Adaptive Meshing in Two-Variable Static Problems with Field Based Error Estimators Using Edge and Facet Elements

P. Girdinio, P. Molfino, G. Molinari, M. Nervi

Dipartimento di Ingegneria Elettrica, Universita' di Genova
11a, Via Opera Pia - I-16145 Genova, Italy

Abstract— **In this paper a family of field-based error estimators for Finite Element analysis of electrostatic and magnetostatic problems in plane and axisymmetric geometries is presented. For the error estimation in magnetostatics, each element is divided in three sub-elements using an edge element approach, whereas for electrostatic problems a subdivision using facet elements is used. The methods and the numerical techniques are described, comparisons with known solutions are performed, some examples of application in cases of practical interest are reported and the obtained results are briefly discussed.**

I. Introduction

Techniques for error estimation in Finite Element solutions of field problems and for automatic mesh modification to guarantee a user-defined error level have been proposed for many years, in all areas of engineering analysis [1]. Today they are becoming more and more interesting, particularly in Electromagnetic Analysis, because of their strategic importance in allowing reliable Finite Element solutions without specific user skills, in turn essential for automation of design environment, device optimization and inverse problem applications, increasingly required in designing advanced electromagnetic devices [2]. Many techniques for the estimation of errors have been proposed, but it has also been shown that the efficiency of each technique is significantly dependent on the specific problem to be solved [3-6].

In this paper a family of error estimators, resulting among the most efficient for electrostatic and magnetostatic problems in the range developed and tested by the authors, is presented. All error estimators are developed for first order, triangular meshes, operate on a single element at a time, and are available both for plane and axisymmetric geometries. The estimators present some analogies with the "Local Error Problem" approach, developed by the authors, that has been found very efficient with respect to other approaches [5]. The error estimators have been developed casting the errors directly in terms of fields, the quantity of more direct physical interest, defining a "Local Field Error Problem" [7].

The distinctive feature of the family of error estimators presented in the paper is the representation of the field over the element, that is related to the type of potential, scalar or vector, used to derive the field. This ensures the ability to capture effectively the biggest error contribution, connected to the normal derivative of the potential from which the field is derived. The representation of the error variables is then in terms of "edge" or "facet" elements for solutions derived from vector or scalar potentials, respectively [8]. To ensure a practical representation of these variables, a "Whitney forms" description has been used [9].

The error estimators defined in this way have then been used to build up an adaptive meshing strategy, based on the subdivision of elements with high errors, usually termed "h refinement" [10,11]. All algorithms have been implemented in the two-dimensional Finite Element development environment CEDEF, also used for other error estimation and adaption procedures developed in the authors' group [12].

In the paper the structure of error estimators is described, the adaptive meshing strategy used is outlined, the validation of their performance in cases with known solutions is performed, and results of usage in cases of practical industrial interest are reported and briefly discussed.

II The Local Field Error Approach

The "Local Field Error Problem" for the electrostatic and magnetostatic cases is derived from the proper subset of Maxwell equations, defining a governing set in terms of curl and divergence of the numerical error [6,7,13]. The estimation is based on the solution of a differential problem over each element using as "error sources" the jump in normal derivatives of potential along element edges. In a similar way, it is possible to derive a formulation defining again a local problem over each element; this will assume as unknowns the errors in the evaluation of field quantities, with "error sources" derived from the jumps in the normal derivatives of potential applying Ampere's and Gauss' laws.

The development of an "a posteriori" error estimate based on a "Local Field Error" approach requires the definition of the error estimate unknown in term of fields, the use of Maxwell equations in differential form and the definition of a closed domain where Dirichlet-like boundary conditions are applied. This implies that the unknown vector entity is uniquely defined by Helmholtz's theorem.

A. Electrostatic Problems

The evaluation of the estimate of numerical errors in FEM solutions of electrostatic problems can be carried out by defining an adjoint problem, in terms of errors in electric field evaluation, where the unknowns are the components of the error vector $\vec{e}$, defined as difference between the "true" electric field $\vec{E}_t$ and the computed one $\vec{E}_c$, that is:

$$\vec{e} = \vec{E}_t - \vec{E}_c \tag{1}$$

The error equations are derived from the electrostatic subset of Maxwell equations applied to the "true" electric field $\vec{E}_t$. This leads to the set of vector equations in terms of the error $\vec{e}$:

$$\nabla \cdot \vec{D} = \delta \Rightarrow \nabla \cdot (\varepsilon \vec{e}) = \delta - \nabla \cdot \left(\varepsilon \vec{E}_c\right) \tag{2}$$

$$\nabla \times \vec{E} = 0 \Rightarrow \nabla \times \vec{e} = -\nabla \times \vec{E}_c \tag{3}$$

where ε is the permittivity of materials and δ is the free charge density. The RHS term of (2) can be expressed in terms of a fictitious charge density δ_f by applying Gauss' law as:

$$\nabla \cdot \vec{D}_c - \delta = \nabla \cdot \left(\varepsilon \vec{E}_c\right) - \delta = \delta_f \tag{4}$$

The fictitious charge density δ_f is the volume source of the problem in terms of error and must be derived from the numerical solution in terms of the electric potential V.

B. Magnetostatic Problems

For magnetostatic problems, the evaluation of the estimate of numerical errors in Finite Element solutions can be carried out defining an adjoint problem where the unknowns are the components of the error vector $\vec{e}$, defined as the difference between the "true" magnetic induction $\vec{B}_t$ and the computed one $\vec{B}_c$, that is:

$$\vec{e} = \vec{B}_t - \vec{B}_c \tag{5}$$

The governing equations of the error problem are derived from the magnetostatic subset of Maxwell equations applied to the "true" magnetic induction $\vec{B}_t$. This leads to the set of vector equations for the numerical vector error $\vec{e}$:

$$\nabla \cdot \vec{B}_t = 0 \Rightarrow \nabla \cdot (\vec{e}) = -\nabla \cdot \left(\vec{B}_c\right) \tag{6}$$

$$\nabla \times \vec{H} = \vec{J} \Rightarrow \nabla \times (\nu \vec{e}) = \vec{J} - \nabla \times \left(\nu \vec{B}_c\right) \tag{7}$$

where ν is the reluctivity of materials and J is the applied current density. The RHS term of (7) can be expressed in terms of a fictitious current density J_f by applying Ampere's law as:

$$\nabla \times \vec{H}_c - \vec{J} = \nabla \times \left(\nu \vec{B}_c\right) - \vec{J} = \vec{J}_f \tag{8}$$

Those fictitious current densities are the volume sources of the problem in terms of error and must be derived from the numerical solution in terms of magnetic vector potential.

C. Solution Strategy

Equations (2) plus (3) and (6) plus (7) define the sets of vector equations for the error problems over a generic open domain, in the electrostatic and magnetostatic cases, respectively. This general form could be applied, in principle, to the whole domain of a problem discretized and solved with FEM, defining an adjoint problem where the unknowns are the error vectors: this problem would be of the same size, in term of unknowns, as the original FEM solution. However, this solution is in general considered too expensive, since it requires at each iteration the solution of two problems roughly of the same size. The solution strategy generally used to overcome this problem is based on the definition of the error problem on "patches" of a limited number of elements, considering the FEM discretized domain as a set of subdomains in each of which, if appropriate boundary conditions are applied, equations (2) and (3) or (6) and (7) can be defined [1,3,4].

In order to cope better with complex geometries with many interfaces, very likely to be of interest in industrial electromagnetic design, the authors have always chosen to restrict the "patch" to a single element [5-7].

III. Error Sources

As previously pointed out, the problem is restricted to the solution of a "local problem", over each element, consisting of a set of two equations: one in terms of "divergence" (eq. 2 or 6) and one in terms of "curl" (eq. 3 or 7). In order to obtain a unique solution, the definition of appropriate boundary conditions on each element is required. As it is well known, the derivatives computed by a FEM solution in terms of scalar potential, or of vector potential with a single component, are continuous in the tangential component at each inter element boundary, and are discontinuous on the normal component at the same boundaries. This leads to the assumption that the information relevant to the numerical error associated with the discrete solution are contained in those discontinuities [3-7].

In the standard approach proposed in [3,4], the continuity of the tangential component of the derivatives of the potential leads to the assumption that the error in the node is by definition set to zero, implying that it can be considered negligible with respect to the error along the sides of the element. Similarly, in the "Field Based" approach proposed by the authors it is assumed that the error related to one of the components of the field (the tangential component of electric field or the normal component of magnetic induction) is negligible with respect to the error related to the other component. The consequence is that one of the two equations (the "curl" equation (3) for the electric field and the "divergence" equation (6) for the magnetic field) can be assumed as identically satisfied and can then be neglected in the error problem. With this assumption, the sources for the error problem can be identified only for the "meaningful" component of the error (the normal component for the electric field and the tangential component for the magnetic field) and the governing set of equations is restricted to one equation only.

On each element of the discretized domain the boundary conditions are given on the surface of the element in term of the jump of the normal derivatives of the potential at inter-element boundaries. The computed jump, expressed in terms of field, is split between the two neighbouring elements, i and j, for the electric and magnetic field, respectively, as:

$$e_{i(j)} = \omega_{i(j)}\left(\bar{E}_c^i - \bar{E}_c^j\right)\cdot \bar{n} \quad (9)$$

$$e_{i(j)} = \omega_{i(j)}\left(\bar{B}_c^i - \bar{B}_c^j\right)\times \bar{n} \quad (10)$$

The weight factors $\omega i(j)$ take into account the ratio between the absolute values of the field in the two adjoining elements. At exterior boundaries and interfaces, the conditions on the error are derived by the residual in the evaluation of the relevant condition with respect to the normal (electric field) or tangential (magnetic field) component of the field.

The fictitious charge density δ_f and current densitiy J_f defined by (4) and (8), assumed to be constant over each element, can be evaluated applying Gauss' law, or Ampere's law, respectively, to the exterior boundary $\partial\Delta$ of each element Δ:

$$\oint_{\partial\Delta} \bar{D}_c \cdot \bar{n} ds - \int_{\Delta} \delta d\Omega = \int_{\Delta} \delta_f d\Omega \quad (11)$$

$$\oint_{\partial\Delta} \bar{H}_c \cdot d\bar{\ell} - \int_{\Delta} \bar{J} \cdot d\bar{\Omega} = \int_{\Delta} \bar{J}_f \cdot d\bar{\Omega} \quad (12)$$

IV. Numerical Solution

The adjoint problem in terms of error, defined by (2) for the electrostatic problem or by (7) for the magnetostatic problem over each element of the discretized mesh, with boundary conditions like (9) or (10) and internal sources like (11) or (12), can be numerically solved by discretizing the domain (that is, each generic element) into three sub-elements, by adding a node at the centroid of the element.

On each sub-element the unknown error (the normal component of the error in the evaluation of the electric field, or the tangential component of the error in the evaluation of magnetic field) is represented using a vector interpolation representation expressed using a "Whitney forms" description [8,9] in terms of the nodal basic interpolation functions of first order (N_1, N_2, N_3).

A. Electrostatic problem

In electrostatic Finite Element solutions, the vector interpolation form for the definition of the error problem that has been found more suitable to represent the error in terms of electric field is the "Facet Element" interpolation technique. This technique is particularly useful for this case since it represents very well quantities related to a flux [8], that is the normal component of fields, which is the one more directly linked to the error in this case, as previously outlined. The boundary conditions and the unknowns for the error problem over each element are shown in Fig. 1.

The three normal components applied to the outer sides are the known jumps, given by (9), that are derived from the numerical solution, while the three normal component applied to the inner sides are the unknown values.

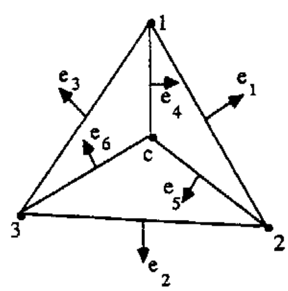

Fig. 1: Discretized "local domain" for the error problem and error unknowns in the electrostatic case.

The error vector on a single sub-element can be defined as:

$$\bar{e} = \sum_k \bar{W}_k e_k \quad \text{where} \quad \bar{W}_k = N_i \nabla \times \bar{N}_j - N_j \nabla \times \bar{N}_i \quad (13)$$

Using a weighted residual approach, the discretized equation is then derived by the integral relation:

$$\int_{\Omega}\left(\nabla \cdot \bar{W}_k\right)\left(\varepsilon \nabla \cdot \bar{e} - \delta_f\right) d\Omega = 0 \quad (14)$$

In this way, on each element a set of three equations is defined, having as three unknowns the normal components of the error vector in the three inner sides of the discretized "local domain", as described in Fig. 1.

B. Magnetostatic problem

In the magnetic case, the vector interpolation form for the definition of the error problem that appears more suitable to represent the vector error in terms of magnetic induction is the "Edge Element" interpolation technique.

This technique is particularly useful for this case since it represents very well quantities related to a circulation [8].

The boundary conditions and the unknowns for the numerical problem are shown in Fig. 2, where the three tangential components applied to the outer sides are the known jumps, given by (10), derived from the numerical solution, while the three tangential components applied to the inner sides are the unknown values.

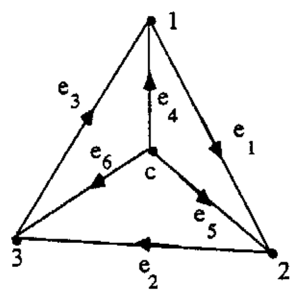

Fig. 2: Discretized "local domain" for the error problem and error unknowns in the magnetostatic case.

The error vector on a single sub-element can be derived as:

$$\bar{e} = \sum_k \bar{W}_k e_k \quad \text{where} \quad \bar{W}_k = N_i \nabla N_j - N_j \nabla N_i \tag{15}$$

Using a weighted residual approach, the discretized equation is then described by the integral relation:

$$\int_\Omega \left(\nabla \times \bar{W}_k\right) \cdot \left(\nu \nabla \times \bar{e} - \bar{J}_f\right) d\Omega = 0 \tag{16}$$

defining on each element a system of three equations in three unknowns, the tangential components of the error vector, defined on the three inner sides of the discretized "local domain", as detailed in Fig. 2.

V. Mesh refinement

The use of the error estimates previously presented in an automatic procedure for mesh refinement requires the identification of an adaption strategy. After a series of initial tests, the authors have identified a procedure that has proven reliable and robust for electromagnetic analysis applications, also in the case of complex geometries of practical industrial interest [10,11].

As defined in a previous paper [11], the procedure is based on the definition of a refinement indicator to guide the subdivision of elements, of a convergence parameter to stop the iterative process of mesh refinement and also provides a final estimate of the local relative error on each element of the final refined mesh. All quantities are computed on the basis of evaluation of quadratic norms over each element of local values. The quadratic norm of a scalar or vector quantity is defined as:

$$\|u\| = \sqrt{\int_\Omega |u|^2 d\Omega} \tag{17}$$

The refinement indicator is defined as :

$$\eta_i = \sqrt{\frac{\|\bar{e}_i\|^2}{\sum_j \|\bar{F}_j\|^2}} \tag{18}$$

where the vector F is the electric field in electrostatic cases or the magnetic induction in magnetostatic problems. The refinement indicator is used to identify the elements to be subdivided. The convergence parameter and the final error estimator are defined by:

$$c = \sqrt{\sum_i \eta_i^2} \quad ; \quad \varepsilon_i = \frac{\|\bar{e}_i\|}{\|\bar{F}\|_{max}} \tag{19}$$

The convergence indicator is used to terminate the iterative procedure, that is stopped when its value falls below a user defined value of "average desired error", in relative or percentual terms. The final error estimate is then evaluated with respect to the maximum field value F (electric or magnetic) computed over the domain. Mesh refinement is realized using the h-refinement procedure detailed in [11].

VI. Implementation and test cases

The proposed method has been implemented in the two-dimensional Finite Element development environment CEDEF, in the interactive module developed for the comparison of adaptive meshing and error estimation techniques [12]. Each solution is obtained with first order triangular elements.

In order to validate the approach and to evaluate the performance of the proposed method, a set of analytically known problems has been analysed. The comparisons between estimated errors and real ones have been realized using in both cases the formula given in (20), but substituting, for the real errors, the exact solution.

All tests performed have indicated a good performance of the method, that has generally provided consistent meshes and a final error estimate close to the real error. To allow a direct evaluation of results, some of the comparisons performed are reported in the following subsections.

A . Dielectric cylinder in uniform field

This model problem is particularly useful to test the performance of the method in the presence of an interface between two materials with different permittivity. An analytical solution for this problem is given in [14].

In Fig. 3 the initial and the refined meshes, with a convergence level set to 1%, are reported. The results obtained, in terms of error estimate and real error on the refined mesh, are plotted in Fig. 4 with reference to the behaviour of errors along a line at y=0 across the interface.

As can be observed from Fig. 4, the procedure shows a good agreement with real errors, as also found in other test cases of similar type, not reported here for the sake of brevity.

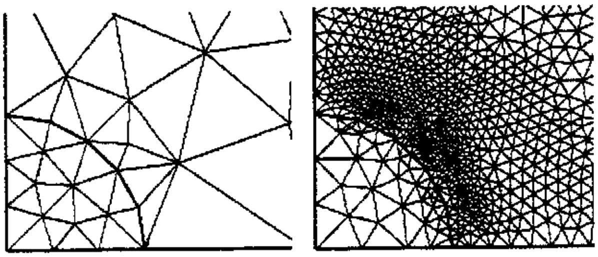

Fig. 3: Initial and refined meshes for the dielectric cylinder problem, showing only a detail of the mesh.

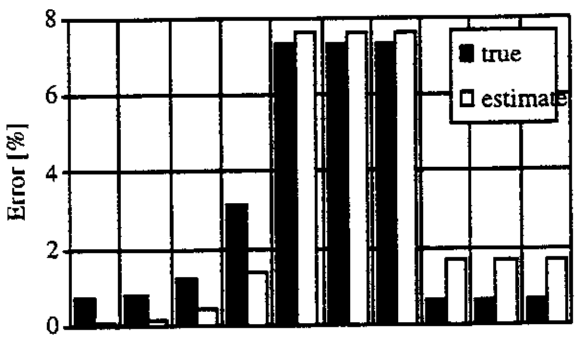

Fig. 4: Plot of the comparisons between true and estimated errors. for the Dielectric cylinder problems along a line at y=0 across the interface.

B. Conductor Bar

Another reference problem that has been used as a test is the evaluation of magnetic induction distribution generated by a conductor bar of infinite length in an ironless domain. The analytic solution is obtained by the integration of Biot-Savart's law [14]. The model for the numerical solution has been obtained using a Dirichlet boundary condition at a sufficient distance from the conductor, computed by means of analytical formulae. In Fig. 5 the initial and the refined mesh, with a convergence level set to 1% are shown. In Fig. 6 the behaviour of the real and estimated errors along a line on the symmetry axis (y=0) crossing the conductor is reported.

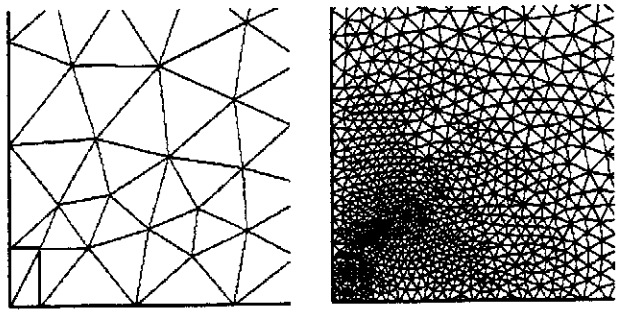

Fig. 5: Initial and refined meshes for the "Conductor Bar" problem, showing only a detail of the mesh.

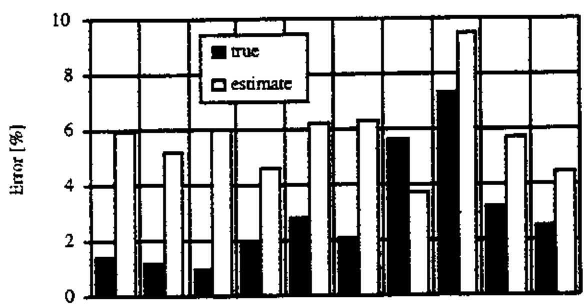

Fig. 6: Behaviour of the real and estimated errors along a line on the symmetry axis of the conductor bar problem.

VII. Rotational symmetry problems

Since the error problems are cast in terms of fields, the error estimates with the "Local Field Error Problem" approach can be also extended to rotational symmetry problems with a limited amount of conceptual changes with respect to the formulation presented above.

This extension has been performed and validated in the CEDEF development environment previously mentioned, and has provided also for this type of geometries very good results. Some examples of this kind are given in the next section.

VIII. Industrial design examples

The adaptive procedures realized on the basis of the "Local Field Error" formulation have been also used in real, industrial level test problems, to evaluate the robustness of the procedure for practical applications.

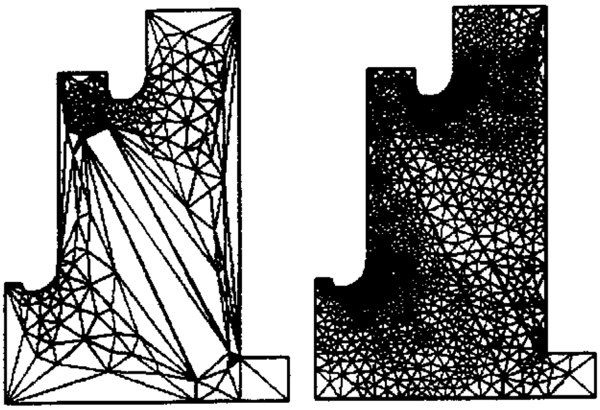

Fig. 7: Initial and final mesh for the evaluation of electrostatic fields in an SF_6 switchgear.

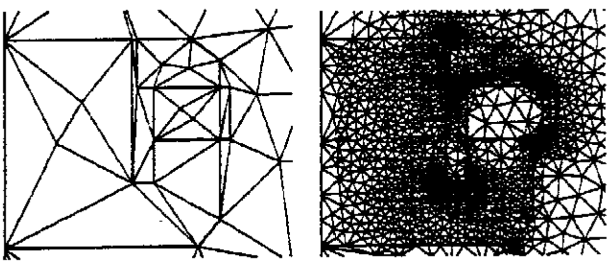

Fig. 8: Initial and final mesh for the evaluation of magnetic induction in a permanent magnet loudspeaker.

In Fig. 7 the initial and final meshes for the evaluation of electrostatic fields in the axial section of an axisymmetric SF_6 switchgear, comprising high voltage electrodes and an insulating cone are given, while in Fig. 8 the initial and final meshes for the evaluation of magnetic fields in a permanent magnet loudspeaker are displayed.

In both cases the initial and final meshes have been produced in complete automation by the procedure, with the user required to define the average accuracy level only, set to one percent for each solution, and the results have proven consistent with those obtained with other codes, run with fine meshes without using an adaptive technique.

IX. Conclusions

The family of error estimators and adaptive algorithms presented in this paper has proven very effective and reliable in the cases tested. They also covered geometries of interest in industrial design, and helped to obtain FEM solutions of practical electrostatic and magnetostatic problems without any effort in the definition of the mesh. The solution quality is under the control of users but independent of their skills.

The algorithms devised appear rather robust and flexible and the authors are carrying on further activity to extend the coverage to other subclasses of electromagnetic analysis.

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References

[1] I Babuska, O.C. Zienkiewicz, J. Gago, E.R. de Oliveira, Eds: *Accuracy estimates and adaptive refinement in Finite Element Computation*, Wiley, New York, 1986.

[2] C. Emson, C.W. Trowbridge, P. Girdinio, G. Molinari, M. Repetto: "Towards an environment for the integration of analysis and design" *Comp. Integrated Manuf. Systems*, Vol. 3, n. 4, Nov. 1990, pp. 216-222.

[3] I. Babuska, W. C. Rheinboldt: "A-posteriori error estimates for the finite element method", *Int. J. for Numerical Methods in Engineering*, Vol. 12, 1978, pp. 1597-1615.

[4] R.E. Bank, A. Weiser: "Some a-posteriori error estimators for elliptic partial differential equations", *Mathematics of Computation*, Vol. 44, Apr. 1985, pp. 283-381.

[5] P. Fernandes, P. Girdinio, P. Molfino, M. Repetto: "Local error estimates for adaptive mesh refinement" *IEEE Trans on Magnetics*, Vol. 24, n. 1, Jan. 1988, pp. 299-302.

[6] P. Barabaschi, G. Drago, P. Girdinio, P. Molfino, M. Repetto, G. Secondo: "A local error 'field based' method for error estimation and adaptive meshing for finite element analysis", *Proc. of 4th IGTE Symposium*, Graz (Austria), Oct. 1990, pp. 47-51.

[7] G. Drago, P. Molfino, M. Nervi, M. Repetto, "A 'Local Field Error Problem' Approach for Error Estimation in Finite Element Analysis", *IEEE Trans. on Magnetics*, vol. 28, n. 2, March 1992, pp. 1743-1746,

[8] A. Bossavit: "A rationale for 'Edge Elements' in 3-D fields computations", *IEEE Trans. on Magnetics*, Vol. 24, n. 1, Jan. 1988, pp. 74-79.

[9] A. Bossavit: "Whitney forms: a class of finite elements for three-dimensional computations in electromagnetism" *IEE Proceedings*, Vol. 135, Pt. A, n. 8, Nov. 1988, pp.493-500.

[10] P. Fernandes, P. Girdinio, P. Molfino, G. Molinari, M. Repetto: " A Comparison of Adaptive Strategies for Mesh Refinement Based on 'A Posteriori' Local Error Estimation Procedure" *IEEE Trans. on Magnetics*, Vol. 26, n. 2, March 1990, pp. 795-798.

[11] P. Fernandes, P. Girdinio, M. Repetto, G. Secondo: "Refinement Strategies in Adaptive Meshing" *IEEE Trans. on Magnetics*, Vol. 28, n. 2, March 1992, pp. 1739-1742.

[12] G. Molinari *et al.* : "A modular Finite Element package for research in electromagnetic analysis developed in a group of italian universities" *Proc. of Int. Conf. BISEF 88*, Beijing, China, Sept. 1988, pp. 625-628.

[13] P. Girdinio, A. Manella, P. Molfino, M. Nervi, "A 'Facet Element' Error Estimate for Adaptive Meshing in Electrostatic FEM Solutions", *Proc. of 6th. IGTE Symposium*, Graz (Austria), Sept. 1994, pp. 138-143.

[14] E. Durand, *Electrostatique et magnetostatique*, Paris: Masson et C.ie, 1953.