Studying and Analysis of a Novel RK-Sinc Scheme

Min Zhu

School of Electronic and Information Engineering
Jingling Institute of Technology, Nanjing, 211169, China
zomi@jit.edu.cn

Abstract **─** In this paper, a novel high-order method, Runge-Kutta Sinc (RK-Sinc), is proposed. The RK-Sinc scheme employs the strong stability preserving Runge-Kutta (SSP-RK) algorithm to substitute time derivative and the Sinc function to replace spatial derivates. The computational efficiency, numerical dispersion and convergence of the RK-Sinc algorithm are addressed. The proposed method presents the better numerical dispersion and the faster convergence rate both in time and space domain. It is found that the computational memory of the RK-Sinc is more than two times of the FDTD for the same stencil size. Compared with the conventional FDTD, the new scheme provides more accuracy and great potential in computational electromagnetic field.

Index Terms **─** Convergence, dispersion, FDTD, Runge-Kutta, Sinc function, stability.

I. INTRODUCTION

The finite-difference time-domain (FDTD) [1] numerical techniques are widely used today for the analysis of various microwave geometries and for the modeling of electromagnetic wave propagation. However, the method has some significant limitations due to the substantial computer resources required when it involves modeling a complicated problem, which has large stencil size at least 10 cells or more per wavelength. The FDTD has a second order accuracy in spatial-temporal and large significant computational errors. In order to improve the limitations of the FDTD method, lots of methods are proposed, including the HO-FDTD [2-4] and MRTD [5] (Multiresolution Time-Domain). The HO-FDTD is presented by Fang firstly in [2], employing the Taylor series instead of the spatial and temporal derivatives to increasing accuracy. The strong stability Runge-Kutta (SSP-RK) method was first introduced and extended in Refs. [6] and [7]. Compactly supported Nth-order wavelets and *m*th-order *m*th-stage Runge-Kutta are applied in spatial discretization and time integration, respectively.

In this paper, we discuss a new method called RK-Sinc, which is considered has the same convergence level for the time and space domains. The remainder of this paper is organized as follows. In Section II, the basic theory and algorithm of the RK-Sinc method is introduced. The stability, dispersion and convergence of the method are studied in Section III. Computational cost and memory requirements are discussed in Section IV. Numerical example is given in Section V. Conclusions are summarized in Section VI.

Table 1: Coefficients $c(v)$ for the RK-Sinc(2, 2v) method (0≤v≤10)

v	***c*(*v*)**
1	1.27323954
2	-0.14147106
3	0.05092958
4	-0.02598448
5	0.01570901
6	-0.01052264
7	0.00753396
8	-0.00565884
9	0.00440567
10	-0.00352697

II. THE RK-SINC METHOD

A. High-order Sinc method

Considering an arbitrary function f, using the Sinc function as the basis function to derive one update equation of the high-order Sinc method as follows:

$$E^{x,n+1}_{i+1/2,j,k} = E^{x,n}_{i+1/2,j,k} + \frac{1}{\varepsilon}\sum_{v=1}^{m} c(v)\cdot\left(H^{z,n+1/2}_{i+1/2,j+m+1/2,k}\frac{\Delta t}{\Delta y} - H^{y,n+1/2}_{i+1/2,j,k+m+1/2}\frac{\Delta t}{\Delta z}\right), \quad (1)$$

where $E^{x,n}_{i+1/2,j,k}$, $E^{x,n}_{i+1/2,j,k}$, $H^{x,n+1/2}_{i,j+m+1/2,k}$, $H^{y,n+1/2}_{i+1/2,j,k+m+1/2}$ are electric field and magnetic field coefficients, m, ε, μ, Δt, Δy and Δz are the spatial stencil size, the permittivity, the permeability, the temporal step size, and the spatial step size in the x-, y- and z-direction, respectively. The coefficients $c(v)$ [8] for different spatial stencil sizes are given in Table 1 for the stencil size m is 10.

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B. RK-Sinc method

For simplicity and without loss of generality, in rectangular coordinate system, using the Sinc function to expand the spatial electric and magnetic field components in Maxwell's equation as described in equation (1) and the SSP-RK method to replace the time derivative of the electric field and magnetic field components in the left part of the equation (2), one update equation of the RK-Sinc(2, 2*v*) which is based on the Sinc function can be written as follows:

$$\frac{\partial E^x_{i+1/2,j,k}(t)}{\partial t}=\frac{1}{\varepsilon}\sum_{v=1}^{m}c(v)\left[\frac{1}{\Delta y}\left(H^z_{i+1/2,j\text{-}1/2+v,k}(t)\text{-}H^z_{i+1/2,j+1/2\text{-}v,k}(t)\right)-\frac{1}{\Delta z}\left(H^y_{i+1/2,j,k\text{-}1/2+v}(t)\text{-}H^y_{i+1/2,j,k+1/2\text{-}v}(t)\right)\right], \tag{2a}$$

$$\frac{\partial E^y_{i,j+1/2,k}(t)}{\partial t}=\frac{1}{\varepsilon}\sum_{v=1}^{m}c(v)\left[\frac{1}{\Delta z}\left(H^x_{i,j+1/2,k\text{-}1/2+v}(t)\text{-}H^x_{i,j+1/2,k+1/2\text{-}v}(t)\right)-\frac{1}{\Delta x}\left(H^z_{i\text{-}1/2+v,j+1/2,k}(t)\text{-}H^z_{i+1/2\text{-}v,j+1/2,k}(t)\right)\right], \tag{2b}$$

$$\frac{\partial E^z_{i,j,k+1/2}(t)}{\partial t}=\frac{1}{\varepsilon}\sum_{v=1}^{m}c(v)\left[\frac{1}{\Delta x}\left(H^y_{i\text{-}1/2+v,j,k+1/2}(t)\text{-}H^y_{i+1/2\text{-}v,j,k+1/2}(t)\right)\text{-}\frac{1}{\Delta y}\left(H^x_{i,j\text{-}1/2+v,k+1/2}(t)\text{-}H^x_{i,j+1/2\text{-}v,k+1/2}(t)\right)\right], \tag{2c}$$

$$\frac{\partial H^x_{i,j+1/2,k+1/2}(t)}{\partial t}=\frac{1}{\mu}\sum_{v=1}^{m}c(v)\left[\frac{1}{\Delta z}\left(E^y_{i,j+1/2,k+v}(t)\text{-}E^y_{i,j+1/2,k+1\text{-}v}(t)\right)-\frac{1}{\Delta y}\left(E^z_{i,j+v,k+1/2}(t)\text{ - }E^z_{i,j+1\text{-}v,k+1/2}(t)\right)\right], \tag{2d}$$

$$\frac{\partial H^y_{i+1/2,j,k+1/2}(t)}{\partial t}=\frac{1}{\mu}\sum_{v=1}^{m}c(v)\left[\frac{1}{\Delta x}\left(E^z_{i+v,j,k+1/2}(t)\text{-}E^z_{i\text{-}v+1,j,k+1/2}(t)\right)\text{ - }\frac{1}{\Delta z}\left(E^x_{i+1/2,j,k+v}(t)\text{-}E^x_{i+1/2,j,k\text{-}v+1}(t)\right)\right], \tag{2e}$$

$$\frac{\partial H^z_{i+1/2,j+1/2,k}(t)}{\partial t}=\frac{1}{\mu}\sum_{v=1}^{m}c(v)\left[\frac{1}{\Delta y}\left(E^x_{i+1/2,j+v,k}(t)\text{-}E^x_{i+1/2,j\text{-}v+1,k}(t)\right)-\frac{1}{\Delta x}\left(E^y_{i+v,j+1/2,k}(t)\text{ - }E^y_{i\text{-}v+1,j,k+1/2}(t)\right)\right]. \tag{2f}$$

III. NUMERICAL PROPERTIES

In this section, the stability, dispersion and convergence of the RK-Sinc method are discussed.

A. Stability

Refer to the Yee's FDTD stability relation [1] and using the Fourier transforms to the Eq. (2) [9], and considering uniform spatial step size ($\Delta x = \Delta y = \Delta z = \Delta$), the general form of the stability condition for the Sinc (2, 2*v*) can be derived as:

$$\Delta t \le \frac{\Delta}{\mathrm{c}(\sum_{v=1}^{m}|c(v)|)\sqrt{d}}, \tag{3}$$

where c is the speed of light in free space, *d* is the spatial dimension.

According to [10], the stability of the RK-Sinc method can be derived as:

$$\Delta t \le \frac{L\Delta}{\mathrm{c}(2\sum_{v=1}^{m}|c(v)|)\sqrt{d}}, \tag{4}$$

$$s=\frac{\mathrm{c}\Delta t}{\Delta}\le\frac{L}{\sqrt{d}\,2\sum_{v=1}^{m}|c(v)|}=s_{\max}, \tag{5}$$

where *L* is a constant dependent on the order *p* of SSP-RK, and can be derived as $\sqrt{3}$ and $2\sqrt{2}$ [6],[11] for the RK_3-Sinc and RK_4-Sinc, respectively.

The maximum stability factor $S_{\max}$ is listed in Table 2. It is show that the RK_4-Sinc is stricter than the RK_3-Sinc.

Table 2: The maximum CFL number for the RK-Sinc method

	RK_3-Sinc	**RK_4-Sinc**
$S_{\max}$	0.3317	0.5353

B. Dispersion

Defining the stability factor $s = (\mathrm{c}\Delta t)/\Delta = 0.25$, the number of cells per wavelength $N = \lambda_c/\Delta$, *u* is the ratio of the theoretic wavelength λ_c to the numerical wavelength λ_n, and V_n is the numerical phase velocity. With the wave propagation angle $\theta = 90°$ and $\Phi = 0°$, Figs. 1-3 show the dispersion errors V_n /c versus *N* for the FDTD and RK-Sinc methods in 3D case.

Figure 1 shows that the RK-Sinc method presented the better dispersion characteristics at the same spatial stencil size. Figure 2 shows the dispersion errors versus wave propagation angle Φ for different methods with $\theta = 90°$ and $\Delta x = \Delta y = \Delta z = \lambda/5$. The FDTD method for different stencil size all presents lower dispersion errors. From Fig. 3, we can see that the RK_4-Sinc is a little fluctuation to that of the RK_3-Sinc method, the RK_4-Sinc has the better dispersion error.

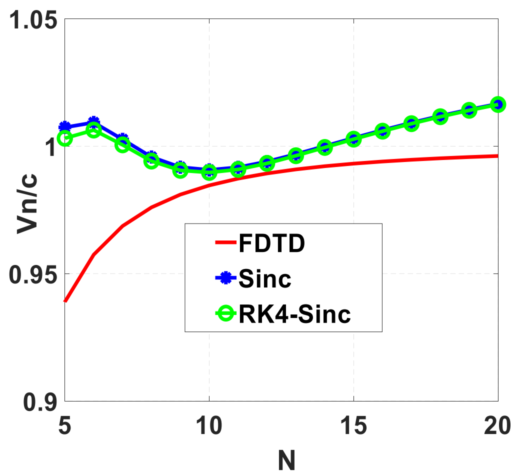

Fig. 1. Dispersion errors of the different methods.

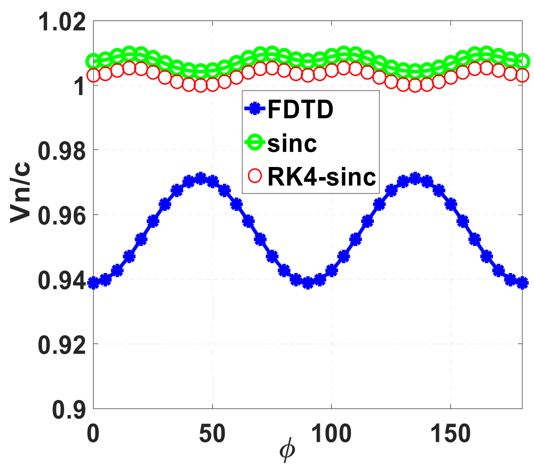

Fig. 2. Dispersion errors versus Φ of the different methods.

C. Convergence

Similar to the RK-MRTD (Runge-Kutta Multiresolution Time-Domain) method in [12], the convergence relation of the RKp-Sinc(2, 2v) can be written as:

$$\text{RK-Sinc-error} \le C_t \Delta t^p + C_x \Delta x^v . \tag{6}$$

When $v > p$ and $\Delta t = s\Delta/c$ ($s \le 1$), it can be simplified as:

$$\text{RK-Sinc-error} \le C_t \Delta t^p + C_v \Delta t^v \le C_p \Delta t^p , \tag{7}$$

where C_t, C_x, C_v and C_p are coefficients.

The convergence properties of the FDTD, RK3-Sinc and RK4-Sinc are described in Fig. 4 and it can be easily found that the order is 2, 2.2 and 3.6 respectively, the δ is uniform spatial step size ($\Delta x = \Delta y = \Delta z = \delta$). The results demonstrate that the RK-Sinc has the faster convergence rate than the FDTD method at the same stencil size.

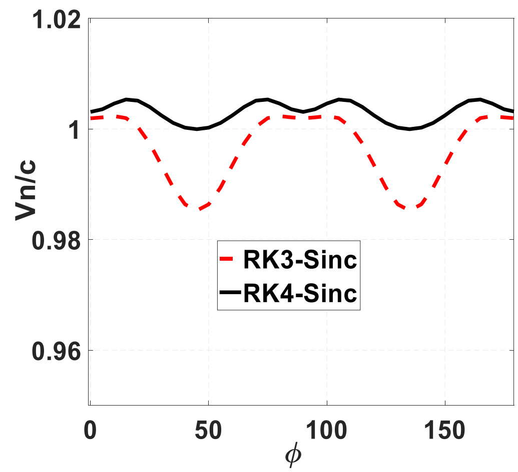

Fig. 3. Dispersion errors versus Φ of the RK-Sinc method.

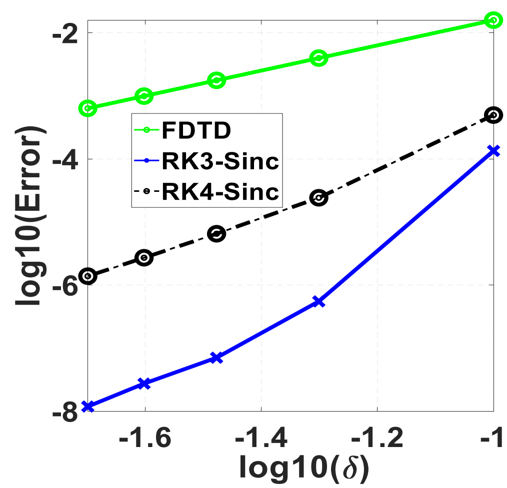

Fig. 4. Convergence of the FDTD, RK_3-Sinc and RK_4-Sinc methods.

D. Computational cost and memory requirements

According to the RK-MRTD method in 3D case [12-14], at each time step for FDTD method only E and H fields at a final time need to be stored, while the memory requirements of the RK-Sinc is more than two times of the FDTD for the same mesh size in [11]. The cost of the FDTD can be written as Cost $_{\text{FDTD}} = 2\,(M_1)^3 \times 2$. It is obviously that the RK_p-Sinc needs largely more than p times computational cost than the FDTD method for the same mesh size.

In 3D case, for a single time step, the computational cost of the high order Sinc(2, 2v) method can be written as follows:

$$\text{Cost}_{\text{Sinc}} = 2\,(M_1)^3 \times 2v, \tag{8}$$

where M_1 is the number of cells in a single direction, 2m is the size of the stencil.

According to [12], the cost of the RK-Sinc can be written as follows:

$$\text{Cost}_{\text{RK}p\text{-Sinc}} = 2\,(M_2)^3 \times p \times (2v+1), \tag{9}$$

where M_2 is the number of cells in a single direction.

If the computational domain is unit size, Δx_1 and Δx_2 are the cell sizes of the Sinc(2, 2v) and RK$_p$-Sinc(2, 2m) methods, then $M_1 = 1/\Delta x_1$, $M_2 = 1/\Delta x_2$; if Δt_1 and Δt_2 are the maximal stable time step for the Sinc(2, 2v) and RK$_p$-Sinc(2, 2v) methods, and Δt_2 chosen as $2\Delta t_1/p$. If the total computational time is 1, then the cost of the two methods as follows:

$$\text{Cost}_{\text{Sinc}} = 2(M_1)^3 \times 2v \times \frac{1}{\Delta t_1}, \tag{10}$$

$$\begin{aligned}\text{Cost}_{\text{RK-Sinc}} &= 2\left(M_2\right)^3 \times p \times \left(2v+1\right) \times \frac{1}{\Delta t_2} \\ &= \left(M_2\right)^3 \times p^2 \times \left(2v+1\right) \times \frac{1}{\Delta t_1}.\end{aligned} \tag{11}$$

IV. NUMERICAL EXAMPLE

In this section, the RK-Sinc is applied to solve a dielectric material slab in one dimension. The thickness of the slab is 8mm, the relative permittivity of the slab is 3.8. The spatial step size is Δx, and the total computational domain is discretized as $\Delta x = \lambda/5$, which λ is the wavelength in the media and corresponds to the concerned maximum frequency f_{max}=10GHz.

The analytical solution is the Mie series solution and the reflection coefficients of the FDTD and RK$_4$-Sinc are shown in Fig. 5. From the Fig. 5, we can see that the RK$_4$-Sinc is the better than FDTD method. Figure 6 describes the errors between the methods and the analytical solution, we can obtain that the RK$_4$-Sinc has the better dispersion error than FDTD Scheme.

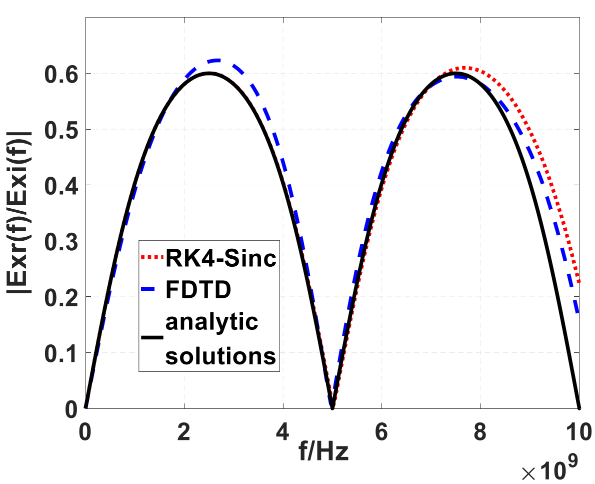

Fig. 5. Magnitude of reflection coefficients for RK$_4$-Sinc FDTD method and analytic solution.

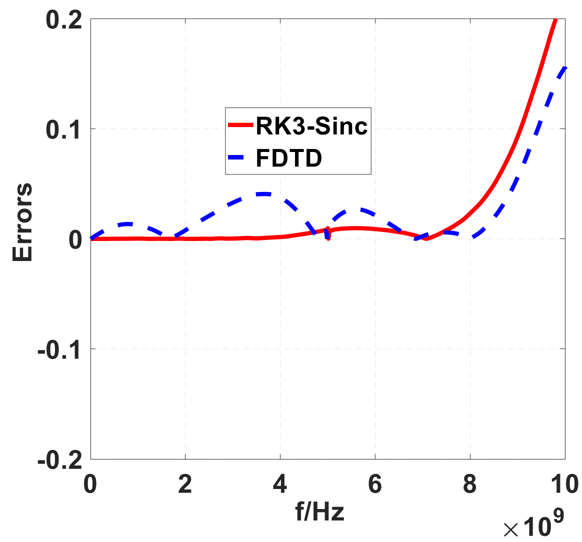

Fig. 6. Errors of the FDTD and RK$_4$-Sinc method.

VI. CONCLUSION

In this paper, a novel method RK-Sinc based on the SSP-RK and the Sinc function has been presented. The characteristic including stability, dispersion and convergence is analyzed and discussed. It can be easily found that the RK-Sinc method has the better dispersion and the faster convergence rate than FDTD method with the same spatial and temporal stencil size. The computational cost and memory requirements are studied and show that the RK$_p$-Sinc needs largely more memory requirements and computational cost of the FDTD for the same mesh size. The dielectric material slab example shows that the RK-Sinc is more accuracy than FDTD method. Therefore, the RK-Sinc scheme can reduce the numerical dispersion and has a fast convergence rata, we can draw a conclusion that the new method is more accurate and efficient, and has well potential applications in some certain computational electromagnetic filed.

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