape of the array factor. Fig. 3 (left) demonstrates this fact, providing evidence that the fourth order FDTD produced good results for an antenna array problem. Fig. 3 (left) also demonstrates that larger arrays can be simulated using the presented fourth order FDTD implementation. The larger the array, the more computational advantage the fourth order formulation will have over the second order formulation.

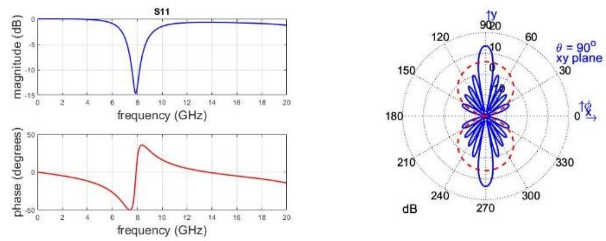

Fig. 3. The simulation results for an array of dipole antennas, (left) is the amplitude and phase of the reflection coefficient of the center element, (right) is the directivity pattern in the x-y plane for a ten-element array (solid blue) and the two-element array (dashed red).

VI. Conclusions

This paper demonstrates a practical and straightforward implementation of a fourth order FDTD formulation. The formulation allows for the integration of thin wire, active and passive circuit elements, CPML, and other special formulations as presented in a second order formulation. In order to confirm the validity and accuracy of the developed formulation the radiation from a single element and arrays of dipole antennas are performed leading to the expected results.

Future work will include implementing fourth order simulations of circuit elements such as inductors, capacitors, and diodes. Additionally, fourth order handling of perfect electric conductor (PEC) objects will be explored.

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