Combining the CORS and BiCORSTAB Iterative Methods with MLFMA and SAI Preconditioning for Solving Large Linear Systems in Electromagnetics

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***Abstract*—** We report on experiments with a novel family of Krylov subspace methods for solving dense, complex, non-Hermitian systems of linear equations arising from the Galerkin discretization of surface integral equation models in Electromagnetics. By some experiments on realistic radar-cross-section calculation, we illustrate the numerical efficiency of the proposed class of algorithms also against other popular iterative techniques in use today.

***Index Terms*—** Krylov subspace methods, Lanczos biconjugate A-orthonormalization methods, multilevel fast multipole method, scattering problems, sparse approximate inverse preconditioning.

I. INTRODUCTION

Mathematical models based on surface integral equations are becoming very popular in computational electromagnetics. They require a simple description of the surface of the target by means of triangular facets, thus simplifying considerably the mesh generation especially in the case of moving objects. Upon discretization they typically yield smaller systems to solve in comparison with finite difference or finite element techniques applied to the same problem [1]. The potential drawback of using integral methods is that they require to solve large dense complex systems of linear equations. Therefore, robust matrix solvers are urgently needed for this problem class [2].

The Maxwell's equations can be reformulated as a set of integral equations defined in the frequency domain as the following variational problem:

Find the surface current j such that for all tangential test functions j^t, we have

$$\begin{aligned} &\iint_\Gamma G(x,y)\left(j(x)\cdot j^t(y)\right)dxdy \\ -\frac{1}{k^2}&\iint_\Gamma G(x,y)\left(div_\Gamma j(x)\cdot div_\Gamma j^t(y)\right)dxdy \\ &= \frac{i}{kZ_0}\int_\Gamma E_{inc}(x)\cdot j^t(x)dx. \end{aligned} \quad (1)$$

We denote by $G(x,y) = \dfrac{e^{ik|y-x|}}{4\pi|y-x|}$ the Green's function of Helmholtz equation, Γ the boundary of the object, k the wave number and $Z_0 = \sqrt{\mu_0/\varepsilon_0}$ the characteristic impedance of vacuum (ε_0 is the electric permittivity and μ_0 the magnetic permeability), and $\mathrm{div} j(x)$ is the divergence operator of a continuously differentiable vector field $j(x)$ defined on a 3D Euclidean space. Equation (1) expresses the electric currents in terms of the electric field and is known as electric field integral equation (EFIE). It is applied to model arbitrary geometries like objects with cavities, disconnected parts, breaks

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on the surface [3,4]. For its generality, the EFIE model is very popular in industrial environment. However, it is tough to solve by iterative methods, compared to other surface integral formulations of electromagnetic scattering problems.

On discretizing Equation (1) in space by the MoM over a mesh containing n edges, the surface current j is expanded into a set of basis functions $\{\varphi_i\}_{1\leq i\leq n}$ with compact support (the Rao-Wilton-Glisson basis [5] is a popular choice), then the integral equation is applied to a set of tangential test functions j^t. Selecting $j^t = \varphi_j$, we are led to compute the set of coefficients $\{\lambda_i\}_{1\leq i\leq n}$ such that

$$\sum_{i=1}^{n} \lambda_i \iint_\Gamma G(x,y)\,(\varphi_i(x)\cdot\varphi_j(y))dxdy$$
$$-\frac{1}{k^2}\sum_{i=1}^{n} \lambda_i \iint_\Gamma G(x,y)\,(div_\Gamma\varphi_i(x)\cdot div_\Gamma\varphi_j(y))dxdy$$
$$= \frac{i}{kZ_0}\int_\Gamma E_{inc}(x)\cdot\varphi_j(x)dx, \tag{2}$$

for each $1 \leq i \leq n$. The set of equations (2) can be recast in matrix form as

$$A\lambda = b, \tag{3}$$

where $A = [A_{ij}]$ and $b = [b_i]$ have elements

$$A_{ij} = \iint_\Gamma G(x,y)\,(\varphi_i(x)\cdot\varphi_j(y))dxdy$$
$$-\frac{1}{k^2}\iint_\Gamma G(x,y)\,(div_\Gamma\varphi_i(x)\cdot div_\Gamma\varphi_j(y))dxdy, \tag{4}$$

$$b_j = \frac{i}{kZ_0}\int_\Gamma E_{inc}(x)\cdot\varphi_j(x)dx. \tag{5}$$

In Equation (3), the set of unknowns are associated with the vectorial flux across an edge in the mesh. The coefficient matrix A generated by MoM is dense complex non-Hermitian; hence the pertinent linear system cannot be solved using the conjugate gradient (CG) algorithm. The restarted generalized minimal residual (GMRES) method, its flexible variant FGMRES, and some of the short-recurrence methods such as BiCG, BiCGStab, and TFQMR are popular options, see e.g. [6,7].

In this study, we illustrate experiments with two recently developed algorithms: the conjugate A-orthogonal residual squared (CORS) and the biconjugate A-orthogonal residual stabilized (BiCORSTAB) methods for non-Hermitian linear systems, sketched in Algorithms 1-2. They compute the approximate solution x_m that belongs to the Krylov subspace $x_0 + K_m(A; v_1)$ by projecting the residual orthogonally to the constraints subspace $\mathscr{L}_m \equiv A^H K_m(A^H; w_1)$. Throughout this paper, we denote by the superscript H the Hermitian (conjugate transpose) of a vector or a matrix and the standard inner product of two complex vectors $u, v \in \mathbb{C}^n$ by

$$\langle u, v\rangle = u^H v = \sum_{i=1}^{n} \bar{u}_i v_i.$$

For the sake of conciseness, we point the reader to [8,9] for a thorough mathematical derivation of the BiCORSTAB and CORS methods.

II. NUMERICAL EXPERIMENTS

For the numerical experiments, we consider some selected scattering problems described in Table 1. We report the number of iterations required by several Krylov methods (listed in Table 2) to reduce the initial residual by five orders of magnitude, starting from the zero vector. The sequential tests are compiled with the Portland Group Fortran 90 compiler (version 9) and run on a cluster of nodes equipped with quad core Intel CPU (2.8 GHz) and 16 GB of physical RAM.

In our sequential experiments, the CORS method was the most effective non-Hermitian solver with respect to CPU time, as it is shown in Table 3. Unrestarted GMRES may outperform all other Krylov methods and should be used when memory is not a concern. We selected a value of 50 for the restart parameter in the GMRES method in our runs on small problems, reported in Table 3, and a value of 100 in the runs on large problems, reported in Tables 4-5. In Figure 1, we illustrate the convergence history of CORS and GMRES(50) on Examples 3 to show the different numerical behaviors of the two families of solvers. The residual reduction is much smoother for GMRES along the iterations. The BiCORSTAB method also

Table 1: Characteristics of the model problems

Example	Description	Size	Frequency (MHz)
1	Open cylinder	6268	362
2	Sphere	12000	535
3	Satellite	1699	57
4	Cavity	727120	300
5	Paraboloid	857862	300

Table 2: List of solvers used and relative cost. We denote by n the problem size, by i the iteration number and by m the restart value in GMRES

Solver	Products by A/A^H	Memory
CORS	2/0	matrix+$14n$
BiCORSTAB	2/0	matrix+$13n$
GMRES	1/0	matrix+$(m+3)n$
QMR	2/1	matrix+$11n$
TFQMR	4/0	matrix+$10n$
BiCGSTAB	2/0	matrix+$7n$

shows fast convergence and may be an appropriate choice. Both CORS and BiCORSTAB are based on short-term recurrences and therefore, they are very cheap in memory (see Table 2).

Finally, methods based on Lanczos biconjugation are also considered in many scattering analysis, due to their simplicity (they are parameter-free) and low memory requirements, see e.g. [10–12]. In our experiments, as shown in Table 3, BiCGSTAB and QMR-like methods are less efficient than CORS.

A. MLFMA and SAI

A straightforward implementation of Krylov methods requires $\mathcal{O}(n^2)$ memory storage, where n is the number of unknowns, to compute a solution for one excitation. The solution cost may be reduced to $\mathcal{O}(n \log n)$ algorithmic and memory complexity using the multilevel fast multipole algorithm (MLFMA) for the M-V operation. Recent progress in the developments of parallel multipole codes, provably scalable to several million discretization points, are urging the quest of robust iterative algorithms for this problem class [13,14]. In this study, we solved the two largest problems *i.e.* Examples 4 and 5, using MLFMA and a sparse approximate inverse (SAI) preconditioner. The SAI preconditioner was computed by minimizing the Frobenius-norm of

Table 3: Number of iterations and CPU time (in seconds) required by Krylov methods to reduce the initial residual to $\mathcal{O}(10^{-5})$. For each example, asterisk "*" indicates the fastest run

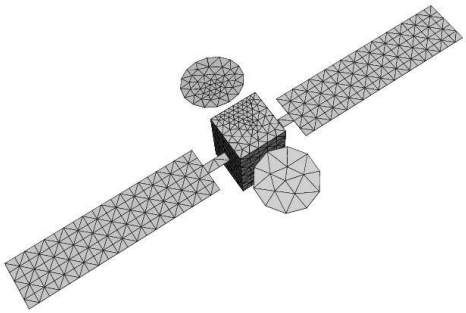
Solver/Example	1	2	3
CORS	601 (253*)	294 (451*)	371 (11*)
BiCORSTAB	941 (614)	423 (1099)	775 (37)
GMRES(50)	2191 (469)	1803 (1397)	871 (17)
QMR	878 (548)	430 (1045)	452 (24)
TFQMR	482 (398)	281 (863)	373 (27)
BiCGSTAB	1065 (444)	680 (1031)	566 (18)

the error matrix

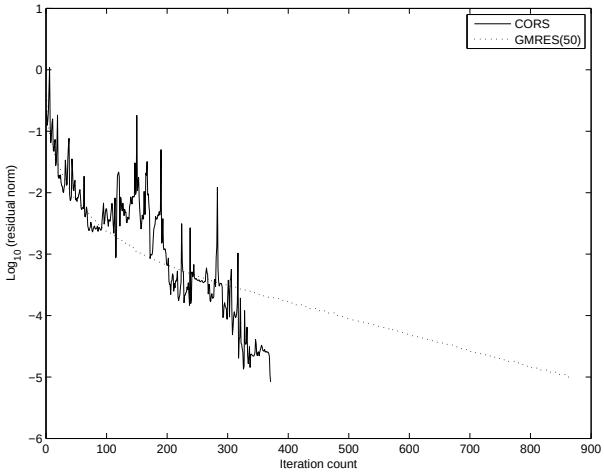
$$\min_{M \in S} \left\| I - M\widehat{A} \right\|_F ,$$

where S is the set of matrices with a given sparsity pattern. We chose $\widehat{A}$ to be sparse and equal to the multipole matrix, and we selected the sparsity pattern of M equal to the nonzero structure of $\widehat{A}$. Details of the SAI preconditioner and of the highly efficient parallel implementation of MLFMA that we used in this study are found in [15] and in [16], respectively.

The first model is a cavity of size $10\lambda \times 10\lambda \times 50\lambda$, discretized with 727,120 nodes and illuminated at an incident angle $(\theta, \phi) = (45°, 0°)$. The second model is a paraboloid of radius λ and focal depth 12λ, discretized with 857,862 nodes and illuminated at an incident angle $(\theta, \phi) = (45°, 0°)$. Besides the GMRES method, in these experiments we compare CORS and BiCORSTAB also against the FGMRES method preconditioned by an inner GMRES solver. This combination of Krylov methods is reported to be amazingly effective on this problem class [6,7]. In Tables 4-5, we report the number of iterations and CPU solution time necessary to achieve convergence on 16 processors. The solution process was declared a solver failure when the initial residual was not reduced by at least four orders of magnitude after 2000 M-V products (or 50 outer iterations for FGMRES). This level of accuracy on the final residual enabled us to calculate a correct radar-cross-section, which is shown in Figure 3 for BiCORSTAB and CORS. We notice again the remarkable robustness and efficiency of the two algorithms.



(a)



(b)

Fig. 1. Example 3: a satellite. (a) The discretized mesh. (b) The convergence history of the CORS vs. the restarted GMRES methods.

III. ENHANCING THE ROBUSTNESS OF CORS BY DEFLATION

It is known that the convergence of an iterative method is mostly dictated by the distribution of the eigenvalues of the coefficient matrix MA. For GMRES, the residual reduction after k iterations writes

$$\|r_k\|_2 / \|r_0\|_2 \leqslant \kappa(V) \min_{p_k} \max_{i=1,\ldots,n} |p_k(\lambda_i)|, \quad (6)$$

where $\kappa(V)$ is the condition number of the eigenvector matrix. If $\kappa(V) \not\gg 1$, the problem of describing the convergence of GMRES reduces

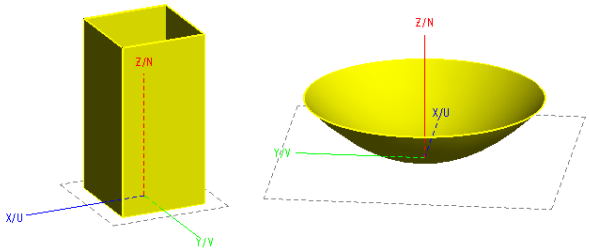


Fig. 2. Geometries of the largest model problems: (left) Example 4: a cavity, and (right) Example 5: a paraboloid.

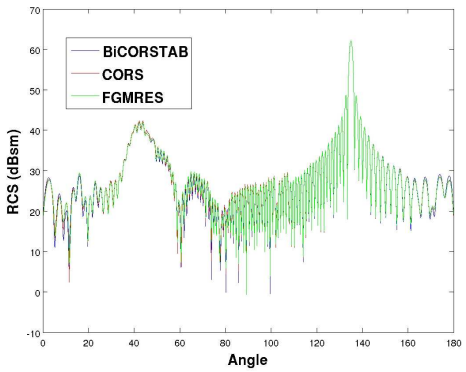
Table 4: Iterations count for the experiments with MLFMA and SAI

Example 4: Cavity $10\lambda \times 10\lambda \times 50\lambda$ Size: 727,120, Setup time SAI: 57s, Nr. procs: 16				
	GMRES(100)	CORS	BiCORSTAB	FGMRES
No prec.	>2000	239	280	21
SAI	798	64	56	10
Example 5: Paraboloid of radius 25λ and focal depth of 12λ Size: 857,862, Setup time SAI: 148s, Nr. procs: 16				
	GMRES(100)	CORS	BiCORSTAB	FGMRES
No prec.	1065)	367	410	11
SAI	112	42	36	5

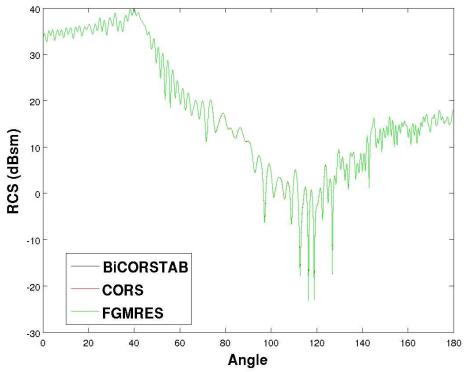
to a problem in approximation theory: how well can one approximate zero on the set of complex eigenvalues using a kth-degree polynomial with value 1 at the origin. From Eq. (6), we see that the presence of small eigenvalues close to the origin in the spectrum of the coefficient matrix of the preconditioned linear system may lead to highly oscillatory polynomials with high degree k, and therefore may increase the number of iterations

Table 5: CPU time for the experiments with MLFMA and SAI. For each example, asterisk "*" indicates the fastest run

Example 4: Cavity $10\lambda \times 10\lambda \times 50\lambda$ Size: 727,120, Setup time SAI: 57s, Nr. procs: 16				
	GMRES(100)	CORS	BiCORSTAB	FGMRES
No prec.	>651s	144s*	168s	2076s
SAI	325s	54s	51s*	898s
Example 5: Paraboloid of radius 25λ and focal depth of 12λ Size: 857,862, Setup time SAI: 148s, Nr. procs: 16				
	GMRES(100)	CORS	BiCORSTAB	FGMRES
No prec.	693s	449s*	495s	1071s
SAI	89s	72s	60s*	160s



(a)



(b)

Fig. 3. Comparative curves of the radar-cross-section (RCS) calculation for (a) the cavity problem, and (b) the paraboloid problem.

of GMRES to obtain convergence. In this section, we show how to enhance the robustness of the CORS method by dumping the slowly converging components of the residuals associated to the smallest eigenvalues. This may finally result in considerably faster convergence.

Let $MA = V\Lambda V^{-1}$ be an eigendecomposition of the preconditioned matrix MA, with $\Lambda = diag(\lambda_i)$, $|\lambda_1| \leq \ldots \leq |\lambda_n|$ are the eigenvalues of MA and V is the matrix collecting the associated right eigenvectors of MA. We denote by V_ε the matrix of the right eigenvectors of MA associated to the eigenvalues λ_i such that $|\lambda_i| \leq \varepsilon$. We also denote by $A_c = V_\varepsilon^H (MA) V_\varepsilon$ the projection of MA in the eigenspace spanned by V_ε, and by $M_c = V_\varepsilon A_c^{-1} V_\varepsilon^H$ its prolongation back to the original space.

Then, the following result holds.

Theorem 1: Let

$$\tilde{A}_c = V^H A V_\varepsilon \text{ has full rank,}$$

$$\tilde{M}_c = V_\varepsilon \tilde{A}_c^{-1} V^H,$$

and

$$\tilde{M} = M + \tilde{M}_c.$$

Then $\tilde{M}A$ is similar to a matrix whose eigenvalues are

$$\begin{cases} \eta_i = \lambda_i & \text{if} \quad |\lambda_i| > \varepsilon, \\ \eta_i = 1 + \lambda_i & \text{if} \quad |\lambda_i| \leq \varepsilon. \end{cases}$$

A_c represents the projection of the matrix MA on the coarse space defined by the approximate eigenvectors associated with its smallest eigenvalues.

Proof

We first remark that $A_c = diag(\lambda_i u_i^H v_i)$ with $|\lambda_i| \leq \varepsilon$ and so A_c is nonsingular. A_c represents the projection of the matrix MA on the space spanned by the approximate eigenvectors associated with its smallest eigenvalues.

Let $V = (V_\varepsilon, V_{\bar{\varepsilon}})$, where $V_{\bar{\varepsilon}}$ is the set of $(n - k)$ right eigenvectors associated with eigenvalues $|\lambda_i| > \varepsilon$.

Let $D_\varepsilon = diag(\lambda_i)$ with $|\lambda_i| \leq \varepsilon$ and $D_{\bar{\varepsilon}} = diag(\lambda_j)$ with $|\lambda_j| > \varepsilon$.

The following relations hold: $\tilde{M}AV_\varepsilon = V_\varepsilon(D_\varepsilon + I_k)$ and, $\tilde{M}AV_{\bar{\varepsilon}} = V_{\bar{\varepsilon}} D_{\bar{\varepsilon}} + V_\varepsilon C$ with $C = A_c^{-1} W^H A V_{\bar{\varepsilon}}$; then we have

$$\tilde{M}AV = V \begin{pmatrix} D_\varepsilon + I_k & C \\ 0 & D_{\bar{\varepsilon}} \end{pmatrix}.$$

For right preconditioning, that is $AMy = b$, similar results hold. Observe that the effect of applying the low-rank correction is to completely removed the effect of the k smallest eigenvalues from the spectrum of MA. Therefore, we may expect that an iterative method may converge faster on the transformed linear system.

The spectral corrections may be implemented in the CORS algorithm as follows.

1) Compute an approximation of the invariant subspace V_ε associated to the smallest eigenvalues of MA.

2) Construct the projected matrix $A_c = V_\varepsilon^H (MA) V_\varepsilon$ and the prolongation matrix $M_c = V_\varepsilon A_c^{-1} V_\varepsilon^H$. The matrix A_c is an $k \times k$ matrix, where k is the number of small eigenvalues that we want to remove, while M_c is $n \times n$.
3) Update the preconditioned vector quantities p in CORS as $p \leftarrow p + M_c p$

The same idea and computational scheme may be extended to the BiCORSTAB algorithm.

We applied deflated CORS (or, shortly DCORS) to a complete RCS calculation, which requires to solve linear systems with multiple right-hand sides. Take as incident field a plane wave of general form in spherical coordinates

$$\vec{E}_{inc}(x, \varphi, p_\theta, p_\varphi) = p_\theta \hat{u}_\theta e^{ikx \cdot \hat{u}_r \varphi} + p_\varphi \hat{u}_\varphi e^{ikx \cdot \hat{u}_r \varphi},$$

where p_θ, p_φ are two complex numbers and $\hat{u}_r, \hat{u}_\theta, \hat{u}_\varphi$ are the unitary vectors:

$$\hat{u}_r = \begin{pmatrix} \cos\varphi \cos\theta \\ \sin\varphi \cos\theta \\ \sin\theta \end{pmatrix}, \hat{u}_\theta = \begin{pmatrix} -\cos\varphi \sin\theta \\ -\sin\varphi \sin\theta \\ \cos\theta \end{pmatrix},$$
$$\hat{u}_\varphi = \begin{pmatrix} -\sin\varphi \cos\theta \\ -\cos\varphi \cos\theta \\ \sin\theta \end{pmatrix}.$$

Choose $\theta = 0$ and increase φ of one degree each time from 0 to π. Then, we obtain a sequence of 180 linear systems, each of them having the same coefficient matrix and a different right-hand side, associated to the following expression for the incident field

$$\vec{E}_{inc}(x) = \hat{z} e^{ikx \cdot \hat{u}_r(\varphi)} = \hat{z} e^{ik(x_1 \cos\varphi + x_2 \sin\varphi)}.$$

In the experiments reported in Table 6, the preconditioner M is the SAI method computed using sixty nonzeros per column, and we deflate an approximate invariance eigenspace V_ε of dimension 10. The invariant subspace V_ε is computed using the ARPACK library [17]. In our runs, the extra cost to setup DCORS is quickly amortized using only three right-hand sides for the satellite problem and four for the cylinder problem.

Table 6: Experiments with deflated CORS

On the satellite: SAI = 60 - AV = 3		
	CORS	DCORS(10)
Avg Its on 180 RHS	62.8	38.2
Total solution time (sec)	275.8	198.9
On the cylinder - SAI = 60		
	CORS	DCORS(10)
Avg Its on 180 RHS	193.4	123.7
Total solution time (sec)	365.5	249.4

IV. CONCLUSIONS

We have analyzed the performance of two novel Krylov projection methods computed from the Lanczos biconjugate A-orthonormalization method for solving dense complex non-Hermitian linear systems in radar-cross-section calculation. This family of solvers shows good convergence properties, is cheap in memory as it is derived from short-term vector recurrences, is parameter-free and does not suffer from the restriction to require a symmetric preconditioner. Additionally, it does not necessitate of matrix multiplication by A^H that might be tricky to implement in some integral application codes combined with MLFMA. Finally, we have illustrated how to possibly enhance the efficiency of the methods for solving linear systems with multiple right-hand sides, a typical scenario arising in realistic RCS calculation in industry.

The numerical results indicate that the proposed solvers may be an efficient alternative to other popular methods especially when robustness and memory are concerns.

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Algorithm 1 *Left preconditioned BiCORSTAB method.*

Compute $r_0 = b - Ax_0$ for some initial guess x_0.
Choose $r_0^* = P(A)r_0$ such that $\langle r_0^*, Ar_0\rangle \neq 0$, where $P(t)$ is a polynomial in t. (For example, $r_0^* = Ar_0$).
for $j = 1, 2, \ldots$ **do**
  solve $Mz_{j-1} = r_{j-1}$
  $\hat{z} = Az_{j-1}$
  $\rho_{j-1} = \langle r_0^*, \hat{z}\rangle$
  if $\rho_{j-1} = 0$, **method fails**
  if $j = 1$ **then**
    $p_0 = r_0$
    solve $Mzp_0 = p_0$
    $q_0 = \hat{z}$
  else
    $\beta_{j-2} = (\rho_{j-1}/\rho_{j-2}) \times (\alpha_{j-2}/\omega_{j-2})$
    $p_{j-1} = r_{j-1} + \beta_{j-2}\left(p_{j-2} - \omega_{j-2}q_{j-2}\right)$
    $zp_{j-1} = z_{j-1} + \beta_{j-2}\left(zp_{j-2} - \omega_{j-2}zq_{j-2}\right)$
    $q_{j-1} = \hat{z} + \beta_{j-2}\left(q_{j-2} - \omega_{j-2}\hat{z}q_{j-2}\right)$
  end if
  solve $Mzq_{j-1} = q_{j-1}$
  $\hat{z}q_{j-1} = Azq_{j-1}$
  $\alpha_{j-1} = \rho_{j-1}/\left\langle r_0^*, \hat{z}q_{j-1}\right\rangle$
  $s = r_{j-1} - \alpha_{j-1}q_{j-1}$
  if $\|s\|_2$ is small, set $x_j = x_{j-1} + \alpha_{j-1}zp_{j-1}$ and stop
  $zs = z_{j-1} - \alpha_{j-1}zq_{j-1}$
  $t = \hat{z} - \alpha_{j-1}\hat{z}q_{j-1}$
  $\omega_{j-1} = \langle t, s\rangle / \langle t, t\rangle$
  $x_j = x_{j-1} + \alpha_{j-1}zp_{j-1} + \omega_{j-1}zs$
  $r_j = s - \omega_{j-1}t$
  check convergence; continue if necessary and $\omega_{j-1} \neq 0$
end for

Algorithm 2 *Left preconditioned CORS method.*

Compute $r_0 = b - Ax_0$ for some initial guess x_0.
Choose $r_0^* = P(A)r_0$ such that $\langle r_0^*, Ar_0\rangle \neq 0$, where $P(t)$ is a polynomial in t. (For example, $r_0^* = Ar_0$).
for $j = 1, 2, \ldots$ **do**
  solve $Mz_{j-1} = r_{j-1}$
  $\hat{r} = Az_{j-1}$
  $\rho_{j-1} = \langle r_0^*, \hat{r}\rangle$
  if$\rho_{j-1} = 0$, **method fails**
  if $j = 1$ **then**
    $e_0 = r_0$
    solve $Mze_0 = e_0$
    $d_0 = \hat{r}$
    $q_0 = \hat{r}$
  else
    $\beta_{j-2} = \rho_{j-1}/\rho_{j-2}$
    $e_{j-1} = r_{j-1} + \beta_{j-2}h_{j-2}$
    $ze_{j-1} = z_{j-1} + \beta_{j-2}zh_{j-2}$
    $d_{j-1} = \hat{r} + \beta_{j-2}f_{j-2}$
    $q_{j-1} = d_{j-1} + \beta_{j-2}\left(f_{j-2} + \beta_{j-2}q_{j-2}\right)$
  end if
  solve $Mq = q_{j-1}$
  $\hat{q} = Aq$
  $\alpha_{j-1} = \rho_{j-1}/\langle r_0^*, \hat{q}\rangle$
  $h_{j-1} = e_{j-1} - \alpha_{j-1}q_{j-1}$
  $zh_{j-1} = ze_{j-1} - \alpha_{j-1}q$
  $f_{j-1} = d_{j-1} - \alpha_{j-1}\hat{q}$
  $x_j = x_{j-1} + \alpha_{j-1}\left(2ze_{j-1} - \alpha_{j-1}q\right)$
  $r_j = r_{j-1} - \alpha_{j-1}\left(2d_{j-1} - \alpha_{j-1}\hat{q}\right)$
  check convergence; continue if necessary
end for

REFERENCES

[1] W. C. Chew, J. M. Jin, C. C. Lu, E. Michielssen, and J. M. Song, "Fast Solution Methods in Electromagnetics," *IEEE Transactions on Antennas and Propagation*, vol. 45, no. 3, pp. 533–543, 1997.

[2] B. Carpentieri, "Fast Iterative Solution Methods in Electromagnetic Scattering," *Progress in Electromagnetic Research*, vol. 79, pp. 151–178, 2007.

[3] F. Bilotti and C. Vegni, "MoM Entire Domain Basis Functions for Convex Polygonal Patches," *Journal of Electromagnetic Waves and Applications*, vol. 17, no. 11, pp. 1519–1538, 2003.

[4] J. Li, L. Li, and Y. Gan, "Method of Moments Analysis of Waveguide Slot Antennas using the EFIE," *Journal of Electromagnetic Waves and Applications*, vol. 19, no. 13, pp. 1729–1748, 2005.

[5] S. Rao, D. Wilton, and A. Glisson, "Electromagnetic Scattering by Surfaces of Arbitrary Shape," *IEEE Trans. Antennas Propagat.*, vol. AP-30, pp. 409–418, 1982.

[6] B. Carpentieri, I. Duff, L. Giraud, and G. Sylvand, "Combining Fast Multipole Techniques and an Approximate Inverse Preconditioner for Large Electromagnetism Calculations," *SIAM J. Scientific Computing*, vol. 27, no. 3, pp. 774–792, 2005.

[7] T. Malas, Ö. Ergül, and L. Gürel, "Sequential and Parallel Preconditioners for Large-Scale Integral-Equation Problems," in *Computational Electromagnetics Workshop, August 30-31, 2007, Izmir, Turkey*, pp. 35–43, 2007.

[8] Y.-F. Jing, T.-Z. Huang, Y. Zhang, L. Li, G.-H. Cheng, Z.-G. Ren, Y. Duan, T. Sogabe, and B. Carpentieri, "Lanczos-Type Variants of the COCR Method for Complex Nonsymmetric Linear Systems," *Journal of Computational Physics*, vol. 228, no. 17, pp. 6376–6394, 2009.

[9] B. Carpentieri, Y.-F. Jing, and T.-Z. Huang, "The BiCOR and CORS Algorithms for Solving Nonsymmetric Linear Systems," *SIAM J. Scientific Computing*, vol. 33, no. 5, pp. 3020–3036, 2011.

[10] J. Song, C.-C. Lu, and W. Chew, "Multilevel Fast Multipole Algorithm for Electromagnetic Scattering by Large Complex Objects," *IEEE Transactions on Antennas and Propagation*, vol. 45(10), pp. 1488–1493, 1997.

[11] J. Rahola and S. Tissari, "Iterative Solution of Dense Linear Systems Arising from the Electrostatic Integral Equation in MEG," *Physics in Medicine and Biology*, vol. 47(6), pp. 961–975, 2002.

[12] J. Lee, C.-C. Lu, and J. Zhang, "Sparse Inverse Preconditioning of Multilevel Fast Multipole Algorithm for Hybrid Integral Equations in Electromagnetics," *IEEE Transactions on Antennas and Propagation*, vol. 52, no. 9, pp. 2277–2287, 2004.

[13] Ö. Ergül and L. Gürel, "Efficient Parallelization of the Multilevel Fast Multipole Algorithm for the Solution of Large-Scale Scattering Problems," *IEEE Transactions on Antennas and Propagation*, vol. 56, no. 8, pp. 2335–2345, 2008.

[14] X. Pan and X. Sheng, "A Highly Efficient Parallel Approach of Multi-Level Fast Multipole Algorithm," *Journal of Electromagnetic Waves and Applications*, vol. 20, no. 8, pp. 1081–1092, 2006.

[15] X.-M. Pan and X.-Q. Sheng, "An Efficient Parallel SAI Preconditioner for Multilevel Fast Multipole Algorithm for Scattering by Extremely Large Complex Targets," in *Microwave Technology and Computational Electromagnetics, 2009. ICMTCE. International Conference on*, pp. 407 –410, Nov. 2009.

[16] X.-M. Pan and X.-Q. Sh, "A Sophisticated Parallel MLFMA for Scattering by Extremely Large Targets [EM Programmer's Notebook]," *Antennas and Propagation Magazine, IEEE*, vol. 50, no. 3, pp. 129 –138, June 2008.

[17] R. Lehoucq, D. Sorensen, and C. Yang, *ARPACK User's Guide: Solution of Large-Scale Eigenvalue Problems with Implicitly Restarted Arnoldi Methods*. Society for Industrial and Applied Mathematics, Philadelphia, 1998.

[18] N. Nakashima, S. Fujino, and M. Tateiba, "Performance Evaluation of State-of-the-Art Linear Iterative Solvers Based on IDR Theorem for Large Scale Electromagnetic Multiple Scattering Simulations," *Applied Computational Electromagnetics Society (ACES) Journal*, vol. 26, no. 1, pp. 37–44, 2011.

[19] Y. Zhang and Q. Sun, "Complex Incomplete Cholesky Factorization Preconditioned Bi-conjugate Gradient Method," *Applied Computational Electromagnetics Society (ACES) Journal*, vol. 25, no. 9, pp. 750–754, 2010.

[20] D. Ding, J. Ge, and R. Chen, "Well-Conditioned CFIE for Scattering from Dielectric Coated Conducting Bodies above a Half-Space," *Applied Computational Electromagnetics Society (ACES) Journal*, vol. 25, no. 11, pp. 936–946, 2010.

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