Robust Optimization of Electromagnetic Design Using Stochastic Collocation Method

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Abstract ─ The way to handle the uncertainty of design parameters has attracted wide attention in the optimization of electromagnetic designs. The Monte Carlo Method works well when dealing with uncertainties but it consumes too much time and computational resources. This paper proposes a computationally efficient way to achieve robust optimization based on Stochastic Collocation Method and the TEAM 22 problem is used as a verification example. It is demonstrated that the approach combining Stochastic Collocation Method and a genetic algorithm provides high computational efficiency, without losing accuracy compared with the Monte Carlo Method.

Index Terms ─ Monte Carlo Method, robust, Stochastic Collocation Method, TEAM 22, uncertainty.

I. INTRODUCTION

It is a common knowledge that the performance of the electromagnetic (EM) design is influenced by uncertainties [1]. The uncertainties may be caused by, for example, manufacturing tolerances in geometric variables or the inevitable variation of material parameters. These uncertainties will jeopardize the robustness of the design. Robustness refers to the insensitivity of the final optimization results to parameter perturbations [4]. A robust design should not be very sensitive to slight changes in its design parameters as this would either seriously impact production costs or make the physical machine behave differently from its optimized computer simulation model [5].

In response to this requirement, many scholars have used a variety of methods to obtain a robust device or control method. Some examples of the state-of-the-art optimization methods used in the field comprise a new algorithm based on the Climb method [4], a possibility-based optimal design algorithm [5], a surrogate modeling technique based on a second-order equation [6], and the space-time kriging surrogate model [8].

The previous methods require a lot of time and computational resources to obtain a robust solution, which significantly affect computational efficiency of EM optimization [9] .Therefore, some measures are taken such as evaluating the robustness of both performance and constraints under uncertainty by the worst-case optimization [9], reusing the global surrogate model to perform sensitivity analyses of generated designs [10], and proposing an efficient reliability-based robust design optimization method [11]. However, these methods more or less show limitations when they are applied to other optimization problems.

To improve the numerical efficiency of EM optimization while maintaining design robustness, an optimization strategy based on Stochastic Collocation Method (SCM) [12] is proposed in this paper. The basic concepts of SCM are given in Section II. Section III shows the process of obtaining the robust optimal solution based on SCM. Section IV introduces the TEAM 22 problem [17] and compares the robustness and time consumption of the optimization techniques, with the results obtained by the different methods. Finally, the paper's conclusions are drawn in Section V.

II. OUTLINE OF THE STOCHASTIC COLLOCATION METHOD

It can be helpful to apply uncertainty analysis methods to computational electromagnetics (CEM) [20] in order to take account of practical complexity and unpredictability within a simulation. To this end, design parameters of EM simulation are presented by random variables with properly assigned distributions.

The Stochastic Collocation method (SCM) is a popular choice for the stochastic processing of complex systems where well-established deterministic codes exist. SCM is supposed to be universal because it only

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deals with the input and output of the problem without changing the solver itself. The applicability of SCM method is not affected by the complexity of the original problem so long as reliable deterministic solver is developed. By utilizing the SCM method, the relationship between the uncertainty of output and the input variables is approximated by the sum of some specific polynomials. One way is to use a Lagrange interpolation approach that is given by (1):

$$y(x) = Lag(f(x)) = \sum_{j=0}^{n} f(x_j) l_j(x), \tag{1}$$

where $y(x)$ is a polynomial approximation of the true solution $f(x)$. n is the number of collocation points. x_j and $f(x_j)$ stand for the collocation points and the corresponding deterministic solutions of these points, respectively. $l_j(x)$ are the Lagrange interpolation polynomials structured by the collocation points:

$$l_j(x) = \prod_{i=0, i\neq j}^{n} \frac{(x - x_i)}{(x_j - x_i)}. \tag{2}$$

As for the SCM, the collocation points are given by the zero points of the generalized Polynomial Chaos [12]. The orthogonal polynomial basis is selected according to the probability distribution of the random variables, as shown in Table 1. Specially, if the variables are multidimensional, the interpolating points are the tensor product form of interpolating points in every dimension [16]. The accuracy of the SCM had been elaborated in [22]. It was shown that the longer each single CEM simulation lasted, the more efficient and thus more desirable SCM could be.

Table 1: The correspondence between the type of generalized Polynomial Chaos and Random Variables

Random Variables	**Wiener-Askey Chaos**	**Support**
Gaussian	Hermite-chaos	$(-\infty, +\infty)$
Gamma	Laguerre-chaos	$[0, +\infty)$
Beta	Jacobi-chaos	$[a, b]$
Uniform	Legendre-chaos	$[a, b]$

III. SCM BASED ROBUST OPTIMAL DESIGN

The traditional optimal design problem is constructed as:

$$\begin{gathered} \min f(\boldsymbol{P}_{\mathrm{d}}) \\ s.t.\ g_i(\boldsymbol{P}_{\mathrm{d}}) \le 0, \quad i = 1, ..., m, \end{gathered} \tag{3}$$

where $f(\boldsymbol{P}_{\mathrm{d}})$ is the objective function for design variable set $\boldsymbol{P}_{\mathrm{d}}$; $g_i(\boldsymbol{P}_{\mathrm{d}})$ are the constraint functions for $i=1,\ldots,m$.

When the uncertainties are taken into account, the random design variable set $\boldsymbol{P}$ is defined as:

$$\boldsymbol{P} = \overline{\boldsymbol{P}} + \xi,$$

where $\overline{\boldsymbol{P}}$ is the mean value of $\boldsymbol{P}$ according to the statistical definition, while ξ is the random variable whose distribution can be assumed to be uniform in order to obtain a robust solution to the optimization problem. Thus, $\boldsymbol{P}$ can be redefined by normalization as:

$$\boldsymbol{P} = \overline{\boldsymbol{P}} + \boldsymbol{\eta} \cdot x, \tag{4}$$

where $\boldsymbol{\eta}$ is half of the range of the distribution and x a random number between [-1,1].

The incorporation of the robustness analysis with formulation (4) incurs high computational time. For the traditionally used Monte Carlo method (MCM), a serious computational burden is imposed by the required large sample size, as well as the iterative nature of the design optimization process. In contrast, the SCM is similar to MCM in the sense that it involves only the solution of a sequence of deterministic calculations at given collocation points in the stochastic space [23].

By applying the SCM to robust optimal design problem, $f(\boldsymbol{P})$ can be approximated by Lagrange interpolation polynomial according to (1):

$$f(\boldsymbol{P}) \approx y(\boldsymbol{P}) = \sum_{i=1}^{n} f(\boldsymbol{P}_i) l_i(\boldsymbol{P}), \tag{5}$$

where $\boldsymbol{P}_i$ (i=1,2,…,n) is the collocation points for univariate (or tensor product form of interpolating points for multivariate) problems, with a similar form to (4):

$$\boldsymbol{P}_i = \overline{\boldsymbol{P}} + \boldsymbol{\eta} \cdot x_i.$$

Generally, the zero points of generalized Polynomial Chaos in Table I are chosen to be x_i. For instance, the Legendre polynomials are orthogonal with respect to the uniform distribution. Its expression is as follows:

$$\begin{gathered} L_0(x) = 1, \\ L_n(x) = \frac{1}{2^n n!} \frac{d^n}{dx^n} \left\{ \left(x^2 - 1 \right)^n \right\}, \quad n = 1, 2, ... \end{gathered} \tag{6}$$

Let the zero points of the n-dimensional Legendre polynomial be $x_1, x_2, ..., x_n$, and the Lagrange basis polynomial $l_i(x)$ are given by (2).

After the $y(\boldsymbol{P})$ is constructed (equation (5)), the mean of the obtained n values is defined as the objective function value of $\boldsymbol{P}$ as MCM does. It is denoted as:

$$\hat{f}_s(\boldsymbol{P}) = \frac{1}{n} \sum_{i=1}^{n} y(\boldsymbol{P}_i) \tag{7}$$

The SCM is not subject to the number of sampling points but to the number of variables. For problems where the solution is a smooth function of the random input variables and the dimension of the stochastic space is moderate, SCM has been shown to converge much faster than MCM [23].

IV. APPLICATION TO TEAM 22

The TEAM workshop problem 22 [14-16] is an optimization case of the Superconducting Magnetics

Energy Storage (SMES) that has been used as a benchmark problem in magneto statics. The TEAM 22 system is composed of two coils with opposite current densities. The configuration of the TEAM 22 is presented in Fig. 1, and its parameters are summarized in Table 2. The goal of TEAM 22 is to find the best configuration in SMES device to maintain the stored energy while minimizing the stray field. The stray field is represented by magnetic flux density $\boldsymbol{B}_{\text{stray}}$ and it is evaluated in 21 equidistant points marked on lines *a* and *b* in Fig. 1:

$$\boldsymbol{B}_{\text{stray}}^{2}=\frac{\sum_{i=1}^{21}\boldsymbol{B}_{\text{stray},i}^{2}}{21}. \tag{8}$$

Furthermore, to keep the superconductivity characteristic, the restriction is given by the inequality (9):

$$\|J\|\leq(-6.4\|\boldsymbol{B}_{\max}\|+54)\ \ (\text{A/mm}^2). \tag{9}$$

Since the current density of both coils is fixed in the TEAM 22, the inequality (9) can also be expressed as (10):

$$\|\boldsymbol{B}_{\max}\|\leq 4.92\text{T}. \tag{10}$$

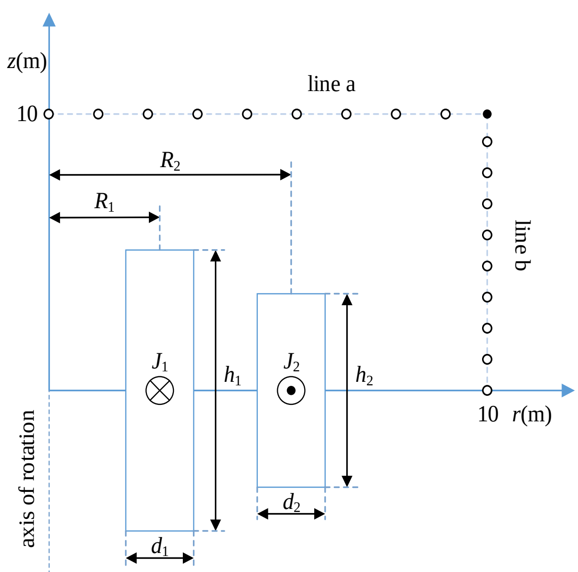


Fig. 1. The configuration of the TEAM 22.

When the robustness is taken into account in the optimization of TEAM 22, it is assumed that the adjustable parameters R_2, h_2, d_2 in the outside coil suffer undesirable and unavoidable presence of the uncertainties. The goals of the robustness-considered case remain the same as the classical problem. By defining the design variables set $\boldsymbol{P}=\{R_2,h_2,d_2\}$, the robust TEAM 22 problem can be formulated with the objectives and the restriction above as:

$$\min\ f(\boldsymbol{P})=\omega_1\frac{\boldsymbol{B}_{\text{stray}}^2}{\boldsymbol{B}_{\text{norm}}^2}+\omega_2\frac{|E-E_0|}{E_0}\ , \tag{11}$$

$$s.t.\ \ \|J\|\leq(-6.4\|\boldsymbol{B}_{\max}\|+54)(\text{A/mm}^2)$$

where $\boldsymbol{B}_{\text{norm}}$=200μT; $\boldsymbol{B}_{\max}$ is the maximum magnetic flux density; E is the energy that is actually stored in the designed device; E_0 is the target stored energy with a fixed value 180MJ; ω_1, ω_2 are the barycentric weights. Take ω_1=0.001, ω_2=1 in this case to keep the relative value of the leakage flux in the same order of magnitude as the relative error of the stored energy:

$$\min\ f(\boldsymbol{P})=0.001\times\frac{\boldsymbol{B}_{\text{stray}}^2}{\boldsymbol{B}_{\text{norm}}^2}+\frac{|E-E_0|}{E_0}\ .$$

$$s.t.\ \ \|B_{\max}\|\leq 4.92\text{T}$$

The genetic algorithm (GA) is used to optimize the TEAM22, and the robustness is achieved by using MCM and SCM respectively at the fitness function which is the criterion for selection operators. The process of finding the optimal solution is shown in Fig. 2. Following the process of survival of the fittest in nature, the objective function is chosen as a selection criterion. Once the genetic representation and the fitness function are defined, the GA proceeds to initialize a population of solutions and then to improve it through repetitive application of the mutation, single point crossover, inversion and selection operators until a termination condition has been reached. The population is set to 200, and the termination condition is set to iterate 200 generations. The workstation used in this paper is Dell T7610 with Intel(R) Xeon(R) E5-2687W v2 3.4GHz and 128G RAM.

Table 2: Team 22 parameters

	R_1[m]	R_2[m]	h_1[m]	h_2[m]	d_1[m]	d_2[m]	J_1[A/mm^2]	J_2[A/mm^2]
Min	-	2.6	-	0.408	-	0.1	-	-
Max	-	3.4	-	2.2	-	0.4	-	-
Step	-	0.01	-	0.007	-	0.003	-	-
Fixed	2.0	-	1.6	-	0.27	-	22.5	-22

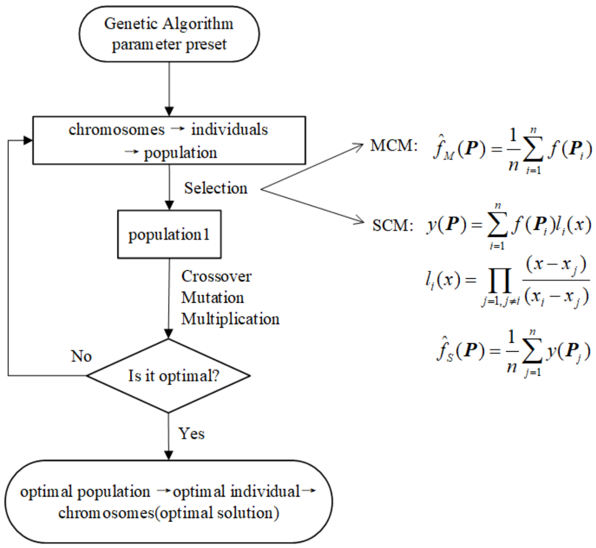


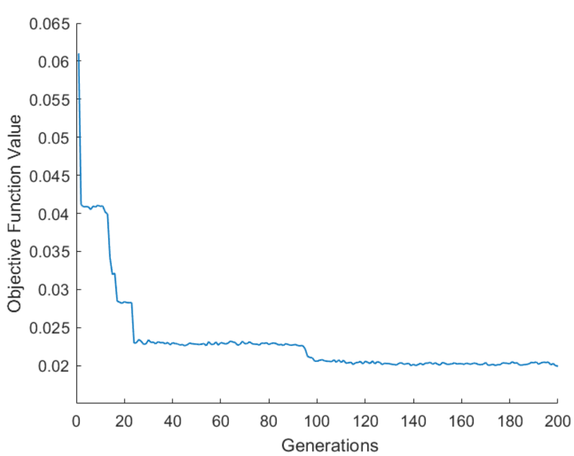
Fig. 2. The process of finding the optimal solution through GA.

A. Robustness achieved by MCM

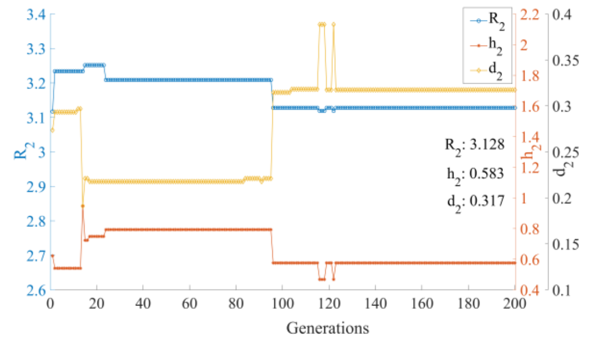
Let $\boldsymbol{\eta} = \{\eta_{R_2}, \eta_{h_2}, \eta_{d_2}\} = \{0.01, 0.007, 0.003\}$ be the set of variable step values given by Table 2. 100 points are randomly sampled in the range of $\overline{\boldsymbol{P}} \pm \boldsymbol{\eta}$ and denoted as $\boldsymbol{P}_1, \boldsymbol{P}_2, \ldots, \boldsymbol{P}_{100}$. Assume the notation $\hat{f}_M(\boldsymbol{P})$ for the mean of objective function value of the selected 100 points. Its expression is given by (12):

$$\hat{f}_M(\boldsymbol{P}) = \frac{1}{100}\sum_{i=1}^{100} f(\boldsymbol{P}_i). \tag{12}$$

Every individual is treated with above process in the GA. And the result of optimization is shown in Fig. 3. The optimal solution is $\boldsymbol{P}$ = {3.128, 0.583, 0.317}.



(a)



(b)

Fig. 3. The result of optimization with robustness considered by MCM: (a) objective value and (b) parameters' value.

B. Robustness achieved by SCM

The cubic polynomials are selected in SCM to diminish the deviation of the interval. The three-dimensional form of the Lagrange interpolation formula is given by (13):

$$f(x,y,z) \approx y(x,y,z) = \sum_{i=1}^{3}\sum_{j=1}^{3}\sum_{k=1}^{3} f(x_i, y_j, z_k) l_k(z) l_j(y) l_i(x). \tag{13}$$

As for the TEAM 22, the interpolation formulation is formed as:

$$y(\boldsymbol{P}) = \sum_{i=1}^{3}\sum_{j=1}^{3}\sum_{k=1}^{3} f(R_{2(i)}, h_{2(j)}, d_{2(k)}) l_k(z) l_j(y) l_i(x), \tag{14}$$

where

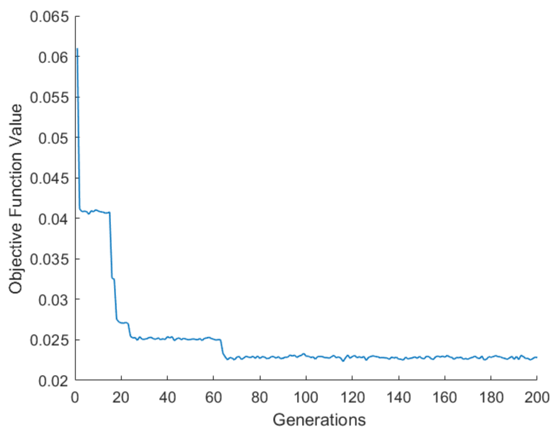
$$\begin{cases} R_{2(i)} = \overline{R_2} + \eta_{R_2} \cdot x_i \\ h_{2(j)} = \overline{h_2} + \eta_{h_2} \cdot y_j \\ d_{2(k)} = \overline{d_2} + \eta_{d_2} \cdot z_k \end{cases}, \tag{15}$$

with x, y, z random numbers between [-1,1]. The interpolating points are $\{I_1, I_2, I_3\} = \left\{-\frac{\sqrt{15}}{5}, 0, \frac{\sqrt{15}}{5}\right\}$ where I can be x, y or z, and the tensor product form is $\left\{-\frac{\sqrt{15}}{5}, 0, \frac{\sqrt{15}}{5}\right\} \otimes \left\{-\frac{\sqrt{15}}{5}, 0, \frac{\sqrt{15}}{5}\right\}$. Thus, equation (14) can provide the answer.

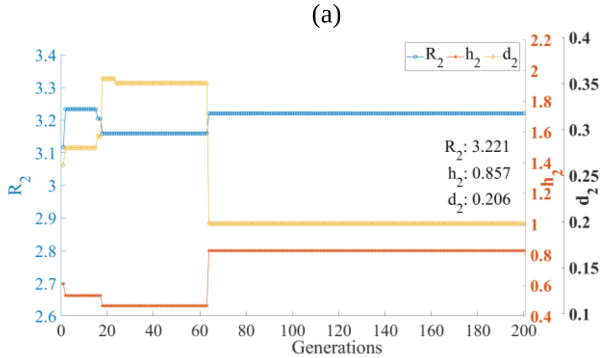
Accordingly, 100 values of x, y and z are each randomly taken and substituted into (14) to obtain the corresponding objective function value .The notation $\hat{f}_S(\boldsymbol{P})$ is defined as the mean of the 100 obtained values to represent the objective function value of $\boldsymbol{P}$ given by (16):

$$\hat{f}_s(\boldsymbol{P}) = \frac{1}{100}\sum_{i=1}^{100} y(\boldsymbol{P}_i)\,. \tag{16}$$

The result of optimization is shown in Fig. 4. The optimal solution is $\boldsymbol{P}$ = {3.221,0.857,0.206}.



(a)



(b)

Fig. 4. The result of optimization with robustness considered by SCM: (a) objective value and (b) parameters' value.

C. Results and discussion

The rate of change in function value in the robust interval $\boldsymbol{D} = 0.6\boldsymbol{\eta}$ is defined as:

$$\delta = \frac{std(f)}{mean(f)},$$

where *std*(*f*) is the standard deviation of objective function *f* in the input parameters' interval while *mean*(*f*) is the mean of *f*.

The "normal" solution that does not consider robustness is also calculated. And it is compared with the solutions obtained above, as outlined in Table 3. Although the "normal" method obtains the smallest objective function value, the robustness of results is much lower than the other two methods.

The δ value that does not consider robustness is the largest when subjected to uncertainty perturbations, suggesting that the EM device considering robustness is more stable. In contrast, MCM and SCM reaches the same level of robustness. At the same time, it is noted that the time required by the optimal solution considering robustness is greatly increased. But the efficiency of achieving robustness by using SCM is improved by nearly 72.06%. Thus the SCM is an effective technique in terms of accuracy and computational efficiency.

For the SCM method, the order of the interpolating polynomials is set to be cubic in this paper. The cubic polynomials can lead to convergence considering both computational efficiency and accuracy according to [19]. To further justify this setting, several orders of interpolating polynomial are applied in the SCM based optimization. The results are compared with that of MCM, as shown in Fig. 5. The coincidence degree between probability distributions of MCM and SCM for different orders of interpolation is calculated, as outlined in Table 4, which means the accuracy of SCM is improved with the rise of order. Meanwhile, the consumed time Tc increases in the form of formula (17) with the rise of interpolating order where n stands for the number of order and p for the number of parameters (p is set to 3 in our case). If the time taken for two-point fitting is assumed for one unit, the relative times consumed by other orders are outlined in Table 4:

$$T_c = n^p\,. \tag{17}$$

It is noted that the accuracy of SCM decreases when interpolating order is higher than five. It is attributed to oscillation at the edges of an interval when using polynomial interpolation of high degree over a set of equispaced interpolation points, which is known as Runge's phenomenon. Therefore, the cubic order is selected to balance accuracy in the approximation of the objective function against Runge's phenomenon and computational requirements.

Table 3: Comparison of robust optimization results obtained by different methods

Method	*R*[m]	*h*[m]	*d*[m]	δ	Consumed Time[s]
Normal	3.127	0.548	0.336	7.58%	40078.79
MCM	3.128	0.583	0.317	6.85%	4709168.56
SCM	3.221	0.857	0.206	5.04%	1315861.82

Table 4: Comparison of robust optimization results

Orders of interpolation	2	3	4	5
Coincidence degree of probability distributions (taking the probability distribution of MCM as reference)	80.16%	90.41%	93.98%	95.46%
T_c (taking the time of two-point interpolation as reference)	1	3.375	8	15.625

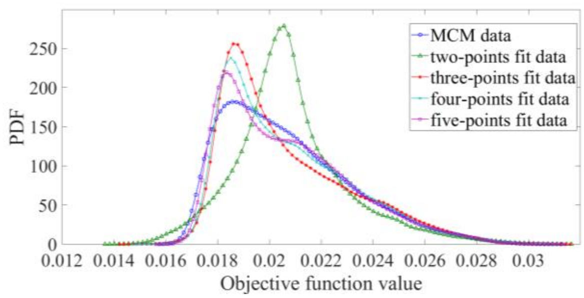


Fig. 5. Comparison of accuracy for different orders of interpolating polynomial.

It is indicated by (13) that the calculation process of SCM is related to the number of variables. It may take much more time to obtain a robust solution when the number of variables increases, which is known as the 'curse of dimensionality' problem. Therefore, SCM needs improvements and can be considered in combination with dimension reduced method [24] to solve the problem of multivariate uncertainty, which is an interesting issue to be investigated in future.

V. CONCLUSION

The uncertainties which exist in design variables and design process are taken into account in the optimization of EM devices. The SCM is applied to the robustness optimization by combing the GA algorithm, which exhibits comparable calculation accuracy to MCM with much less consumed time. For the TEAM 22 optimization, the computational efficiency is improved by nearly 72.06% using SCM. As a compromise between accuracy and efficiency, the order of interpolating polynomials for the SCM method is set to cubic. Meanwhile, the 'curse of dimensionality' problem of SCM caused by multivariate uncertainty still needs further investigation.

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