Effect of Sparse Array Geometry on Estimation of Co-array Signal Subspace

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***Abstract*—This paper considers the effect of sparse array geometry on the co-array signal subspace estimation error for Direction-of-Arrival (DOA) estimation. The second largest singular value of the signal covariance matrix plays an important role in controlling the distance between the true subspace and its estimate. For a special case of two closely-spaced sources impinging on the array, we explicitly compute the second largest singular value of the signal covariance matrix and show that it can be significantly larger for a nested array when compared against a uniform linear array with same number of sensors.**

***Index Terms*—Davis Kahan, Difference co-arrays, Sparse arrays, Subspace Estimation.**

I. Introduction

Sparse array geometries have recently gained significant research interest [1]–[4] owing to several attractive benefits over traditionally used uniform linear arrays (ULA), such as the ability to identify $O(P^2)$ uncorrelated sources using only P sensors [1], [2], as well as lower Cramér-Rao bounds and higher spatial resolution than ULAs with the same number of spatial and temporal measurements [4], [5]. These enhanced abilities are attributed to the fact that the difference co-arrays of these sparse arrays contain a ULA segment (consisting of consecutive lags around 0) of size $O(P^2)$ that can be exploited by algorithms such as co-array MUSIC [1], [4], [6] to resolve more sources than sensors. However, the performance of co-array based DOA estimation algorithms can potentially deteriorate when the so-called signal subspace is not identified properly. In this paper, we study how the geometry of sparse arrays can influence the distance between the true signal subspace and its estimate for the special case of two narrowband sources. We show that the second largest singular value of the signal covariance matrix controls the mismatch between the true and estimated subspaces. In general, greater the second largest singular value, smaller the mismatch. Given a sparse array and ULA with same number of sensors, we show that the second largest singular value of the sparse can be significantly larger than that of the ULA. We will use this result in future to understand non asymptotic performance of sparse arrays with closely spaced sources.

II. Effect of Sparse Array Geometry on Signal Subspace Perturbation

Consider a linear array of P antennas whose physical locations are given by $\{d_p\lambda/2, p = 1, 2, \cdots, P\}$, where d_p belongs to an integer set $\mathbb{S}, |\mathbb{S}| = P$, and λ is the wavelength of far-field narrowband sources impinging on the array. In this paper we consider the special case of two narrowband sources with Direction-of-Arrival (DOA) $\theta_1, \theta_2 \in (-\pi/2, \pi/2]$. The data vector $\mathbf{y} \in \mathbb{C}^P$ consisting of measurements collected at the P antennas is given by:

$$\mathbf{y} = \mathbf{A}\mathbf{x} + \mathbf{n}. \tag{1}$$

Here $\mathbf{A} = [\mathbf{a}(\theta_1)\ \mathbf{a}(\theta_2)]$ is the array manifold matrix where $\mathbf{a}(\theta_i) = [e^{j\pi d_1 sin(\theta_i)}\ \ldots\ e^{j\pi d_P sin(\theta_i)}]^T$ represents the array steering vector corresponding to the direction θ_i. Furthermore $\mathbf{x} = [x_1, x_2]^T$ is the vector of complex amplitude of the two sources and $\mathbf{n}$ represents the additive noise at the P antennas.
Role of Difference Co-array: If we assume that the source amplitudes x_1, x_2 are zero-mean statistically uncorrelated random variables, and the noise is zero-mean white and statistically independent of $\mathbf{x}$, the correlation matrix of $\mathbf{y}$ is given by:

$$\mathbf{R_{yy}} = \mathbf{A}\mathbf{P}\mathbf{A}^H + \sigma_n^2\mathbf{I}_P, \tag{2}$$

where $\mathbf{P} = \text{diag}(p_1, p_2)$ is a diagonal matrix with $p_1 = \mathbb{E}(|x|_1^2), p_2 = \mathbb{E}(|x|_2^2)$ and $\mathbb{E}\left(\mathbf{n}\mathbf{n}^H\right) = \sigma_n^2\mathbf{I}$. Notice that the cross-correlation between the measurement at the mth and kth antennas is given by:

$$[\mathbf{R_{yy}}]_{m,k} = \sum_{i=1}^{2} e^{j\pi \sin\theta_i(d_m - d_k)} p_i + \sigma_n^2\delta[m-k]. \tag{3}$$

In other words, the cross-correlation depends on the pairwise difference $\{d_m - d_k\}$ between the sensor locations. This naturally leads to the notion of a difference set [1].

Definition II.1. The difference set of a set of integers $\mathbb{S}$ is defined as $\mathbb{D}_{\mathbb{S}} = \{d_i - d_j | d_i, d_j \in \mathbb{S}\}$.

Given a sensor array $\mathbb{S}$, we can associate a "virtual difference co-array" whose element locations are given by the set $\mathbb{D}_{\mathbb{S}}$. We can also define an integer $N_{\max}$ as the largest integer such that $\{0, 1, \ldots, N_{\max}\} \subset \mathbb{D}_{\mathbb{S}}$. In this case, the set $\mathbb{U}_{\mathbb{S}} = \{0, 1, \cdots, N_{\max}\}$ is called the "Non-negative ULA segment" of the difference co-array. We can extract $N_{\max}+1$ entries $\mathbf{R}_{m,n}$ of the data correlation matrix such that $\{d_m - d_n\} \in \mathbb{U}_{\mathbb{S}}$ and collect them in a vector $\mathbf{z} \in \mathbb{C}^{N_{\max}+1}$ [1]. Using (3), it can be shown that:

$$\mathbf{z} = \mathbf{A}_{\mathbb{U}_{\mathbb{S}}}\mathbf{p} + \sigma_n^2\mathbf{e}, \tag{4}$$

where $\mathbf{A}_{\mathbb{U}_{\mathbb{S}}} \in \mathbb{C}^{(N_{\max}+1)\times 2}$ is the array manifold matrix corresponding to a ULA with element locations given by the set $\mathbb{U}_{\mathbb{S}}$ and $\mathbf{e} = [1, 0, 0, \ldots, 0]$ is a canonical basis vector in $\mathbb{C}^{N_{\max}+1}$. It is well-known that for specially designed sparse arrays such as nested and coprime arrays [1], [2], $N_{\max} = \Theta(P^2)$ whereas for ULA, $N_{\max} = \Theta(P)$. The large difference set of sparse arrays can be utilized to localize more sources than sensors using the so-called co-array MUSIC

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algorithm [1], [4], [6]. In this paper, we investigate how the large difference co-array of nested arrays helps to characterize subspace estimation errors.

Distance between Perturbed and True Signal Subspaces: Let $\hat{\mathbf{z}}$ be an estimate of $\mathbf{z}$ obtained from the sample covariance matrix $\hat{\mathbf{R}}_{\mathbf{yy}}$. Then, we have:

$$\text{Toep}(\hat{\mathbf{z}}) = \mathbf{A}_{\mathbb{U}_\mathbb{S}}\mathbf{P}\mathbf{A}_{\mathbb{U}_\mathbb{S}}^H + \mathbf{H}, \tag{5}$$

where Toep($\hat{\mathbf{z}}$) represents a Hermitian Toeplitz matrix whose first column is $\hat{\mathbf{z}}$ and $\mathbf{H} = \text{Toep}\left(\sigma_n^2\mathbf{e} + \hat{\mathbf{z}} - \mathbf{z}\right)$ captures the effect of noise and estimation error. The co-array signal subspace is defined as the span of the co-array steering vectors corresponding to the source directions θ_1 and θ_2 and is given by [4]:

$$\mathcal{S}_{\text{ca}} = \text{Range}\left(\mathbf{A}_{\mathbb{U}_\mathbb{S}}\right). \tag{6}$$

Algorithms such as co-array MUSIC [1], [4] aim to obtain an estimate of this subspace, which is subsequently used for identifying the DOAs using similar principle as the classical MUSIC algorithm [7]. A popular practice is to use the following estimate of $\mathcal{S}_{\text{ca}}$:

$$\hat{\mathcal{S}}_{\text{ca}} = \text{Range}\left(\hat{\mathbf{U}}\right), \tag{7}$$

where the columns of $\hat{\mathbf{U}} \in \mathbb{C}^{(N_{\max}+1)\times 2}$ are the singular vectors of Toep($\hat{\mathbf{z}}$) corresponding to the two largest singular values. It is obvious that the accuracy of DOA estimation crucially depends on the subspace estimation error, which in turn, depends on the array geometry. Let $\mathbf{U}$ be an orthogonal basis for $\mathcal{S}_{\text{ca}}$. Let σ_2 be the second largest singular value of $\mathbf{A}_{\mathbb{U}_\mathbb{S}}\mathbf{P}\mathbf{A}_{\mathbb{U}_\mathbb{S}}^H$. If $\sigma_2 > \|\mathbf{H}\|_2$, we can use Davis-Kahan theorem [8] to bound the distance between the true subspace and its estimate as:

$$\text{dist}(\mathbf{U}, \hat{\mathbf{U}}) = \|\mathbf{U}\mathbf{U}^H - \hat{\mathbf{U}}\hat{\mathbf{U}}^H\|_2 \leq \frac{\|\mathbf{H}\|_2}{\sigma_2 - \|\mathbf{H}\|_2}. \tag{8}$$

Here, $\|\mathbf{H}\|_2$ is the spectral norm of the matrix $\mathbf{H}$. The condition $\sigma_2 > \|\mathbf{H}\|_2$ is also related to the assumption of "no subspace swap", which is crucial for analysis of co-array MUSIC [4]. We next explicitly characterize the role of co-array geometry in determining how large σ_2 can be by considering two array geometries: ULA and nested array. Let $L = N_{\max}+1$ and $\beta = (\sin(\theta_1)-\sin(\theta_2))/2$ is the normalized angle distance. Using the fact that the smallest eigenvalue of $\mathbf{A}_{\mathbb{U}_\mathbb{S}}^H\mathbf{P}\mathbf{A}_{\mathbb{U}_\mathbb{S}}$ is equal to the second largest singular value of $\mathbf{A}_{\mathbb{U}_\mathbb{S}}\mathbf{P}\mathbf{A}_{\mathbb{U}_\mathbb{S}}^H$, we explicitly compute the second largest singular value of $\mathbf{A}_{\mathbb{U}_\mathbb{S}}\mathbf{P}\mathbf{A}_{\mathbb{U}_\mathbb{S}}^H$ as:

$$\sigma_2 = \frac{L(p_1+p_2)}{2} - \sqrt{\left(\frac{L(p_1-p_2)}{2}\right)^2 + p_1p_2\frac{1-cos(2\pi L\beta)}{1-cos(2\pi\beta)}}. \tag{9}$$

Notice that σ_2 grows linearly with $L = N_{\max}+1$, when other quantities are held constant. Recall that for a ULA, $N_{\max} = P-1$ and for a nested array, $N_{\max} = \lfloor\frac{P}{2}\rfloor(\lceil\frac{P}{2}\rceil+1)$ [1]. Therefore, for a nested array, σ_2 grows quadratically with the number of sensors P whereas for a ULA, it only grows linearly with P. This is illustrated in Fig. 1. In Fig. 2, we plot σ_2 as a function of β which is the normalized separation between source directions. For most values of the separation between sources, σ_2 for the nested array is significantly larger than that of a ULA.

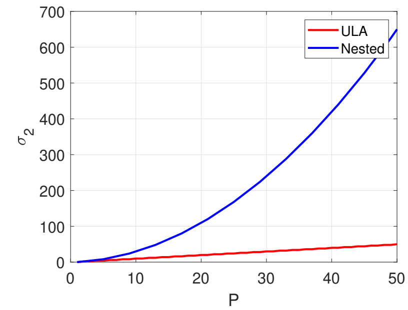

Fig. 1. Dependence of σ_2 on the number of antennas P for nested array and ULA. Here, $p_1 = p_2 = 1$ and $\beta = \frac{1}{2}$.

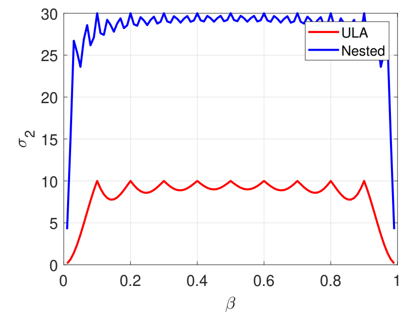

Fig. 2. Dependence of σ_2 on β for nested array and ULA. Here, $P = 10$ for both arrays and $p_1 = p_2 = 1$.

III. Conclusion

We studied the effect of sparse array geometry on co-array signal subspace estimation. For the special case of two sources, we explicitly characterized the second largest singular value and how it controls the distance between the true signal subspace and any estimate. In future, we will use these results to characterize non-asymptotic performance of sparse array based DOA estimation algorithms.

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