Optimal Location for Matching Points for Wire Modelling with MMP

P. Leuchtmann
ETH Zentrum, CH-8092 Zürich, Switzerland

Abstract: *It is a basic premise in electromagnetic field modelling that the tangential electromagnetic field on the surface of ideal conductors must vanish. When point matching is used to impose this condition, the location of the matching points must be chosen. This paper treats the special case of thin wires. It is shown that for common approximations of the wire currents, the optimum locations of the matching points are well defined and that forcing the boundary condition beside these well defined matching points would increase the errors. For both piecewise-linear and staircase approximation of the current, explicit formulae for the optimum location of the matching points are given.*

Introduction

To find the current in a wire one uses an equation of the form

$$E_z = 0, \tag{1}$$

where E_z is the longitudinal tangential component of the electric field on the surface of the wire. If the wire is thin, E_z is small compared to other field components at the same location. Nevertheless it is the only component to be matched. For an approximate solution, E_z will never be zero on the whole surface of the wire, but only in particular points. The most simple procedure for finding an approximate solution of (1) is point-matching, where the wire is segmented and then one matching point is placed in the center of each wire segment. MMP [1] uses an extended point matching technique and lets it up to the user to place any number of matching points. (A minimum of one matching point per segment is necessary.) Equations (1) are then solved in the sense of least squares.

It has been found that the results obtained by the MMP-program are unstable in certain cases. In particular, an increase of the number of matching points did *not* lead to better results [2]. A closer look at the error of (1) along wires shows a great regularity of the error distribution.

In this paper, we shall show that the behavior of the error of E_z along the wire is well defined. Depending on the expansion functions of the current, there are zeros of E_z at certain points. Besides these points, E_z should never be forced to be zero, since such a forcing would affect the current in a highly unintended manner.

The following scheme gives the principal way to find the results:

- Calculate of E_z of an *arbitrary* current $I(z)$ on a wire (analytically)
- Approximate $I(z)$ by a staircase function (like MININEC) or a piecewise-linear function (MMP) and calculate $\tilde{E}_z$ of the approximated current (analytically)
- Define the systematic error as the difference $E_z - \tilde{E}_z$
- Use zeros of the systematic error as optimum locations for matching points

The reason why the zeros of the systematic error are optimum locations for matching points is quite obvious: if one would try to push down the error at a location, where the systematic error is nonzero, the error of the solution is increased, since the systematic error *cannot* be eliminated.

As a principal result, we shall find that these optimum locations are highly independent of the global shape of $I(z)$, but only a consequence of the local 'misfit' of the expansion functions vs. the exact field. As a matter of fact, the optimum locations of the matching points depend on both the expansion functions and the unknown field. Therefore, both of them have to be considered to find good rules for the placing of the matching points. To get rid of this highly contradictory situation — the unknown field must be known — we shall consider that class of unknown current distributions which can be represented by a converging Taylor series. It is worth noting that the behavior of the expansion functions alone is not sufficient to find the optimum locations of the matching points.

Taylor-Representation of the Exact Current- and Charge Distribution

Let us consider a straight (thin) z-oriented wire with radius r which is carrying a current $I_{\text{exact}}(z)$. On a piece of finite length L of this wire, the current may be represented by a power series[1]

$$I_{\text{exact}}(z) \approx I(z) = \sum_{n=0}^{N} \alpha_n z^n \tag{2}$$

We assume, that the finite power series (2) is able to represent the function $I_{\text{exact}}(z)$ within the length L so that no further error considerations are necessary. Later on, we shall divide up the length L into segments and we shall focus on the local effects of this segmentation. From this point of view, the assumptions are reasonable.

Due to the charge conservation principle, the current distribution $I(z)$ is accompanied by the charge distribution[2]

$$q(z) = \frac{1}{i\omega}\frac{\partial I}{\partial z} = \frac{1}{i\omega}\sum_{n=1}^{N} n\alpha_n z^{n-1} \tag{3}$$

As a further assumption, we state, that L is much smaller than the wavelength λ, so that the electric field $\vec{E}$ may be calculated from $q(z)$ by means of electrostatics. Since we are only interested in local effects, we do not consider the field, which is produced by the currents and charges beyond the length L. This field is not necessarily small but almost constant along L. In particular, it does not vary on a segment of L. For this reason, it has no importance for the location of the matching points relative to the segment.

The z-component of the $\vec{E}$-field is obtained from $q(z)$ by Coulomb's integral:

$$E_{z0}(r,z) = \frac{1}{4\pi\varepsilon}\int_{-\frac{L}{2}}^{\frac{L}{2}} \frac{q(z')(z-z')}{\sqrt{r^2+(z-z')^2}^3}dz' = \frac{1}{4\pi i\omega\varepsilon}\sum_{n=1}^{N} n\alpha_n \int_{-\frac{L}{2}}^{\frac{L}{2}} \frac{(z')^{n-1}(z-z')}{\sqrt{r^2+(z-z')^2}^3}dz' \tag{4}$$

[1] Convergence is given, if L is sufficiently small and $I(z)$ is smooth.

[2] Time dependency of $e^{-i\omega t}$ is assumed!

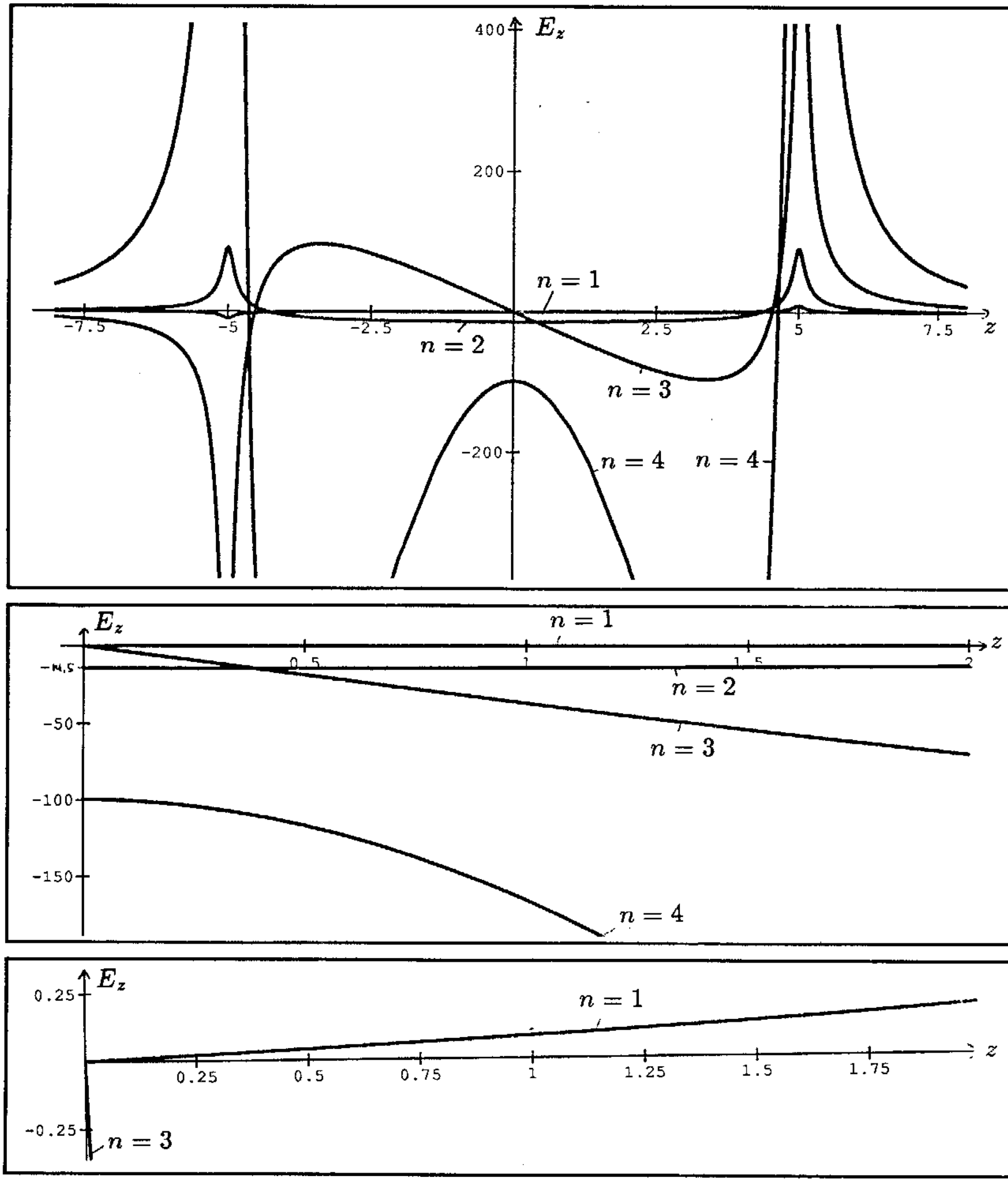


Figure 1 Different current distributions cause different distributions of the z-component of $\vec{E}$. The pictures show the four parts of E_z for $n = 1 \ldots 4$ with $\alpha_n = 1$, $r = 0.1$ and $L = 10$. At the middle and at the bottom, details of the interval $0 \leq z \leq 2$ are shown. The numerical values are obtained from (3), omitting the factor $1/(4\pi i\omega\varepsilon)$. We focus on the area around $z = 0$.

The integral on the right hand side may be solved analytically for any n. These solutions and the plots of these functions have been obtained using Stephen Wolfram's 'mathematica'. [3] (See fig. 1!)

Due to the superposition principle, each term may be treated alone. Provided that the optimum locations for matching points being the same for each n, these locations are optimum for any $I(z)$. This is exactly what we shall find.

At the ends of L, we assume a constant continuation of the current. Otherwise, the steps in the current distributions would cause point charges at the ends, which would disturb the results in producing an additional large electric field.

Staircase Approximation

The wire of length L is cut into J segments of equal length $l = L/J$. Then the function $I(z)$ is approximated by a staircase. (See fig. 2!) The same approximation is used in MININEC [4]. Note that the point-matching technique which is described in this paper is different to the (more sophisticated) procedure used in MININEC.

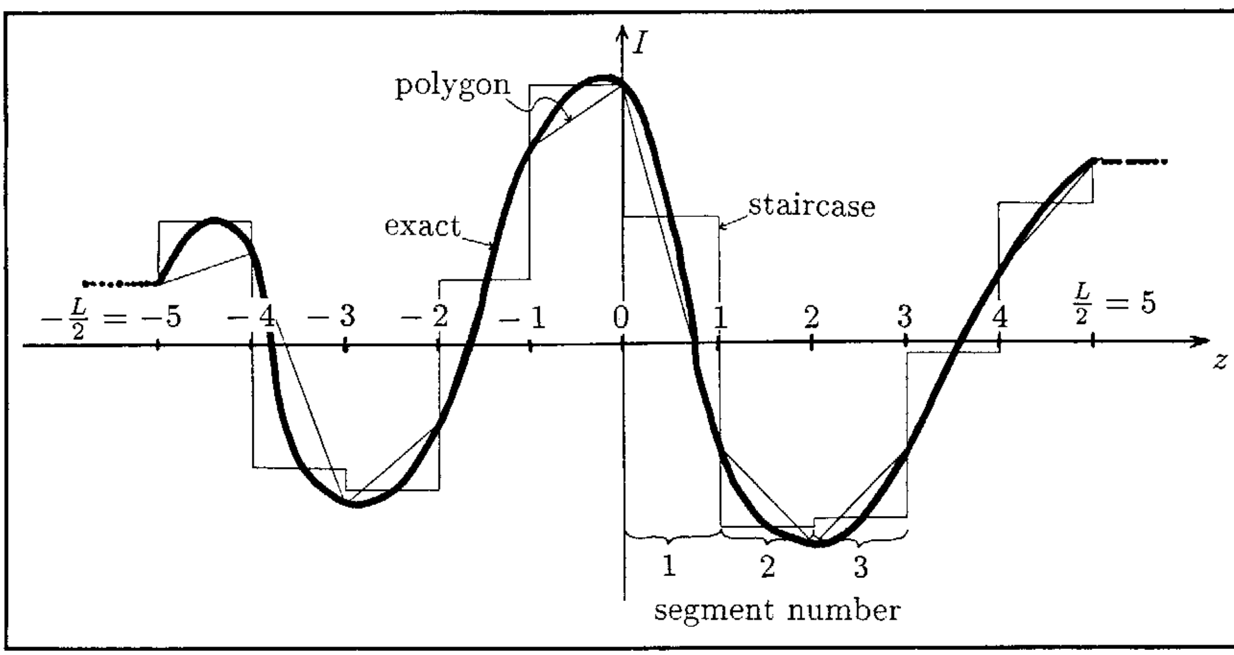


Figure 2 The smooth current $I(z)$ is approximated by staircase (like MININEC) or piecewise-linear functions (MMP). Each type of approximation is accompanied by its typical (erroneous) charge distribution. It is this charge distribution which causes a relatively big error in the local distribution of the longitudinal electric field.

Let[3]

$$z_j := jl; \qquad -\frac{J}{2} \leq j \leq \frac{J}{2} \tag{5}$$

be the z-coordinate of the boundary between the j-th and the $(j+1)$-th segment. In the

[3] For simplicity's sake, J is assumed to be even!

j-th segment the current is constant and equal to

$$I_j := \begin{cases} I\left(z_j - \frac{l}{2}\right) & \text{for } -\frac{J}{2}+1 \leq j \leq \frac{J}{2} \\ I(z_j) & \text{for } j = -\frac{J}{2} \\ I(z_{j-1}) & \text{for } j = \frac{J}{2}+1 \end{cases} \tag{6}$$

I.e., we assume the approximation being exact *in the center of the segment*. The special definition of the last two currents is due to the constant continuation beyond L. This current distribution leads to *point charges* Q_j located at the segment boundaries z_j. One has

$$Q_j = \frac{1}{i\omega}(I_{j+1} - I_j) \tag{7}$$

From that, we obtain for the z-component of the resulting electric field

$$E_{zs}(r,z) = \frac{1}{4\pi\varepsilon} \sum_{j=-\frac{J}{2}}^{\frac{J}{2}} \frac{Q_j \cdot (z - z_j)}{\sqrt{r^2 + (z - z_j)^2}^3} = \frac{1}{4\pi i\omega\varepsilon} \sum_{j=-\frac{J}{2}}^{\frac{J}{2}} \frac{(I_{j+1} - I_j)(z - z_j)}{\sqrt{r^2 + (z - z_j)^2}^3} \tag{8}$$

The sum has $J+1$ terms, because a line of J segments has $J+1$ boundaries. Figure 3 shows the big variation of E_z along the wire and in particular along a single segment.

Let us define now the *systematic error* F_s^n as the difference between the "exact" result (4) and the approximation (8):

$$F_s^n(r,z) = E_{zs}^n(r,z) - E_{z0}^n(r,z) \tag{9}$$

where the upper index n denotes the power of the current: $I(z) = z^n$. It is clear, that F_s^0 must vanish, since the staircase approximation of a constant function is exact. Figure 4 shows the functions F_s^n along the segments 1 and 2. The values around $z = 0$ are not typical, because of the superimposition of a non local effect due to the symmetrical situation: With *odd* powers of the current distribution (i.e., even powers for the charge distribution), the far action of the discretization errors are cancelled due to symmetry. However, with *even* powers of the current distribution (i.e., odd powers for the charge distribution), these actions add together and cause a vertical shift of the curve. For these reasons, the behavior in the second segment is more typical.

In general we can say that *for all powers* n, the systematic error has a relatively flat zero in the middle of the segment. For that reason, it is optimum to place the matching points in the center of the segment. The second zero (close to the segment boundaries) is not useful for this purpose, because the error function is much too steep in this area. Considering the exact values of E_z (see fig. 1!), we find that the error is not negligible at all.

As a preliminary conclusion we state: For staircase approximation it is optimum to place matching points in the center of the segments — and nowhere else. This is due to the fact that the systematic error *cannot be eliminated*. A placing of a matching point at a position with a high systematic error would increase the global error of the field in such a sense, that the sum of the two vanishes.

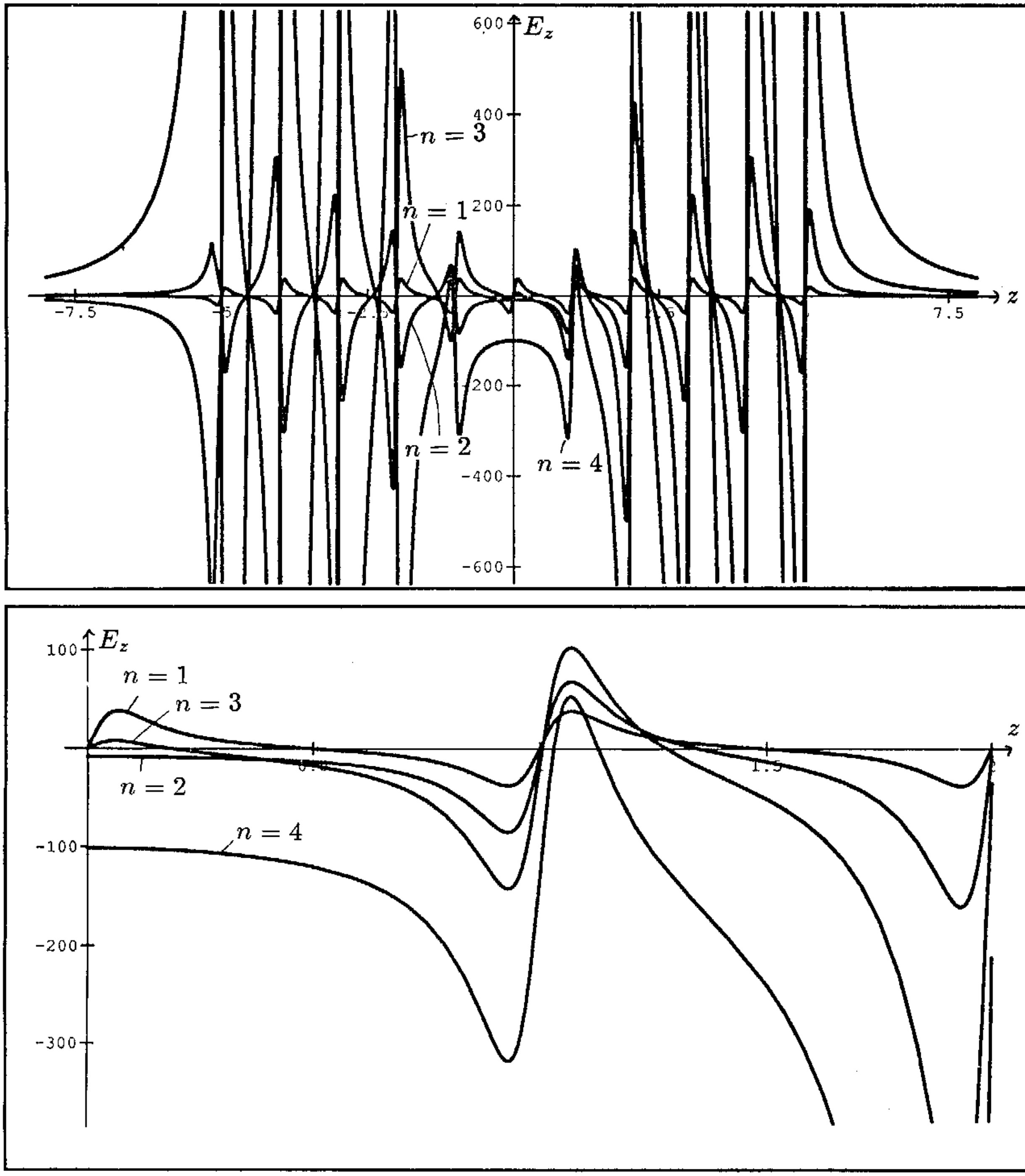


Figure 3 The staircase approximation of the current gives a much less smooth behavior of E_z than the smooth current in fig. 1. See particularly the big values in the center (near $z = 0$)! The picture shows the four parts of E_z for $n = 1 \ldots 4$ with $\alpha_n = 1$, $r = 0.1$, $L = 10$ and $J = 10$ ($\rightarrow$ segment length $l = 1$). At the middle and at the bottom, details of the interval $0 \leq z \leq 2$ are shown. The numerical values are obtained from (9), omitting the factor $1/(4\pi i\omega\varepsilon)$.

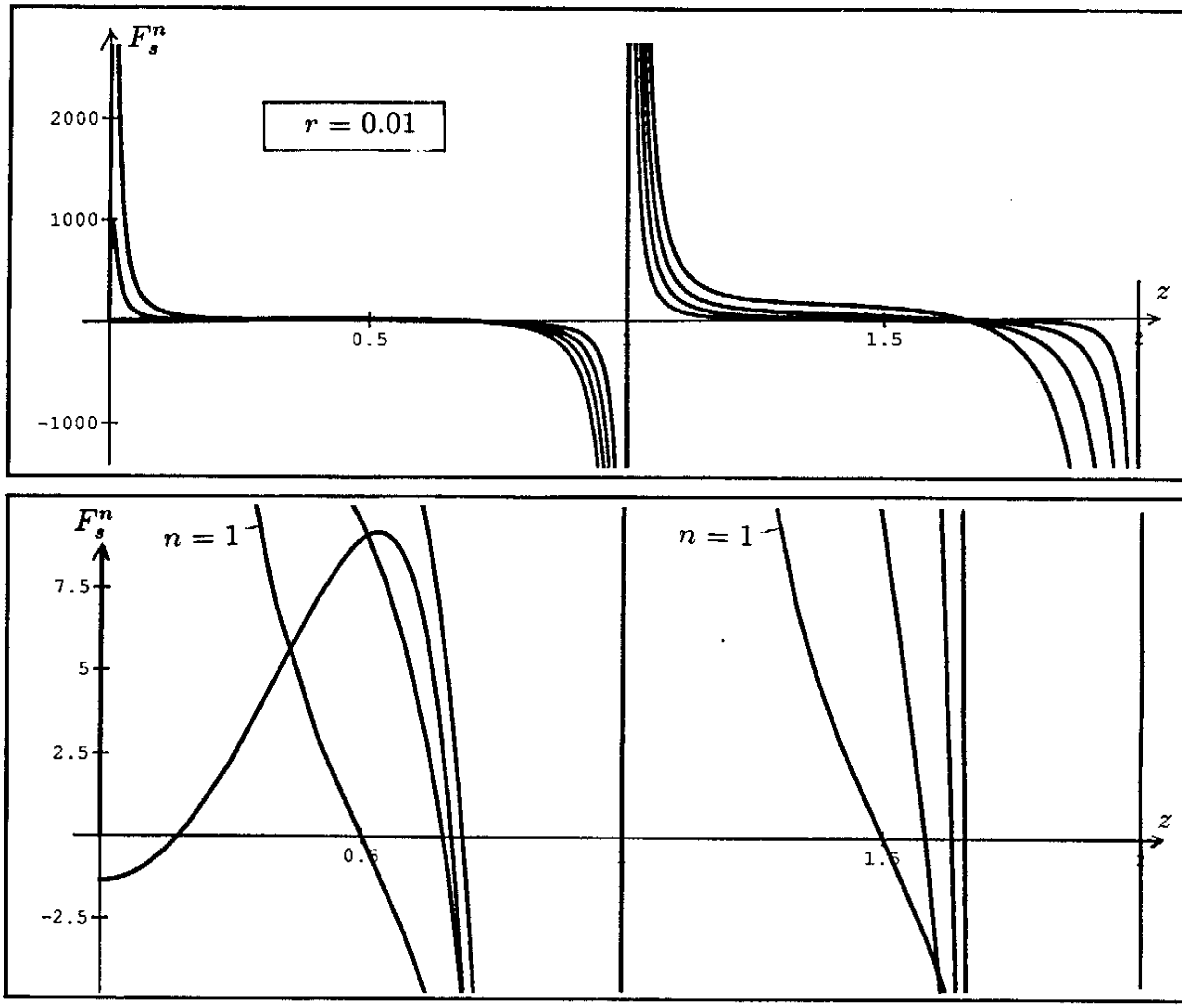
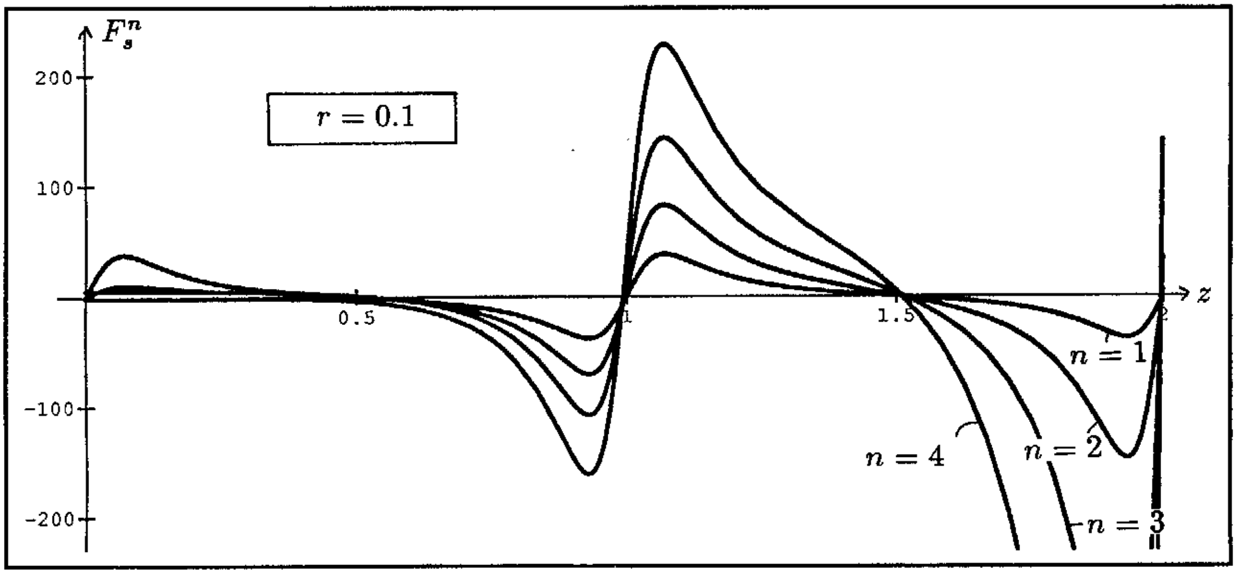


Figure 4 The systematic error F_s^n of E_z (using a staircase approximation of the current) depends on the wire radius r. Since the length of the segments has been kept constant, the ratio length/radius of a segment is actually varied. The change of the vertical scale at the bottom of each picture shows the poor flatness of the zero. As expected, the absolute value of the errors decreases with increasing radius. Note that the numerical values in fig. 1 are related to a wire radius of $r = 0.1$.



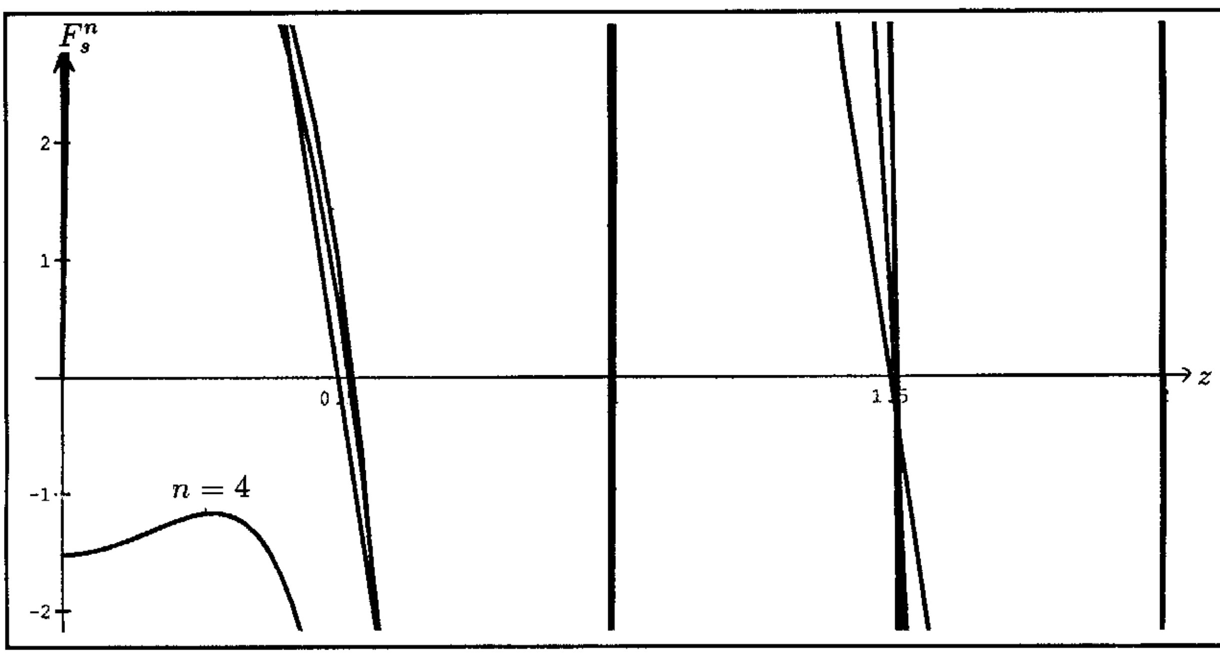


Figure 4 Continued, see above!

Piecewise-linear Approximation (MMP)

As it has been already done in the previous section, the wire of length L is split up into J segments of equal length $l = L/J$. Other than before, the function $I(z)$ is now approximated by a piecewise-linear function[4]. In the j-th segment, the current is a linear function:

$$I_j(z) = I(z_{j-1}) + \frac{I(z_j) - I(z_{j-1})}{l}(z - z_{j-1}) \tag{10}$$

[4] In the MMP-programs, $I(z)$ is approximated using the functions $\cos kz$ and $\sin kz$, where k is the wave number of the surrounding medium [2]. Since the segments are small compared to wavelength, these two functions tend to 1 and z.

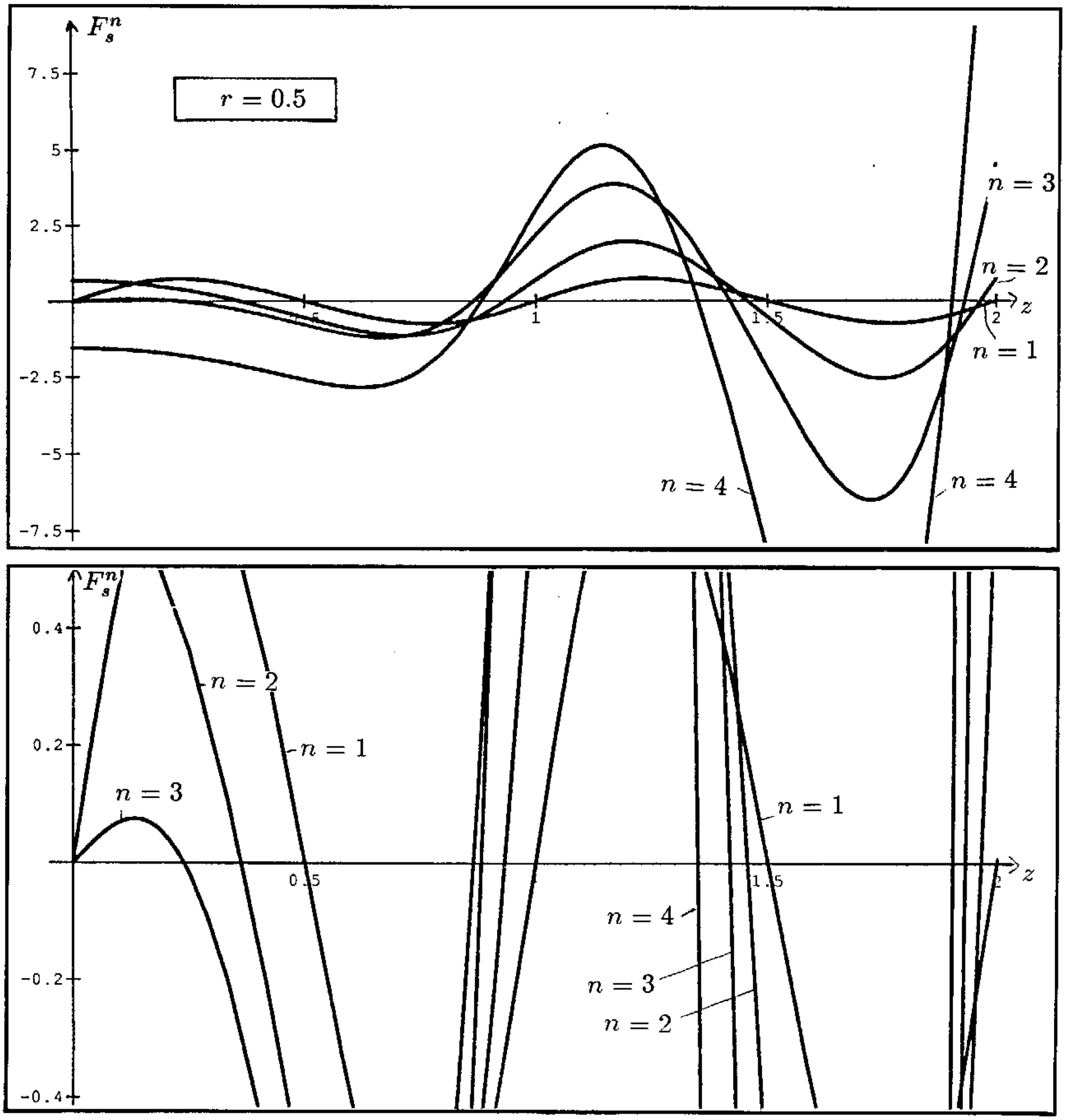


Figure 4 Continued, see above!

i.e., we assume that the approximation is exact at the *segment boundaries*. (See fig. 2!) This approximation of the current distribution leads to a *constant charge distribution* q_j on the j-th segment:

$$q_j = \frac{1}{i\omega}\frac{I(z_j) - I(z_{j-1})}{l} \tag{11}$$

and the z-component of the resulting electric field is

$$E_{zl}(r,z) = \frac{1}{4\pi\varepsilon}\sum_{j=-\frac{J}{2}+1}^{\frac{J}{2}}\int_{z_{j-1}}^{z_j}\frac{q_j\cdot(z-z')}{\sqrt{r^2+(z-z')^2}^3}dz'$$

$$= \frac{1}{4\pi i\omega\varepsilon}\frac{1}{l}\sum_{j=-\frac{J}{2}+1}^{\frac{J}{2}}\Big(I(z_j) - I(z_{j-1})\Big)\int_{z_{j-1}}^{z_j}\frac{z-z'}{\sqrt{r^2+(z-z')^2}^3}dz' \qquad (12)$$

Obviously, this sum has one term less than the one in (8). Figure 5 shows the variation of E_z which is now much smaller than the one shown in fig. 3.

Again, the systematic error F_l^n is defined as the difference between the "exact" result (4) and the approximation (12):

$$F_l^n(r,z) = E_{zl}^n(r,z) - E_{z0}^n(r,z) \qquad (13)$$

where the meaning of the upper index is the same as in (9). It is clear, that both F_l^0 and F_l^1 vanish, since the piecewise-linear approximation of a linear function is exact.

Considering the systematic error we find — as in the staircase approximation — a great regularity, which is qualitatively equal for all powers. On the other hand, there is a fundamental difference: The zeros of the systematic error do not occur in the middle of the segments, but more close to the segment's boundaries. (See figs. 6 and 7!)

Due to the global effect in the middle of L (vertical shift of the even order curves and a zero at $z = 0$ for the odd orders due to symmetry), the first zero after $z = 0$ is not used. (For the numbering of the zeros, see fig. 6 at the bottom.) The exact location of the zeros of the systematic error is a function of the radius r, the order n and the number of the zero. Figure 8 shows this locations as a function of the wire radius r. Note the great regularity!

Due to the fact that the lowest order (here $n = 2$) is most important and, on the other hand, curves for different zero numbers are — at least for small radii, i.e. thin (long) segments — almost equal, it makes sense to use e.g. the third zero of the second order as a reference. The bigger variation at r/l-ratios more than $1/3$ does not matter, since the absolute values of the systematic errors decrease with increasing wire radius.

We conclude: For piecewise-linear approximation, matching points in the center of the segments should be avoided. There are *two optimal locations* for each segment, symmetrically placed at a distance d away from the segment's boundaries.

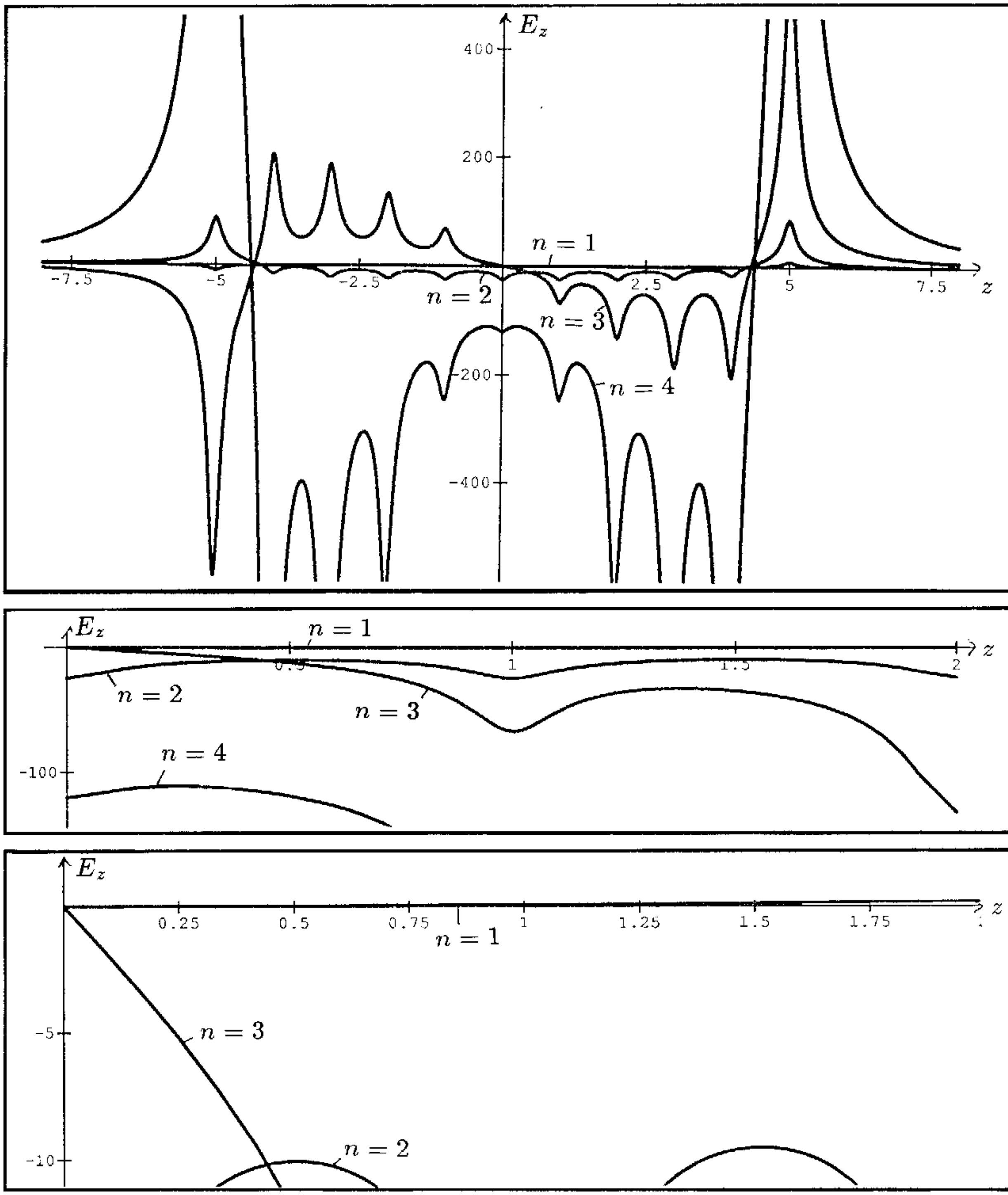


Figure 5 Compared with fig. 1 the piecewise-linear approximation of the current still gives an unquiet behavior of E_z, but the improvement with respect to the staircase approximation (fig. 3) is obvious. The pictures show the four parts of E_z for $n = 1 \ldots 4$ with $\alpha_n = 1$, $r = 0.1$, $L = 10$ and $J = 10$ ($\rightarrow$ segment length $l = 1$). At the middle and at the bottom, details of the interval $0 \leq z \leq 2$ are shown. The numerical values are obtained from (12), omitting the factor $1/(4\pi i\omega\varepsilon)$.

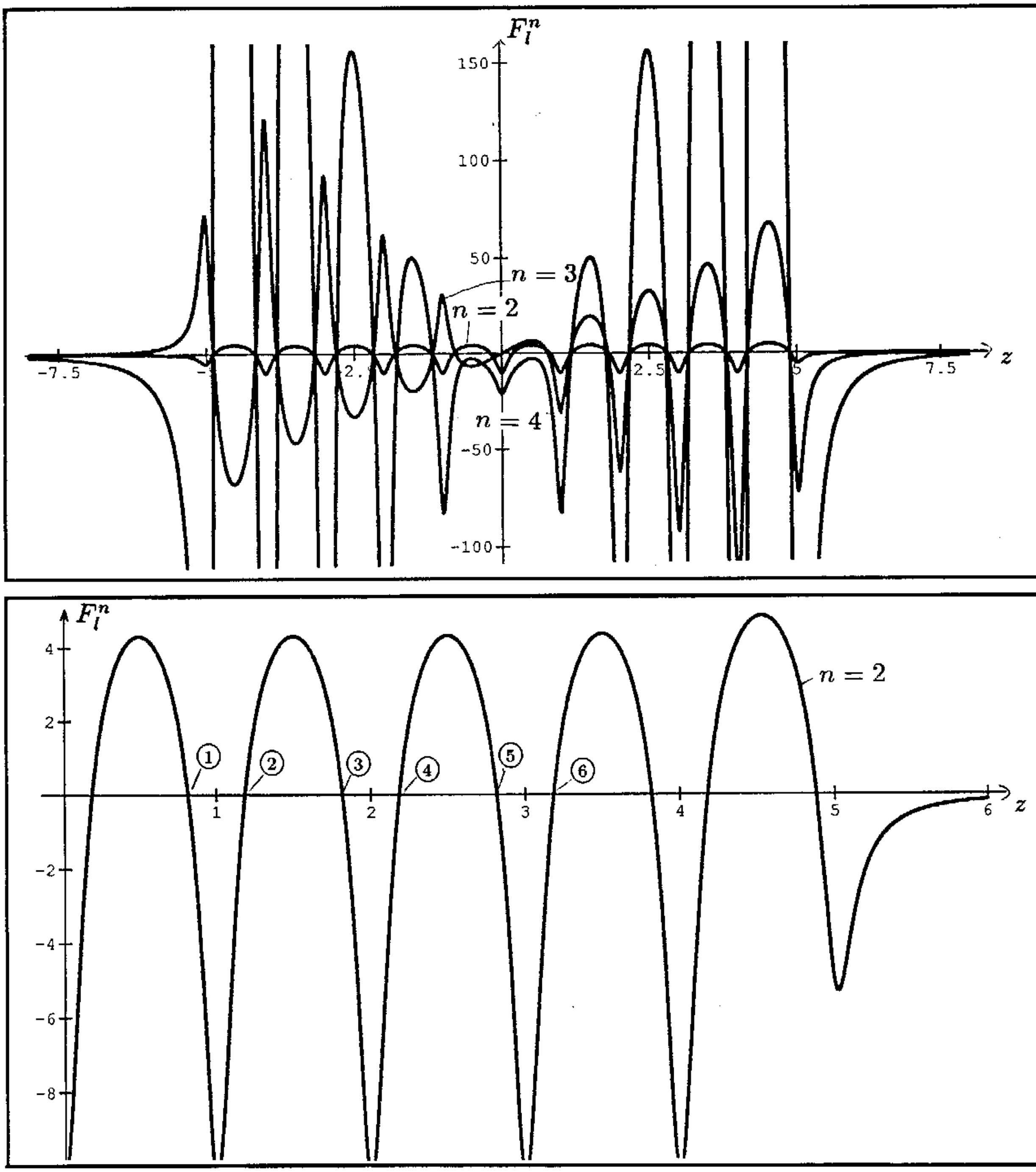


Figure 6 The systematic errors F_l^n (in the case of a piecewise-linear approximation for the current) are very similar for different powers. At the top, the figure shows the systematic errors of the order $n = 2 \ldots 4$. At the bottom, the most important order 2 is given individually. The numbering of the zeros (numbers in circles) starts at the second one. This is due to the symmetry caused fact, that the even curves (from 4-th order on) are vertically shifted down around $z = 0$, while odd curves have a zero at $z = 0$. The distance between the zeros and the segment boundaries (here integer valued z's!) is the important parameter for placing matching points.

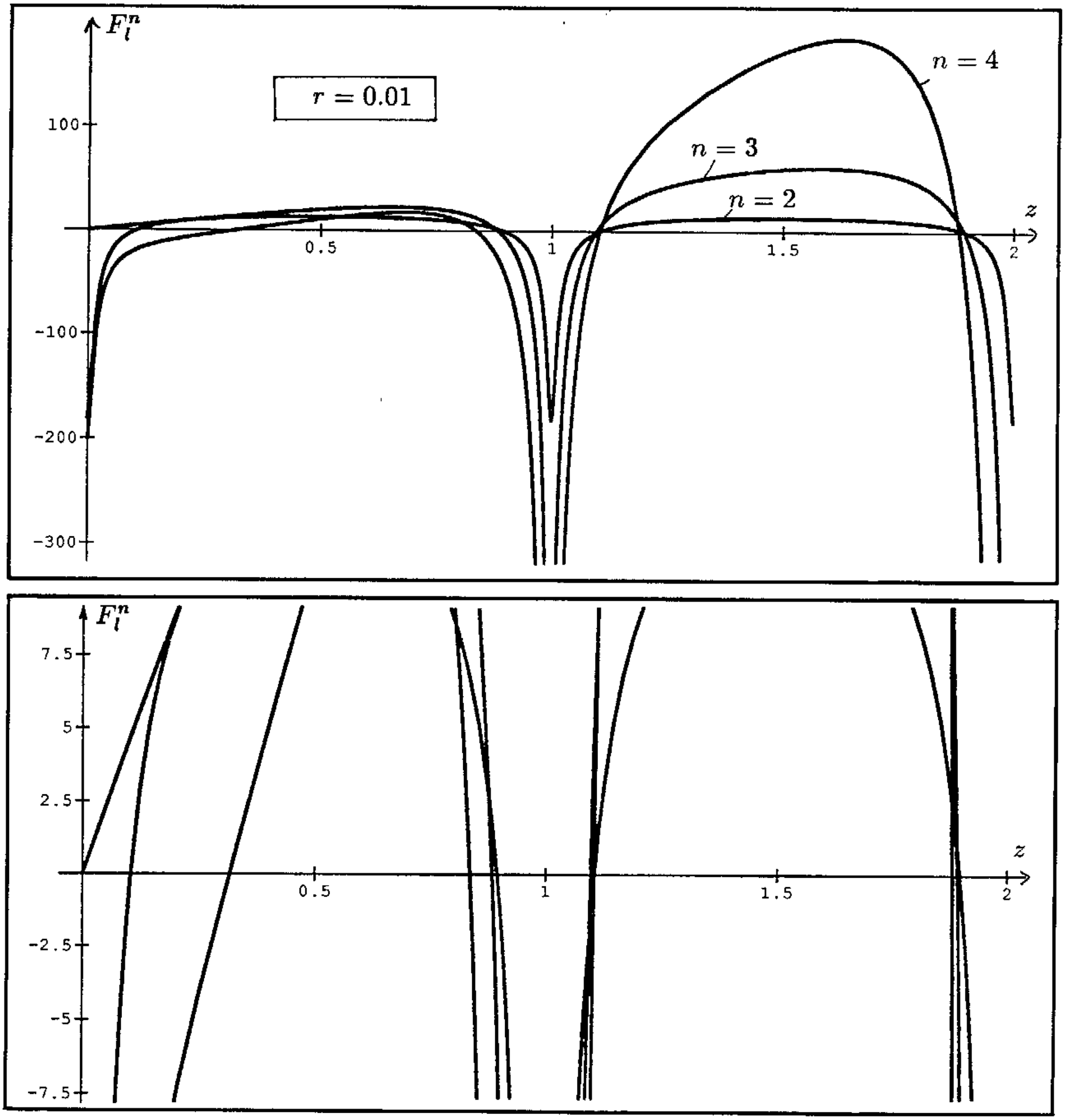
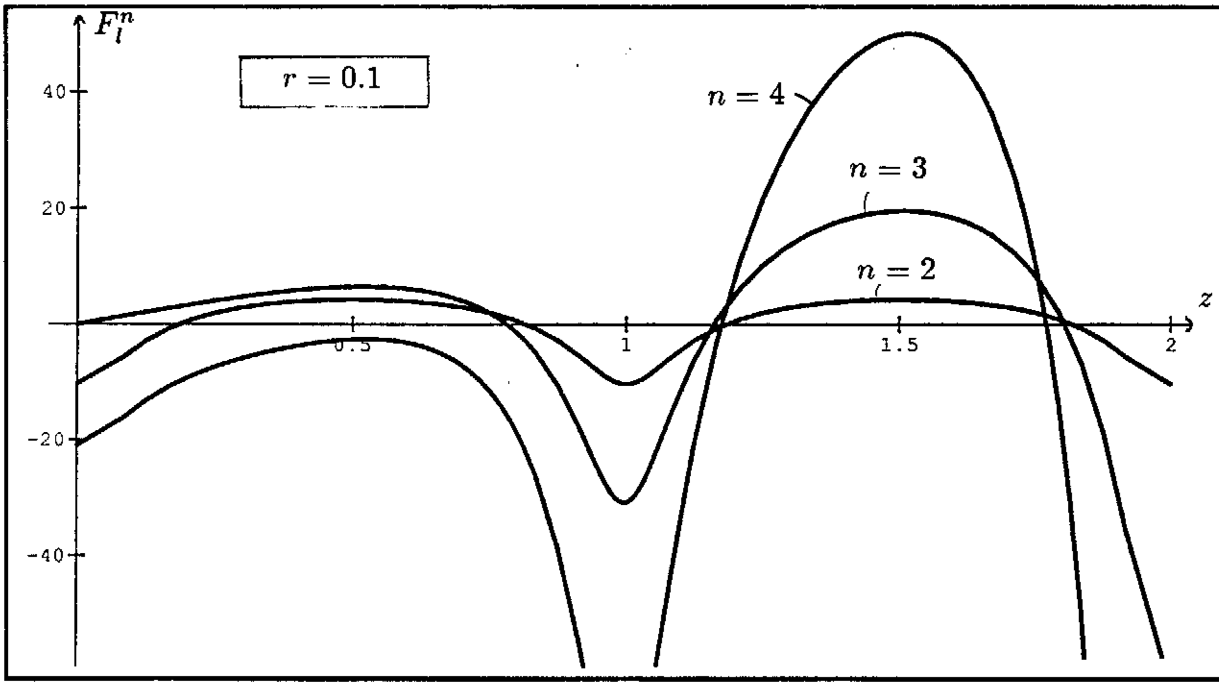


Figure 7 The systematic error F_l^n of E_z when using a piecewise-linear approximation of the current depends on the wire radius r. Since the length of the segments has been kept constant, the ratio length/radius of a segment is actually varied. Note that the numerical values in fig. 1 are related to a wire radius of $r = 0.1$.



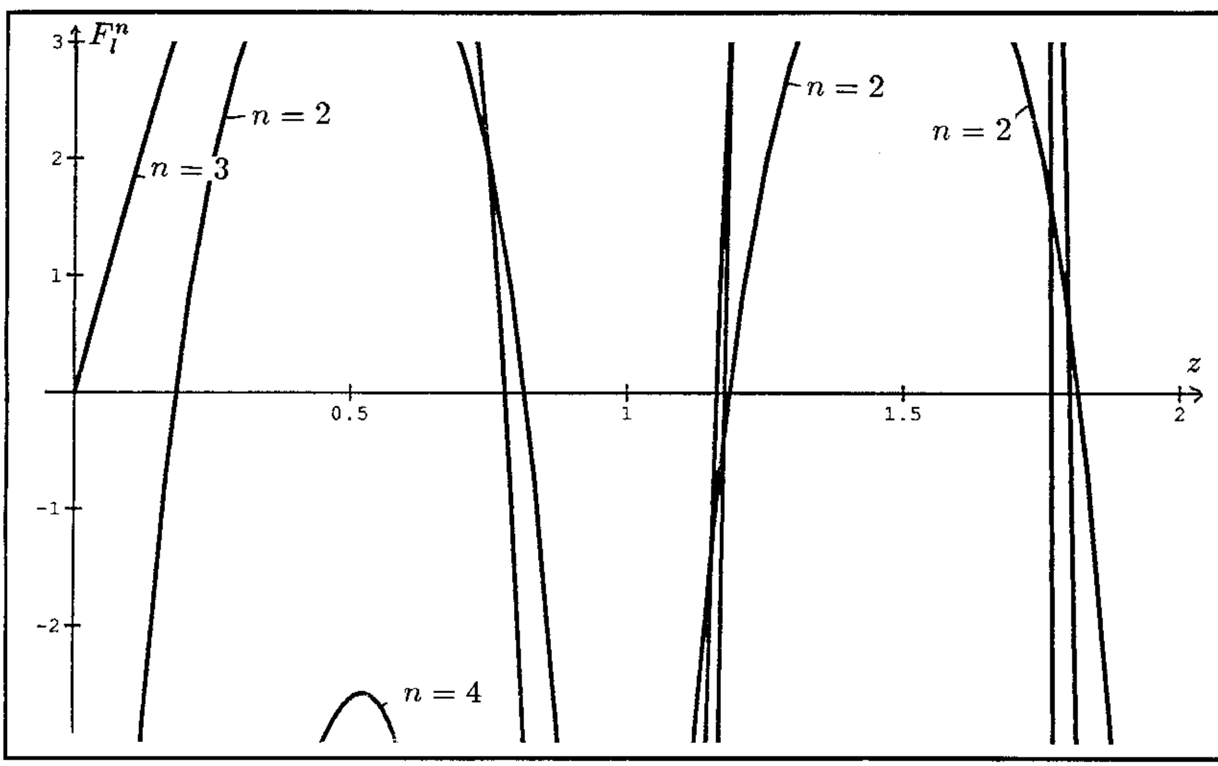


Figure 7 Continued, see above!

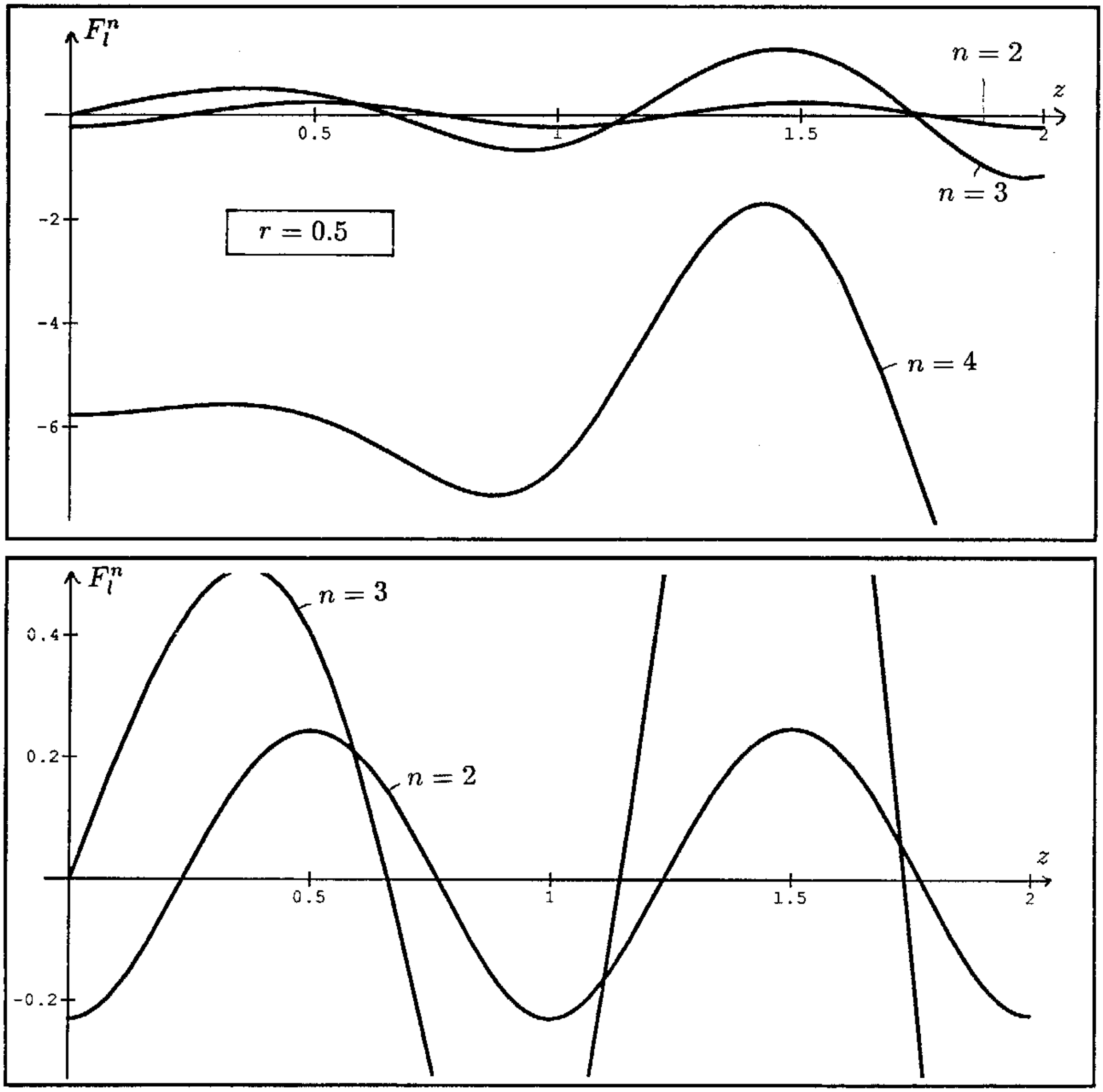


Figure 7 Continued, see above!

Polynomial Approximation of the Zero's Relative Position

We have chosen as a reference the zero number 3 of the quadratic term of the current. It is located at a distance d_3 away from the segment's boundary. This distance d_3 is a function of the wire radius r: $d_3 = d_3(r)$. 'Mathematica' [3] does the job to expand the function $d_3(r)$ into a power series of m-th order. We normalize the segment length to 1 and find:

$$\begin{aligned} d_3^2(r) &= 0.1246832 + 0.572177r - 0.7090428r^2 \\ d_3^3(r) &= 0.1085357 + 0.893254r - 2.043528r^2 + 1.480294r^3 \\ d_3^4(r) &= 0.0978577 + 1.245951r - 4.679222r^2 + 8.298414r^3 - 5.672312r^4 \end{aligned} \tag{14}$$

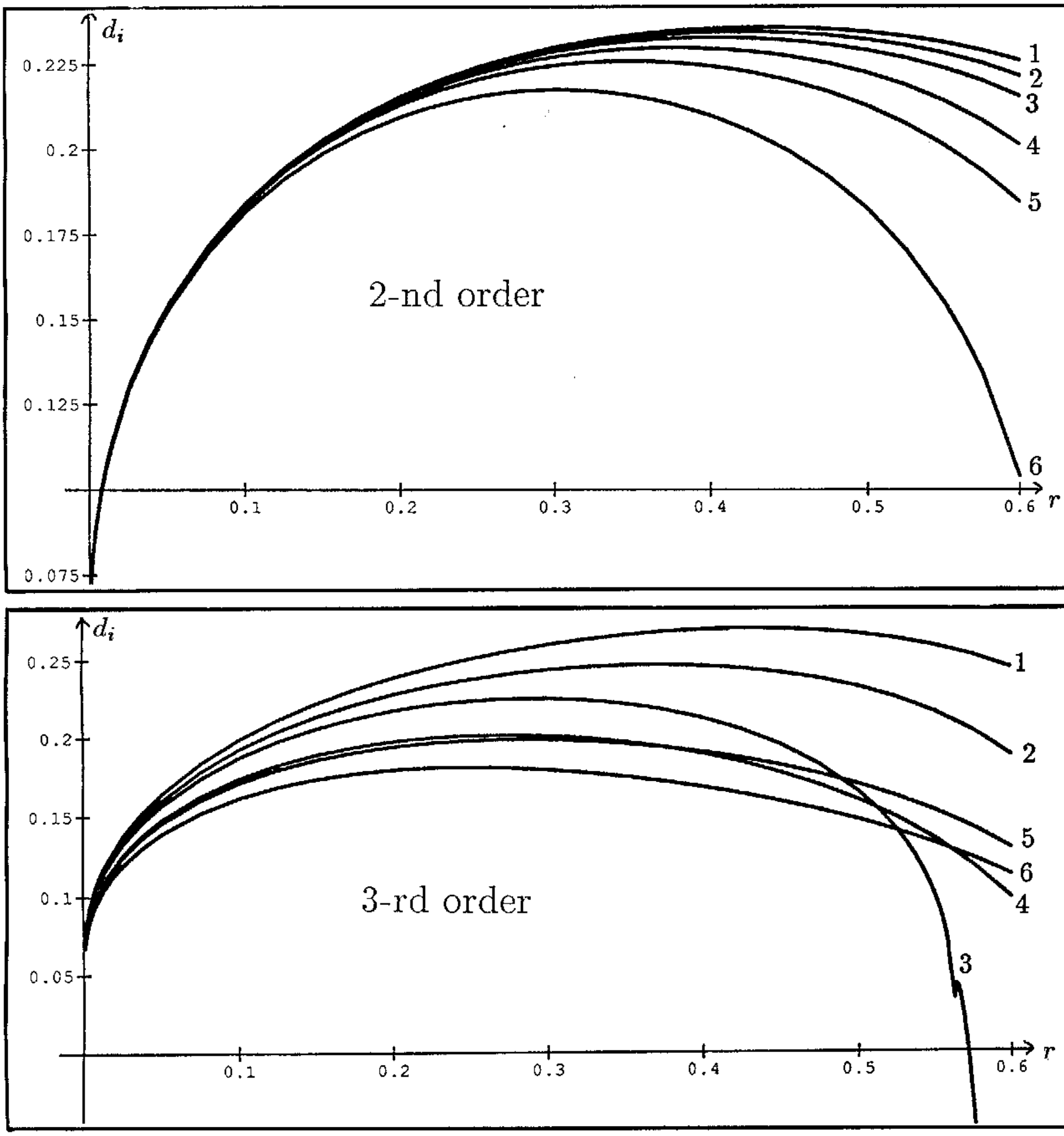


Figure 8 The relative position of the zeros of the systematic error varies only little. The pictures show the distance d_i between the i-th zero and the next segment boundary as a function of the wire radius r. Only the second and the third order is shown. The position of the zeros with higher numbers are inaccurate due to end effects of the model (only 10 segments).

The upper index denotes the order of the approximations, which are valid within the interval $0.001 \leq r \leq 0.6$.

The deviation of the true value is maximum at the border of this interval. We denote the biggest deviation downwards (approximation smaller than true value) with l_m and the biggest deviation upwards (approximation bigger than true value) with u_m. Mathematica delivers

$$
\begin{aligned}
l_2 &= 0.05406; & u_2 &= 0.00926 \\
l_3 &= 0.03824; & u_3 &= 0.00634 \\
l_4 &= 0.02791; & u_4 &= 0.00471 \\
l_5 &= 0.02208; & u_5 &= 0.00396
\end{aligned}
\tag{15}
$$

Comparing these numbers with the true value (≈ 0.2) and considering the variation between the different zeros (see fig. 8!), an approximation of $d_3(r)$ of 4-th order is sufficient.

Summary

We have found that the positions of matching points along thin wires must be chosen carefully. Depending on the segmentation and the type of the approximation of the currents in the wire, optimum matching point positions are different. The results have been found by examining the error produced by a 'best fit' of some expansion set (staircase and piecewise-linear expansion functions) to an *arbitrary* current distribution.

The free choice of matching points in the MMP-code is an error source which could be avoided. Using MMP one should set two matching points for each segment, symmetrically located at a distance d away from the segment's boundaries (see eq. (14)!). This choice is optimum at all locations where the current $I(z)$ is either concave, convex or linear.

As a further result, we find that the point matching technique is preferred to any other method to satisfy boundary condition (1), as long as the current is a) approximated by fixed functions and b) strictly coupled to the electric field through Maxwell's equations. This second point is not fulfilled, e.g., by MININEC: it uses a piecewise-constant approximation of the *charge* distribution, which would be related to a piecewise-linear current distribution.

References

[1] Ch. Hafner, *The generalized multipole technique for computational electromagnetics*, Artech House Books, 1990.

[2] P. Leuchtmann, "Thin wire feature for the MMP-code," in *6th Annual Review of Progress in Applied Computational Electromagnetics (ACES), Conference Proceedings*, (Monterey), Mar. 1990.

[3] S. Wolfram, *Mathematica*, Addison-Wesley Publishing Company, Inc., 1988.

[4] *NEEDS, (Numerical Electromagnetic Engineering Design System)*, distributed by the Applied Computational Electromagnetics Society.