Investigating the Composite Step Biconjugate A-Orthogonal Residual Method for Non-Hermitian Dense Linear Systems in Electromagnetics

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***Abstract*—** An interesting stabilizing variant of the biconjugate A-orthogonal residual (BiCOR) method is investigated for solving dense complex non-Hermitian systems of linear equations arising from the Galerkin discretization of surface integral equations in electromagnetics. The novel variant is naturally based on and inspired by the composite step strategy employed for the composite step biconjugate gradient method from the point of view of pivot-breakdown treatment when the BiCOR method has erratic convergence behaviors. Besides reducing the number of spikes in the convergence history of the norm of the residuals to the greatest extent, the present composite step BiCOR method can provide some further practically desired smoothing behavior towards stabilizing the numerical performance of the BiCOR method in the case of irregular convergence.

***Index Terms*—** Krylov subspace methods, Lanczos biconjugate A-orthonormalization methods, scattering problems, sparse approximate inverse preconditioning.

I. INTRODUCTION

Solution of large linear systems is crucial to many numerical simulations in science and engineering [1,2]. Many real-world applications demand an accurate numerical solution of physical problems arising from fields such as fluid mechanics, structural engineering, computational electromagnetics, design and computer analysis of circuits, power system networks, and economics models [3]. Take scattering problems of determining the diffraction pattern irradiated by an illuminated object for instance. They include medical imaging, electromagnetic compatibility, radar cross section (RCS) calculation of large objects. Krylov subspace methods, which are considered as one of the "Top Ten Algorithms of the 20th Century" [4], are one of the most widespread and extensively accepted techniques for iterative solution of today's large-scale linear systems [5]. The starting point for this work was the investigation of the applicability of a new interesting stabilizing variant of the biconjugate A-orthogonal residual (BiCOR) method [7] to iterative solution of non-Hermitian systems of linear equations in electromagnetism applications.

In recent years, there have been many advances in Krylov subspace methods for solution of large linear systems [5]. Different variants of restarted, augmented, deflated, flexible, nested, and inexact methods are involved in these new developments. Various methods differ in the way they extract information from Krylov spaces [8–10]. Observing from earlier studies on surface scattering problems, different Krylov subspace methods have both advantages and disadvantages [11]. For instance, the GMRES method is robust but expensive due to long recurrence in the underlying Arnoldi procedure. Restarting the GMRES deteriorates convergence significantly. The BiCGSTAB method typically requires many more iterations than the GMRES method, especially on complex

Submitted On: Sept. 25, 2011
Accepted On: Dec. 23, 2011

geometries. QMR-like methods are only slightly more competitive than the BiCGSTAB method, but less robust than the GMRES method.

A novel Lanczos-type biconjugate A-orthonormalization procedure has recently been established to give birth to a new family of efficient short-recurrence methods for large real nonsymmetric and complex non-Hermitian systems of linear equations, named as the Lanczos biconjugate A-orthonormalization methods [7]. As observed from numerous numerical experiments carried out with the Lanczos biconjugate A-orthonormalization methods, it has been numerically demonstrated that this family of solvers shows competitive convergence properties, is cheap in memory as it is derived from short-term vector recurrences, is parameter-free and does not suffer from the restriction to require a symmetric preconditioner like other methods [12–15]. However, the family of Lanczos biconjugate A-orthonormalization methods is often faced with apparently irregular convergence behaviors appearing as "spikes" in the convergence history of the norm of the residuals, possibly leading to substantial build-up of rounding errors and worse approximate solutions, or possibly even overflow. Therefore, it is quite necessary to tackle their irregular convergence properties to obtain more stabilized variants so as to improve the accuracy of the desired physical numerical solutions.

Our main attention in this paper is focused on the demonstration of the straightforward natural enhancement of the BiCOR method, which is the basic underlying variant of the above-mentioned family of Lanczos biconjugate A-orthonormalization methods. The content of this paper can be considered as the natural follow-up to the paper [16]. In particular, we exploit the composite step strategy taken for the composite step biconjugate gradient (CSBCG) method [17,18] from the point of view of pivot-breakdown treatment when the BiCOR method has erratic convergence behaviors. The outline of the paper is organized as follows. In the coming section, the good performance of the BiCOR algorithm in electromagnetics will be illustrated numerically by recalling some introductory comparative experiments. In Section III, we present the interesting stabilizing variant–the composite step BiCOR (CSBiCOR) method with applications on a set of model problems representative of realistic RCS calculation to show the improved numerical performance with respect to the stabilizing effect of the composite step strategy on the BiCOR method. Conclusions and perspectives are finally made with some future research issues.

Throughout the paper, denote the overbar ("–") the conjugate complex of a scalar, vector or matrix and the superscript "T"the transpose of a vector or matrix. For a non-Hermitian matrix $A = (a_{ij})_{N\times N} \in \mathbb{C}^{N\times N}$, the Hermitian conjugate of A is denoted as

$$A^H \equiv \overline{A}^T = (\overline{a}_{ji})_{N\times N}.$$

The standard Hermitian inner product of two complex vectors $u, v \in \mathbb{C}^N$ is defined as

$$\langle u, v\rangle = u^H v = \sum_{i=1}^{N} \overline{u}_i v_i.$$

The nested Krylov subspace of dimension t generated by A from v is of the form

$$\mathcal{K}_t(A, v) = \text{span}\{v, Av, A^2 v, \ldots, A^{t-1} v\}.$$

In addition, e_i denotes the ith column of the appropriate identity matrix.

II. PRELIMINARY REVIEW FOR THE BICOR METHOD

First, we briefly recall a version of the Lanczos biconjugate A-orthonormalization procedure [7] as in Algorithm 1, which can ideally build up a pair of biconjugate A-orthonormal bases for the dual Krylov subspaces $\mathcal{K}_m(A, v_1)$ and $\mathcal{K}_m(A^H, w_1)$, where v_1 and w_1 are chosen initially to satisfy certain conditions.

Observe that the above algorithm is possible to have Lanczos-type breakdown whenever δ_{j+1} vanishes while $\hat{w}_{j+1}$ and $A\hat{v}_{j+1}$ are not equal to $\mathbf{0} \in \mathbb{C}^N$ appearing in line 8. In the interest of counteraction against such breakdowns, we refer the reader to possible remedies proposed in earlier studies, such as so-called look-ahead strategies [19–27] which can enhance stability

Algorithm 1 *Biconjugate A-Orthonormalization Procedure*

Choose v_1, ω_1*, such that* $\langle \omega_1, Av_1 \rangle = 1$
Set $\beta_1 = \delta_1 \equiv 0, \omega_0 = v_0 \equiv \mathbf{0} \in \mathbb{C}^N$
for $j = 1, 2, \ldots$ **do**
  $\alpha_j = \langle \omega_j, A\left(Av_j\right) \rangle$
  $\hat{v}_{j+1} = Av_j - \alpha_j v_j - \beta_j v_{j-1}$
  $\hat{\omega}_{j+1} = A^H \omega_j - \bar{\alpha}_j \omega_j - \delta_j \omega_{j-1}$
  $\delta_{j+1} = |\langle \hat{\omega}_{j+1}, A\hat{v}_{j+1} \rangle|^{\frac{1}{2}}$
  $\beta_{j+1} = \dfrac{\langle \hat{\omega}_{j+1}, A\hat{v}_{j+1} \rangle}{\delta_{j+1}}$
  $v_{j+1} = \dfrac{\hat{v}_{j+1}}{\delta_{j+1}}$
  $\omega_{j+1} = \dfrac{\hat{\omega}_{j+1}}{\bar{\beta}_{j+1}}$
end for

while increasing cost modestly. However, the analysis of these strategies is outside the scope of this paper and we shall not pursue that here. For more details, please refer to [5,10] and the references therein. In the present paper, we suppose there is no such Lanczos-type breakdown encountered during algorithm implementations because most of our considerations concern the exploration of the composite step strategy [17,18] to handle the pivot breakdown occurring in the BiCOR method for solving non-Hermitian linear systems in electromagnetics.

Next, some properties of the vectors produced by Algorithm 1 are reviewed [7] in the following proposition for the preparation of the theoretical basis of the composite step method.

Proposition 1: If Algorithm 1 proceeds m steps, then the right and left Lanczos-type vectors $v_j, j = 1, 2, \ldots, m$ and $w_i, i = 1, 2, \ldots, m$ form a biconjugate A-orthonormal system in exact arithmetic, i.e.,

$$\langle \omega_i, Av_j \rangle = \delta_{i,j}, 1 \leq i, j \leq m.$$

Furthermore, denote by $V_m = [v_1, v_2, \ldots, v_m]$ and $W_m = [w_1, w_2, \ldots, w_m]$ the $N \times m$ matrices and by $\underline{T_m}$ the extended tridiagonal matrix of the form

$$\underline{T_m} = \begin{bmatrix} T_m \\ \delta_{m+1} e_m^T \end{bmatrix},$$

where

$$T_m = \begin{bmatrix} \alpha_1 & \beta_2 & & & \\ \delta_2 & \alpha_2 & \beta_3 & & \\ & \ddots & \ddots & \ddots & \\ & & \delta_{m-1} & \alpha_{m-1} & \beta_m \\ & & & \delta_m & \alpha_m \end{bmatrix},$$

whose entries are the coefficients generated during the algorithm implementation, and in which $\alpha_1, \ldots, \alpha_m, \beta_2, \ldots, \beta_m$ are complex while $\delta_2, \ldots, \delta_m$ are positive. Then with the biconjugate A-orthonormalization procedure, the following four relations hold

$$AV_m = V_m T_m + \delta_{m+1} v_{m+1} e_m^T, \tag{1}$$
$$A^H W_m = W_m T_m^H + \bar{\beta}_{m+1} \omega_{m+1} e_m^T, \tag{2}$$
$$W_m^H A V_m = I_m, \tag{3}$$
$$W_m^H A^2 V_m = T_m. \tag{4}$$

Given an initial guess x_0 to the non-Hermitian linear system $Ax = b$ associated with the initial residual $r_0 = b - Ax_0$, define a Krylov subspace $\mathcal{L}_m \equiv A^H \text{span}(W_m) = A^H \mathcal{K}_m(A^H, w_1)$, where W_m is defined in Proposition 1, $v_1 = \frac{r_0}{||r_0||_2}$ and w_1 is chosen arbitrarily such that $\langle w_1, Av_1 \rangle \neq 0$. But w_1 is often chosen to be equal to $\frac{Av_1}{||Av_1||_2^2}$ subjecting to $\langle w_1, Av_1 \rangle = 1$. It is worthy noting that this choice for w_1 plays a significant role in establishing the empirically observed superiority of the BiCOR method to the BiCR [6] method as well as to the BCG method [7]. Thus running Algorithm 1 m steps, we can seek an mth approximate solution x_m from the affine subspace $x_0 + \mathcal{K}_m(A, v_1)$ of dimension m, by imposing the Petrov-Galerkin condition

$$b - Ax_m \perp \mathcal{L}_m,$$

which can be mathematically written in matrix formulation as

$$(A^H W_m)^H (b - Ax_m) = 0. \tag{5}$$

Analogously, an mth dual approximation x_m^* of the corresponding dual system $A^H x^* = b^*$ is sought from the affine subspace $x_0^* + \mathcal{K}_m(A^H, w_1)$ of dimension m by satisfying

$$b^* - A^H x_m^* \perp A\mathcal{K}_m(A, v_1),$$

which can be mathematically written in matrix formulation as

$$(AV_m)^H (b^* - A^H x_m) = 0, \tag{6}$$

where, x_0^* is an initial dual approximate solution and V_m is defined in Proposition 1 with $v_1 = \frac{r_0}{||r_0||_2}$.

Consequently, the BiCOR iterates $x_j's$ can be computed by the coming Algorithm 2, which is just the unpreconditioned BiCOR method with the preconditioner M there taken as the identity matrix [7] and has been rewritten with the algorithmic scheme of the unpreconditioned BCG method as presented in [10,18].

Algorithm 2 *Algorithm BiCOR*

Compute $r_0 = b - Ax_0$ for some initial guess x_0.
Choose $r_0^* = P(A)r_0$ such that $\langle r_0^*, Ar_0 \rangle \neq 0$, where $P(t)$ is a polynomial in t. (For example, $r_0^* = Ar_0$).
Set $p_0 = r_0, p_0^* = r_0^*, q_0 = Ap_0, q_0^* = A^H p_0^*, \hat{r}_0 = Ar_0, \rho_0 = \langle r_0^*, \hat{r}_0 \rangle$.
for $n = 0, 1, \ldots$ **do**
  $\sigma_n = \langle q_n^*, q_n \rangle$
  $\alpha_n = \rho_n / \sigma_n$
  $x_{n+1} = x_n + \alpha_n p_n$
  $r_{n+1} = r_n - \alpha_n q_n$
  $x_{n+1}^* = x_n^* + \bar{\alpha}_n p_n^*$
  $r_{n+1}^* = r_n^* - \bar{\alpha}_n q_n^*$
  $\hat{r}_{n+1} = Ar_{n+1}$
  $\rho_{n+1} = \left\langle r_{n+1}^*, \hat{r}_{n+1} \right\rangle$
  if $\rho_{n+1} = 0$, **method fails**
  $\beta_{n+1} = \rho_{n+1} / \rho_n$
  $p_{n+1} = r_{n+1} + \beta_{n+1} p_n$
  $p_{n+1}^* = r_{n+1}^* + \bar{\beta}_{n+1} p_n^*$
  $q_{n+1} = \hat{r}_{n+1} + \beta_{n+1} q_n$
  $q_{n+1}^* = A^H p_n^*$
  check convergence; continue if necessary
end for

Before ending this section, we review some introductory comparative experiments to see the good numerical performance of the BiCOR algorithm [12]. The set of linear systems selected for the numerical experiments arise from RCS calculations of realistic targets. They are dense complex non-Hermitian. We report the characteristics of the model problems in Table 1. Although not very large, the selected problems are representative of realistic RCS calculation. Their solution demands considerable computer resources as it can be seen in the table. Larger problems require using the multilevel fast multipole algorithm (MLFMA) [28–31] for the M-V products to reduce the memory requirement and effective preconditioners to accelerate the convergence, and they are out of the scope of this study. We carried out the M-V product at each iteration using dense linear algebra packages, i.e. the `ZGEMV` routine available in the LAPACK library and we did not use preconditioning. In addition to the BiCOR method, we considered the other two evolving variants known as the conjugate A-orthogonal residual squared (CORS) method and the biconjugate A-orthogonal residual stabilized (BiCORSTAB) method, complex versions of iterative algorithms based on Lanczos biorthogonalization, such as BiCGSTAB and QMR, and on Arnoldi orthogonalization, such as GMRES. In Table 2, we list the complete set of solvers used in our experiments and their algorithmic and memory complexity. All the runs were done on one node of the *Entu* cluster facility located at CRS4. Each node features a quad core Intel CPU at 2.8GHz and 16 GB of physical RAM. The codes were compiled in Fortran with the Portland Group Fortran 90 compiler version 9.

Table 1: Characteristics of the model problems

Example	Description	Size	Frequency (MHz)
1	Open cylinder	6,268	362
2	Sphere	12,000	535
3	Satellite	1,699	57

In Table 3, we show the number of iterations and CPU time (in seconds) required by Krylov methods to reduce the initial residual to $\mathcal{O}(10^{-5})$ starting from the zero vector. The right-hand side of the linear system is set up so that the exact solution is

Table 2: Algorithmic cost and memory expenses of the implementation of Krylov algorithms that are used for the experiments. We denote by n the problem size, by i the iteration number and by m the restart value in GMRES

Solver	Products by A/A^H	Memory
BiCOR	1/1	matrix+$10n$
CORS	2/0	matrix+$14n$
BiCORSTAB	2/0	matrix+$13n$
GMRES	1/0	matrix+$(m+3)n$
QMR	2/1	matrix+$11n$
TFQMR	4/0	matrix+$10n$
BiCGSTAB	2/0	matrix+$7n$

Table 3: Number of iterations and CPU time (in seconds) required by Krylov methods to reduce the initial residual to $\mathcal{O}(10^{-5})$; for each example, an asterisk "*" indicates the fastest run

Solver/Example	1	2	3
CORS	601 (253*)	294 (451*)	371 (11*)
BiCOR	785 (334)	338 (525)	431 (15)
BiCORSTAB	941 (614)	423 (1099)	775 (37)
GMRES(50)	2191 (469)	1803 (1397)	871 (17)
QMR	878 (548)	430 (1045)	452 (24)
TFQMR	482 (398)	281 (863)	373 (27)
BiCGSTAB	1065 (444)	680 (1031)	566 (18)

the vector of all ones. We observe the effectiveness of the BiCOR method, that is the the second fastest non-Hermitian solver with respect to CPU time on most selected examples. Its performance is very close to that of the CORS method and may be an appropriate choice.

III. INVESTIGATION OF THE CSBICOR METHOD IN ELECTROMAGNETICS

Suppose Algorithm 2 runs successfully to step n, that is $\sigma_i \neq 0,\ \rho_i \neq 0,\ i = 0, 1, \ldots, n-1$. The BiCOR iterates satisfy the following properties [7].

Proposition 2: Let $R_{n+1} = [r_0, r_1, \ldots, r_n]$, $R^*_{n+1} = [r^*_0, r^*_1, \ldots, r^*_n]$ and $P_{n+1} = [p_0, p_1, \ldots, p_n]$, $P^*_{n+1} = [p^*_0, p^*_1, \ldots, p^*_n]$. We have

(1) $Range(R_{n+1}) = Range(P_{n+1}) = \mathcal{K}_{n+1}(A, r_0)$,

$Range(R^*_{n+1}) = Range(P^*_{n+1}) = \mathcal{K}_{n+1}(A^H, r^*_0)$.

(2) $R^{*H}_{n+1} A R_{n+1}$ is diagonal.

(3) $P^{*H}_{n+1} A^2 P_{n+1}$ is diagonal.

Similarly to the breakdowns of the BCG method [18], it is observed from Algorithm 2 that there also exist two possible kinds of breakdowns for the BiCOR method:

(1) $\rho_n \equiv \langle r^*_n, \hat{r}_n \rangle \equiv \langle r^*_n, A r_n \rangle = 0$ but r^*_n and $A r_n$ are not equal to $\mathbf{0} \in \mathbb{C}^N$ appearing in line 14;

(2) $\sigma_n \equiv \langle q^*_n, q_n \rangle \equiv \left\langle A^H p^*_n, A p_n \right\rangle = 0$ appearing in line 6.

Although the computational formulae for the quantities where the breakdowns reside are different between the BiCOR method and the BCG method, we do not have a better name for them. Therefore, we still call the two cases of breakdowns described above as Lanczos breakdown and pivot breakdown, respectively.

The Lanczos breakdown can be cured using look-ahead techniques [19–27] as mentioned in the previous section, but such techniques require a careful and sophisticated way so as to make them become necessarily quite complicated to apply. This aspect of applying look-ahead techniques to the BiCOR method demands further research.

In this paper, we attempt to resort to the composite step idea employed for the CSBCG method [17,18] to handle the pivot breakdown of the BiCOR method with the assumption that the underlying biconjugate A-orthonormalization procedure depicted as in Algorithm 1 does not breakdown; that is the situation where $\sigma_n = 0$ while $\rho_n \neq 0$.

Suppose Algorithm 2 comes across a situation where $\sigma_n = 0$ after successful algorithm implementation up to step n with the assumption that $\rho_n \neq 0$, which indicates that the updates of $x_{n+1},\ r_{n+1},\ x^*_{n+1},\ r^*_{n+1}$ are not well defined. Taking the composite step idea, we will avoid division by $\sigma_n = 0$ via skipping this $(n+1)$th update and exploiting a composite step update to directly obtain the quantities in step $(n+2)$ with scaled versions of r_{n+1} and r^*_{n+1} as well as with the previous primary search direction vector p_n and shadow search direction vector p^*_n. The following process for deriving the CSBiCOR method is the same as that of the derivation of the CSBCG method [18] except for the different underlying procedures involved to correspondingly generate different Krylov subspace bases.

Analogously, define auxiliary vectors $z_{n+1} \in \mathcal{K}_{n+2}(A, r_0)$ and $z^*_{n+1} \in \mathcal{K}_{n+2}(A^H, r^*_0)$ as follows

$$\begin{aligned} z_{n+1} &= \sigma_n r_{n+1} \\ &= \sigma_n r_n - \rho_n A p_n, \end{aligned} \tag{7}$$

$$\begin{aligned} z^*_{n+1} &= \bar{\sigma}_n r^*_{n+1} \\ &= \bar{\sigma}_n r^*_n - \bar{\rho}_n A^H p^*_n, \end{aligned} \tag{8}$$

which are then used to look for the iterates $x_{n+2} \in x_0 + \mathcal{K}_{n+2}(A, r_0)$ and $x^*_{n+2} \in x^*_0 + \mathcal{K}_{n+2}(A^H, r^*_0)$ in step $(n+2)$ as follows

$$\begin{aligned} x_{n+2} &= x_n + [p_n, z_{n+1}] f_n, \\ x^*_{n+2} &= x^*_n + [p^*_n, z^*_{n+1}] f^*_n, \end{aligned}$$

where, $f_n,\ f^*_n \in \mathbb{C}^2$. Correspondingly, the $(n+2)$th primary residual $r_{n+2} \in \mathcal{K}_{n+3}(A, r_0)$ and shadow residual $r^*_{n+2} \in \mathcal{K}_{n+3}(A^H, r^*_0)$ are respectively computed as

$$r_{n+2} = r_n - A[p_n, z_{n+1}] f_n, \tag{9}$$

$$r^*_{n+2} = r^*_n - A^H [p^*_n, z^*_{n+1}] f^*_n. \tag{10}$$

The biconjugate A-orthogonality condition between the BiCOR primary residuals and shadow residuals shown as Property (2) in Proposition 2 requires

$$\begin{aligned} \left\langle [p^*_n, z^*_{n+1}], A r_{n+2} \right\rangle &= 0, \\ \left\langle [p_n, z_{n+1}], A^H r^*_{n+2} \right\rangle &= 0, \end{aligned}$$

combining with Eqns. (9) and (10) gives rise to the following two 2×2 systems of linear equations for respectively solving f_n and f^*_n

$$\begin{bmatrix} \langle A^H p^*_n, A p_n \rangle & \langle A^H p^*_n, A z_{n+1} \rangle \\ \langle A^H z^*_{n+1}, A p_n \rangle & \langle A^H z^*_{n+1}, A z_{n+1} \rangle \end{bmatrix} \begin{bmatrix} f_n^{(1)} \\ f_n^{(2)} \end{bmatrix} = \begin{bmatrix} \langle p^*_n, A r_n \rangle \\ \langle z^*_{n+1}, A r_n \rangle \end{bmatrix}, \tag{11}$$

$$\begin{bmatrix} \langle A p_n, A^H p^*_n \rangle & \langle A p_n, A^H z^*_{n+1} \rangle \\ \langle A z_{n+1}, A^H p^*_n \rangle & \langle A z_{n+1}, A^H z^*_{n+1} \rangle \end{bmatrix} \begin{bmatrix} f_n^{*(1)} \\ f_n^{*(2)} \end{bmatrix} = \begin{bmatrix} \langle A p_n, r^*_n \rangle \\ \langle A z_{n+1}, r^*_n \rangle \end{bmatrix}. \tag{12}$$

Similarly, the $(n+2)$th primary search direction vector $p_{n+2} \in \mathcal{K}_{n+3}(A, r_0)$ and shadow search direction vector $p^*_{n+2} \in \mathcal{K}_{n+3}(A^H, r^*_0)$ in a composite step are computed with the following form

$$p_{n+2} = r_{n+2} + [p_n, z_{n+1}] g_n, \tag{13}$$

$$p^*_{n+2} = r^*_{n+2} + [p^*_n, z^*_{n+1}] g^*_n, \tag{14}$$

where, $g_n,\ g^*_n \in \mathbb{C}^2$.

The biconjugate A^2-orthogonality condition between the BiCOR primary search direction vectors and shadow search direction vectors shown as Property (3) in Proposition 2 requires

$$\begin{aligned} \left\langle [p^*_n, z^*_{n+1}], A^2 p_{n+2} \right\rangle &= 0, \\ \left\langle [p_n, z_{n+1}], (A^H)^2 p^*_{n+2} \right\rangle &= 0, \end{aligned}$$

combining with Eqns. (13) and (14) results in the following two 2×2 systems of linear equations for respectively solving g_n and g^*_n

$$\begin{bmatrix} \langle A^H p^*_n, A p_n \rangle & \langle A^H p^*_n, A z_{n+1} \rangle \\ \langle A^H z^*_{n+1}, A p_n \rangle & \langle A^H z^*_{n+1}, A z_{n+1} \rangle \end{bmatrix} \begin{bmatrix} g_n^{(1)} \\ g_n^{(2)} \end{bmatrix} = - \begin{bmatrix} \langle A^H p^*_n, A r_{n+2} \rangle \\ \langle A^H z^*_{n+1}, A r_{n+2} \rangle \end{bmatrix}, \tag{15}$$

$$\begin{bmatrix} \langle A p_n, A^H p^*_n \rangle & \langle A p_n, A^H z^*_{n+1} \rangle \\ \langle A z_{n+1}, A^H p^*_n \rangle & \langle A z_{n+1}, A^H z^*_{n+1} \rangle \end{bmatrix} \begin{bmatrix} g_n^{*(1)} \\ g_n^{*(2)} \end{bmatrix} = - \begin{bmatrix} \langle A p_n, A^H r^*_{n+2} \rangle \\ \langle A z_{n+1}, A^H r^*_{n+2} \rangle \end{bmatrix}. \tag{16}$$

Therefore, it could be able to advance from step n to step $(n+2)$ to provide $x_{n+2},\ r_{n+2},\ x^*_{n+2},\ r^*_{n+2},\ p_{n+2},\ p^*_{n+2}$ by solving the above four 2×2 linear systems represented as in Eqns. (11), (12), (15), and (16). With an appropriate combination of 1×1 and 2×2 steps, the CSBiCOR method can be simply obtained with only a minor modification to the usual implementation of the BiCOR method. The pseudocode for the preconditioned CSBiCOR with a left preconditioner B can be represented by Algorithm 3. For full details on the derivation and analysis of the CSBiCOR method, please refer to our recent work [32].

Algorithm 3 *Left preconditioned CSBiCOR method*

Compute $r_0 = b - Ax_0$ for some initial guess x_0.
Choose $r_0^* = P(A)r_0$ such that $\langle r_0^*, Ar_0 \rangle \neq 0$, where $P(t)$ is a polynomial in t. (For example, $r_0^* = Ar_0$). Set $p_0 = r_0, \tilde{p}_0 = \tilde{r}_0, q_0 = Ap_0, \tilde{q}_0 = A^H \tilde{p}_0$.
Compute $\rho_0 = \langle \tilde{r}_0, Ar_0 \rangle$.
Begin LOOP $(n = 0, 1, 2, \dots)$
$\sigma_n = \langle \tilde{q}_n, q_n \rangle$
$s_{n+1} = \sigma_n r_n - \rho_n q_n$
$\tilde{s}_{n+1} = \bar{\sigma}_n \tilde{r}_n - \bar{\rho}_n \tilde{q}_n$
$y_{n+1} = As_{n+1}$
$\tilde{y}_{n+1} = A^H \tilde{s}_{n+1}$
$\theta_{n+1} = \langle \tilde{s}_{n+1}, y_{n+1} \rangle$
$\zeta_{n+1} = \langle \tilde{y}_{n+1}, y_{n+1} \rangle$
if 1×1 step **then**
    $\alpha_n = \rho_n / \sigma_n$
    $\rho_{n+1} = \theta_{n+1} / \sigma_n^2$
    $\beta_{n+1} = \rho_{n+1} / \rho_n$
    $x_{n+1} = x_n + \alpha_n p_n$
    $r_{n+1} = r_n - \alpha_n q_n$
    $\tilde{r}_{n+1} = \tilde{r}_n - \bar{\alpha}_n \tilde{q}_n$
    $p_{n+1} = s_{n+1}/\sigma_n + \beta_{n+1} p_n$
    $\tilde{p}_{n+1} = \tilde{s}_{n+1}/\bar{\sigma}_n + \bar{\beta}_{n+1} \tilde{p}_n$
    $q_{n+1} = y_{n+1}/\sigma_n + \beta_{n+1} q_n$
    $\tilde{q}_{n+1} = \tilde{y}_{n+1}/\bar{\sigma}_n + \bar{\beta}_{n+1} \tilde{q}_n$
    $n \leftarrow n + 1$
else
    $\delta_n = \sigma_n \zeta_{n+1} \rho_n^2 - \theta_{n+1}^2$
    $\alpha_n = \zeta_{n+1} \rho_n^3 / \delta_n$
    $\alpha_{n+1} = \theta_{n+1} \rho_n^2 / \delta_n$
    $x_{n+2} = x_n + \alpha_n p_n + \alpha_{n+1} s_{n+1}$
    $r_{n+2} = r_n - \alpha_n q_n - \alpha_{n+1} y_{n+1}$
    $\tilde{r}_{n+2} = \tilde{r}_n - \bar{\alpha}_n \tilde{q}_n - \bar{\alpha}_{n+1} \tilde{y}_{n+1}$
    solve $Bz_{n+2} = r_{n+2}$
    solve $B^H \tilde{z}_{n+2} = \tilde{r}_{n+2}$
    $\hat{z}_{n+2} = Az_{n+2}$
    $\hat{\tilde{z}}_{n+2} = A^H \tilde{z}_{n+2}$
    $\rho_{n+2} = \left\langle \hat{\tilde{z}}_{n+2}, r_{n+2} \right\rangle$
    $\beta_{n+1} = \rho_{n+2} / \rho_n$
    $\beta_{n+2} = \rho_{n+2} \sigma_n / \theta_{n+1}$
    $p_{n+2} = z_{n+2} + \beta_{n+1} p_n + \beta_{n+2} s_{n+1}$
    $\tilde{p}_{n+2} = \tilde{z}_{n+2} + \bar{\beta}_{n+1} \tilde{p}_n + \bar{\beta}_{n+2} \tilde{s}_{n+1}$
    $q_{n+2} = \hat{z}_{n+2} + \beta_{n+1} q_n + \beta_{n+2} y_{n+1}$
    $\tilde{q}_{n+2} = \hat{\tilde{z}}_{n+2} + \bar{\beta}_{n+1} \tilde{q}_n + \bar{\beta}_{n+2} \tilde{y}_{n+1}$
    $n \leftarrow n + 2$
end if
Check convergence; continue if necessary
End LOOP

This study illustrates the applicability of the CSBiCOR method in electromagnetics to show its improved numerical behaviors in comparison with the BiCOR method. The dense linear systems considered in these experiments arise from RCS calculation of perfectly conducting objects. They are generated by applying the method of moments discretization to the electric field integral equation for surface scattering problems (see e.g. [33]). In all the experiments, we use ten discretization points per wavelength and a physical right-hand for the linear system. We precondition the linear systems using a sparse approximate inverse method based on the minimization of the Frobenius norm. The preconditioner is computed by minimizing the Frobenius-norm of the error matrix

$$\min_{M \in S} \left\| I - M\widehat{A} \right\|_F ,$$

where S is the set of matrices with a given sparsity pattern. We construct the approximate inverse M from a sparse approximation $\widehat{A}$ of the dense coefficient matrix A. The sparsity patterns of $\widehat{A}$ and M are computed in advance by selecting a fixed number of the largest entries in each column of A. Details of the preconditioner are found in [11]. The stopping criterion for solving the linear system consists in reducing the initial residual by six orders of magnitude, starting from the zero vector. In the experiments reported in [11], it was shown that this value of the tolerance is sufficient to enable a correct reconstruction of the RCS signal for engineering purposes. These experiments are run in double precision complex arithmetic in Fortran on a PC equipped with an Intel(R) Core(TM)2 Duo CPU P8700 running at 2.53GHz, and with 4 GB of RAM.

The stabilizing and robust effect of the composite step strategy on the BiCOR method can be observed according to the comparative figures presented in Table 4 and the convergence histories depicted in Fig. 1.

IV. CONCLUSIONS

We have presented an investigation of a new interesting variant of the BiCOR method for solving dense complex non-Hermitian systems of

Table 4: Number of iterations and CPU time required by BiCOR and CSBiCOR to reduce the initial residual by six orders of magnitude on some dense linear systems from electromagnetics

$Problem$	size	CSBiCOR		BiCOR	
		Iter	CPU time	Iter	CPU time
guide	1080	29	0.63	37	0.71
sphere	2430	36	2.97	44	3.46
parallelepipede	2016	32	1.95	41	2.25
cube	1800	53	2.54	67	2.95
paraboloid	1980	39	2.21	49	2.59
satellite	1701	98	4.48	126	5.10

linear equations in electromagnetics. Our approach is naturally based on and inspired by the composite step strategy taken for the CSBCG method [17, 18]. The present CSBiCOR method can be both theoretically and numerically demonstrated to avoid near pivot breakdowns and compute all the well-defined BiCOR iterates stably with only minor modifications with the assumption that the underlying biconjugate A-orthonormalizaion procedure does not break down [32]. Besides reducing the number of spikes in the convergence history of the norm of the residuals to the greatest extent, the CSBiCOR method could provide some further practically desired smoothing behavior towards stabilizing the behavior of the BiCOR method when it has erratic convergence behaviors. Additionally, the CSBiCOR method seems to be superior to the CSBCG method to some extent because of the inherited promising advantages of the empirically observed stability and fast convergence rate of the BiCOR method over the BCG method.

Since the BiCOR method is the most basic variant of the family of Lanczos biconjugate A-orthonormalization methods, its improvement will analogously lead to similar improvements for the CORS and BiCORSTAB methods, which is under investigation.

ACKNOWLEDGMENT

The authors would like to thank Professor Randolph E. Bank for showing us the “PLTMG”software package and for his insightful and beneficial discussions and suggestions.

We gratefully thank the EMC Team at CERFACS in Toulouse for providing us with some test examples used in the numerical experiments.

This research is supported by NSFC (60973015, 61170311, 11126103), Sichuan Province Sci. & Tech. Research Project (2011JY0002), the Fundamental Research Funds for the Central Universities.

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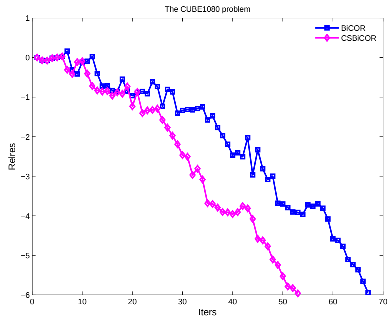
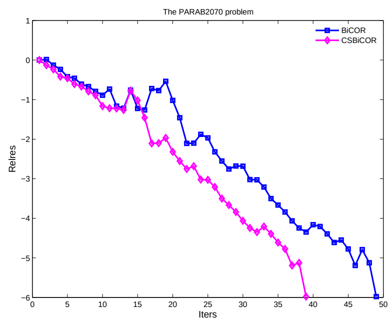
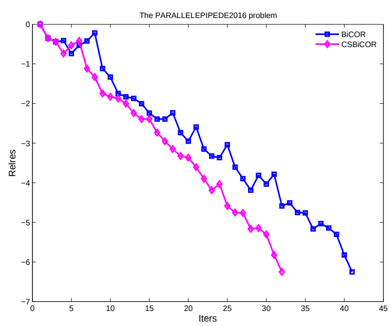


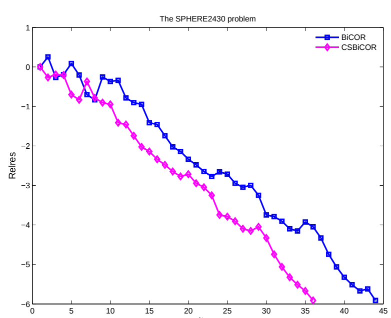
(a)



(b)



(c)



(d)

Fig. 1. Comparative experiments between the BiCOR and CSBiCOR methods on dense linear systems from electromagnetics

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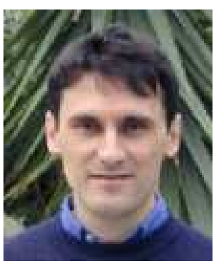

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