ANALYSIS OF THREE DIMENSIONAL DIELECTRIC LOADED CAVITIES WITH EDGE ELEMENTS

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ABSTRACT

In this paper we show that edge elements (a class of mixed finite elements) provide an efficient numerical approach in the determination of resonant modes in three dimensional high frequency cavities. These finite elements avoid "spurious modes", the non-physical numerical fields obtained from the solution of eigenvalue problems.

Here, empty cavities as well as dielectric loaded cavities are analyzed: no "spurious mode" was observed. Moreover, comparisons with analytical results and previously published ones show the great accuracy of the numerical technique.

INTRODUCTION

Electromagnetic resonance is important in the design of particle accelerators, microwave ovens and resonant cavities. For such analysis, numerical techniques including the finite element method have been developed.

The well known finite element method seems very attractive since for several years it has been found to be an efficient tool in low frequency electromagnetic field computations. In high frequency applications, finite elements were used for cavity resonances analysis [1]-[5]; resonant modes and resonant frequencies are obtained as solutions from an eigenvalue problem.

The main serious drawback in these studies is that the computed solutions are plagued by non-physical (or "spurious") solutions: solutions which do not satisfy the divergence free condition implied by the Maxwell's equations. Many attempts were performed to circumvent these unwanted numerical fields (enforcing the divergence condition with a global penalty term [1],[5], reducing the number of unknowns by taking locally into account the divergence condition [3] , using divergence free trial functions ...). It has been observed that discretized fields with continuous tangential components suppress the "spurious modes" (the problem was solved for a scalar function in two dimensions [5] and for the magnetic vector potential [4] in three dimensions); nevertheless no precise argument was put forward to explain the importance of this kind of approximation.

However the choice of finite elements for electromagnetic field computations is essential: A. Bossavit showed that "edge elements" are well adapted for the representation of vector fields since they allow their possible discontinuities [6]. These "edge elements" are also a class of the mixed finite element proposed by J.C. Nedelec [7]. Such elements were successfully used in eddy currents problems [8]-[10] and are well adapted for the approximation of scattering and resonance problems [11] [12] ; the reason for which they would not generate "spurious modes" is explained in [12].

We have developed and applied such a numerical approach for empty and dielectric loaded cavities. In this paper, we present first the variational formulation of the Maxwell's equations in terms of electric field and the reason of the occurence of "spurious modes". Then we detail the numerical discretization and explain the interest of "edge elements". Finally we present the analysis of three dimensional cavities .

VARIATIONAL FORMULATION

We deal with the Maxwell's time-harmonic equations in a bounded region Ω surrounded by a perfect conductor and containing lossless materials :

$$\text{rot } e = - i\omega \, \mu_0 h \quad (1)$$
$$\text{rot } h = i\omega \, \epsilon_0 \epsilon_r \, e \quad (2)$$

where e and h are the complex electric and magnetic fields, ϵ_0 and μ_0 are respectively the permittivity and permeability of vacuum, ϵ_r is the relative permittivity and ω is the angular frequency.

The conditions on the boundary Γ of Ω are those of a perfect conductor:

$$n \wedge e = 0 \quad (3) \qquad\qquad n \cdot h = 0 \quad (4)$$

where n is the outward normal vector.

Substituting (1) into (2), we deduce the following eigenvalue problem expressed in terms of the electric field

$$\begin{cases} \text{rot} (\text{rot } e) - \epsilon_r \epsilon_0 \mu_0 \omega^2 e = 0 & \text{in } \Omega \quad (5) \\ n \wedge e = 0 & \text{on } \Gamma \quad (6) \end{cases}$$

A weak formulation of (5)-(6) holds in E_0 [11]:

$$\int_\Omega \text{rot } e.\text{rot } e' \, d\Omega - k^2 \int_\Omega \epsilon_r \, e.e' \, d\Omega = 0 \quad \forall e' \in E_0 \quad (7)$$

where

$$E_0 = \left\{ e \in [L^2 (\Omega)]^3, \text{rot } e \in [L^2 (\Omega)]^3, n \wedge e \, |_\Gamma = 0 \right\}$$

and where the values of k ($k^2 = \omega^2 \epsilon_0 \mu_0$) are the wavenumbers.

SPURIOUS MODES

The searched resonant fields ($\omega > 0$) are theoretically divergence free since from (5) it follows:

$$\text{div } \epsilon_r e = \frac{1}{k^2} \text{div (rot (rot e)) } = 0 \quad (8)$$

Moreover, for a simply- connected region Ω , the only field corresponding to ω=0 (static field) satisfying equations (1),(3) and the condition div ϵ_re=0 is e=0.

The trouble arises when discretizing (7) with classical finite elements (for example nodal vector elements) : a matrix with many eigenvalues being zero is obtained (0 is a highly degenerate eigenvalue) [3] [5]. The numerical approximations of this value k^2=0 are difficult to isolate from the meaningful lowest non-zero eigenvalues ($k^2 > 0$); especially when the number of degrees of freedom increases. Most of them do not satisfy div ϵ_re=0 and then are unacceptable as solutions of Maxwell's equations. The resulting set of solutions is a mixture of physical modes and numerical spurious ones.

With edge elements, as we shall see in the next section such a situation doesn't occur .

FINITE ELEMENT DISCRETIZATION

Mixed finite elements [7] are used for the numerical approximation of (7). Let Ω_h (with boundary Γ_h) be the discretization of Ω with tetrahedra. The "edge elements" have the following properties:

The degrees of freedom e_a and the trial functions w_a are associated with the mesh edges. For every edge "a" containing the nodes "i" and "j" :

. e_a is the circulation of e along "a".

$$e_a = \int_a e.t \, d\gamma \qquad (9)$$

. w_a can be expressed in terms of the barycentric functions λ_i and λ_j as :

$$w_a = \lambda_i \, \mathrm{grad}\lambda_j - \lambda_j \, \mathrm{grad}\lambda_i \qquad (10)$$

In each tetrahedron e is :

$$e(r) = \alpha + \beta \wedge r \qquad (11)$$

where α and β are three-components constant vectors and r is the vector (x,y,z).

We introduce the space E_{0h} :

$$E_{0h} = \left\{ e \ , \ e = \sum e_a \, w_a \ , \ n \wedge e \ |_\Gamma = 0 \right\} \qquad (12)$$

For every e in E_{0h} the tangential part of e is continuous across tetrahedra interfaces. The approached problem is to find e in E_{0h} so that :

$$\int_{\Omega_h} \mathrm{rot}\, e.\mathrm{rot}\, e' d\Omega_h - k^2 \int_{\Omega_h} \epsilon_r \, e.e' d\Omega_h = 0 \quad \forall e' \in E_{0h} \quad (13)$$

Finally, we have to solve a generalized algebraic eigenvalue problem of the form :

$$A\, u = k^2 \ B\, u \qquad (14)$$

A ("stiffness matrix") and B ("mass matrix") have dimensions $n_a \times n_a$ where n_a is the number of edges in the finite element mesh.

Remark: with "edge elements" all the numerical solutions corresponding to $k^2 > 0$ are "weakly divergence free" : no spurious mode occur. The reason is the following :

Let φ be any linear combination of the barycentric functions λ_i (φ piecewise affine) and $\varphi = 0$ on Γ; it can be showed that all the fields e' of the form e'= grad φ are in E_{0h} [11] [12]. Then they can be chosen as admissible test fields in (13). Rewriting e'= gradφ in (13) leads to :

$$k^2 \int_{\Omega_h} \epsilon_r e. \ \mathrm{grad}\varphi \ d\Omega_h \quad = \quad 0 \qquad (15)$$

Equation (15) is equivalent to say that for every $k^2 > 0$ the solution div $\epsilon_r e = 0$ is verified in the distribution sense. An integration by parts of (15) shows that for every inner node "i" of Ω_h, an integral involving the sum of jumps of ϵ_r e.n through facets of tetrahedra having node "i" in common is zero. This property is important since it gives an average local divergence condition. Such result does not exist in case of classical finite elements (nodal ones) since the fields of the form $e' = \mathrm{grad}\varphi$ are not in the space of the test functions E_{0h} .

NUMERICAL RESULTS

a. Empty rectangular cavity.

An empty rectangular cavity with perfectly conducting walls was modelled with the above developed technique. The cavity has dimensions: a = 0.4 m , b = 0.3 m , c = 1. m (figure 1-a). A quarter of the cavity (figure 1-b) was analyzed using 220 tetrahedra. Symmetry conditions were prescribed on faces x = 0 and z = 0 .

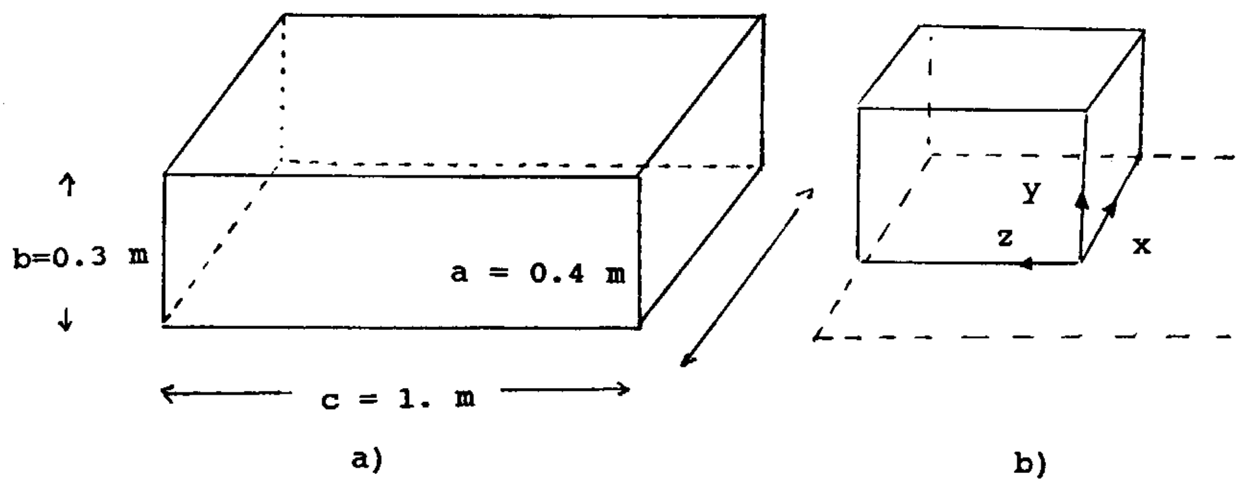


Figure 1 Studied air-filled cavity

The problem is symmetrical in x, y, z; so the fields can be expressed as TE (transverse electric) or TM (transverse magnetic) to any one of these coordinates [13]. It is conventional to choose the longer dimension along the z direction. Analytical solutions are then labelled TE_{mnp} (modes whose electric field has no z-component) and TM_{mnp} (modes whose magnetic field has no z-component). The corresponding resonant wave numbers are :

$$k^2 = \pi^2 \left(\frac{m^2}{a^2} + \frac{n^2}{b^2} + \frac{p^2}{c^2} \right)$$

The algorithm for solving the matricial eigenvalue problem is based on the classical QR method.

Computations were performed on a DN4000 Apollo workstation; about 10 minutes are necessary to solve the entire problem. The six lowest computed modes are shown on Table 1.

Mode	Wavenumber k computed	Wavenumber k analytical	Error (%)
TE_{101}	8.365	8.458	1.
TE_{111} (TM_{111})	13.243	13.461	1.6
	13.488	13.461	0.2
TE_{103}	12.327	12.268	0.5
TE_{113} (TM_{311})	16.273	16.129	0.9
	16.356	16.129	1.4

Table 1
Numerical Results

This simple case is known to give spurious solutions when solved with classical finite elements [2] [14]. Here no spurious mode is observed. Moreover the relative error never exceeds 2%.

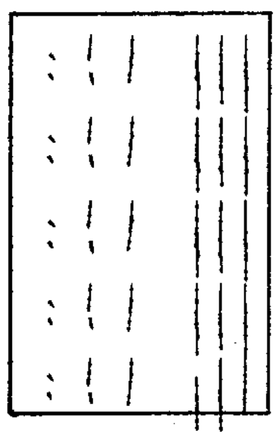


Figure 2
Electric field (TE_{101} mode in (x,y) plane for z=0.5 m)

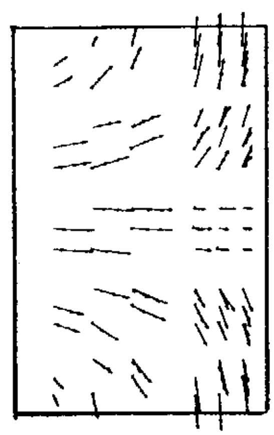


Figure 3
Electric field (TE_{111} mode in (x,y) plane for z=0.5 m)

Figures 2 and 3 show, in the quarter of the cavity, the distribution of vector fields for the TE_{101} mode and the TE_{111} mode respectively. The plane of symmetry is on the right of the figures. On this plane e is tangential and on the others e is normal. In each tetrahedron e is represented with an arrow in the centre of gravity and the length of the arrow is proportional to the magnitude.

b. Inhomogeneous dielectric loaded cavities

Two examples of inhomogeneous loaded cavities were analyzed.

The first one (figure 4) is the preceding cavity with dielectric discontinuities in one direction only. The relative permittivity ϵ_r of the dielectric material is $\epsilon_r = 16$. For a quarter of the cavity 230 tetrahedra were used. The theoretical eigenvalue for the dominant mode (lowest eigenvalue) is known [15] : k = 2.5829. An error of 0.4% was found.

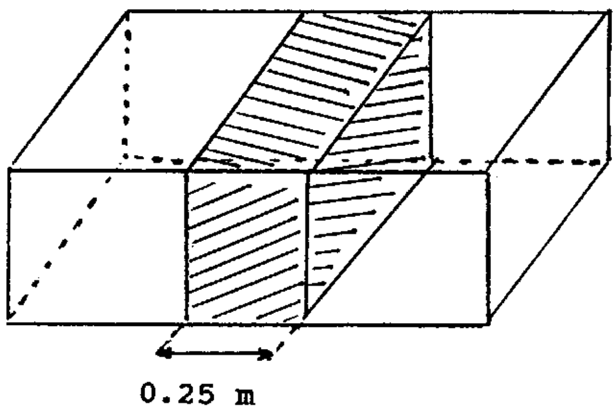
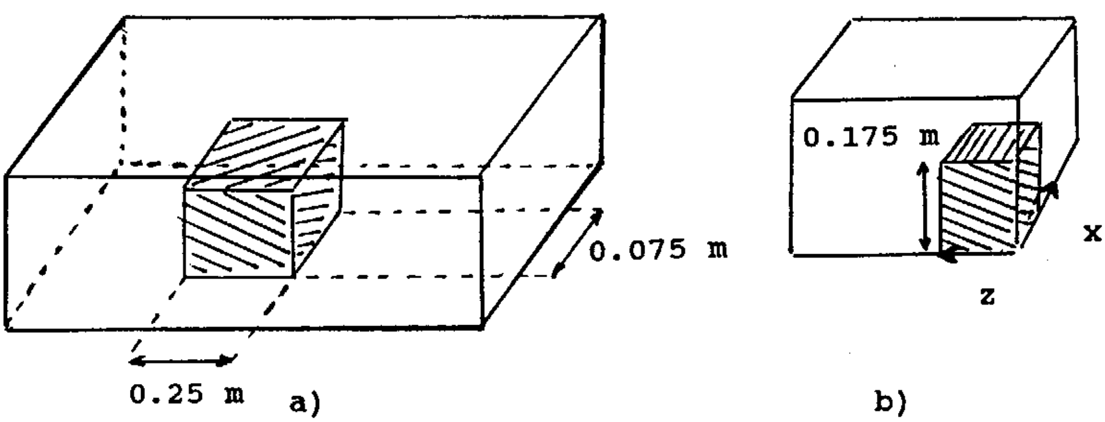
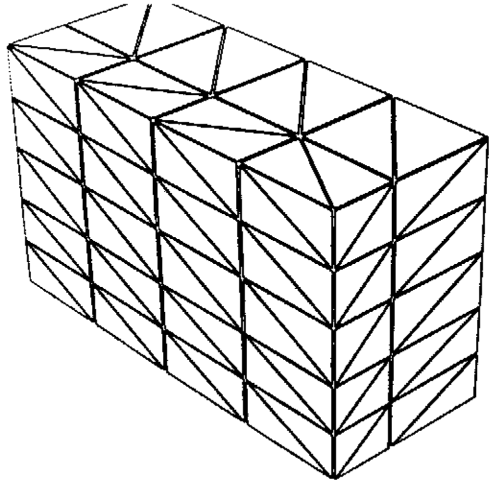


Figure 4
Cavity with dielectric block (example 1)

The second cavity (figure 5-a) is the cavity of a. with dielectric discontinuities in three dimensions ($\epsilon_r = 16$). The quarter (figure 5-b) was modelled with 240 tetrahedra (figure 5-c). Computing time is about 15 minutes.





c)

Figure 5

Cavity with dielectric block (example 2)

No analytical result is available but comparison with already published computed values is possible; results for the dominant mode are the following:

Source	k (computed)
Ref [1]	5.60
Ref [15]	5.529
Ref [16]	4.907
Presented Method	5.102

All these results agree within roughly 10%. Some others structures should be modelled in order to make a comparison more satisfactory between all the methods. However the mixed finite elements used here are well adapted in case of dielectric materials because they imply the tangential continuity of the electric field across interfaces and take account of the discontinuity of the normal component .

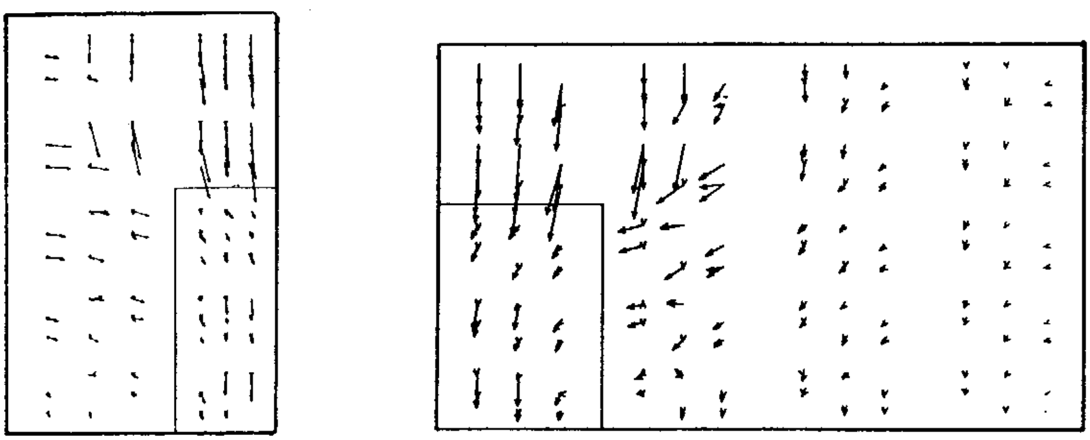


Figure 6

Electric field (dominant mode)

In the (x,y) plane for z=0.5 m

In the (y,z) plane for x=0.2 m

No spurious mode occured; all the computed field correspond to resonant fields. Figure 6 shows electric vector fields in the quarter of the cavity.

CONCLUSION

"Edge elements" (a class of mixed finite element) have been used to model empty and dielectric loaded cavities. The first resonant frequencies were computed; comparison with analytical values or results published in previous papers shows the efficiency of the method.

These elements avoid all the well known "spurious modes" and seem very promising for the study of more complicated problems in high frequency applications

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