High Accuracy Evaluation of the EFIE Matrix Entries on a Planar Patch

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Abstract - A method for the evaluation of the integral of the free-space Green's function on a planar patch, that is exact to machine precision, is developed. The results are used to evaluate two other, commonly used, methods - singularity extraction and singularity cancellation. It was found that these two methods produced unacceptable results. It is shown what steps need to be taken to improve the performance of these methods for patches with varying aspect ratios.

I. INTRODUCTION

The matrix entries arising within numerical solutions of the electric field integral equation, EFIE, for a wide range of applications involve an evaluation of integrals of the form

$$I(x,y) = \iint f(x',y') \frac{e^{-jkR}}{R} dx'dy' \tag{1}$$

where f is usually a bounded, well-behaved function $k = 2\pi/\lambda$ and R is given by

$$R = \sqrt{(x-x')^2 + (y-y')^2} \,. \tag{2}$$

These calculations are most difficult when the test point (x, y) is within or near the source cell over which the integral is performed, due to the $O(1/R)$ behavior of the Green's function, e^{-jkR}/R.

One widely-used method of evaluating (1) is the singularity extraction (SE) procedure, often implemented as

$$\begin{aligned} I(x,y) &= \iint \left\{ f(x',y') \frac{e^{-jkR}}{R} - f(x,y)\frac{1}{R} \right\} dx'dy' \\ &\quad + f(x,y) \iint \frac{1}{R} dx'dy'. \end{aligned} \tag{3}$$

The first integral in (3) is to be evaluated by quadrature, while the second yields an analytical result for triangular or rectangular domains [1]. The first integrand in (3), although bounded, is still not analytic in the vicinity of $R = 0$. Therefore, the accuracy of the result obtained with standard quadrature rules for that integral may be limited.

A second approach for evaluating (1) is the singularity cancellation (SC) method, often known as the Duffy transformation [2]. Suppose that the domain of integration is the rectangle $0 < x' < a$, $0 < y' < b$, and the test point (singularity) is $x = y = 0$. The SC method requires that the domain be divided into two triangles, each of which is transformed into a rectangular domain according to (4),

$$\begin{aligned} I(x,y) &= \int_{x'=0}^{a} \int_{y'=0}^{Kx'} f(x',y') \frac{e^{-jkR}}{R} dx'dy' \\ &\quad + \int_{y'=0}^{b} \int_{x'=0}^{y'/K} f(x',y') \frac{e^{-jkR}}{R} dx'dy' \\ &= \int_{x'=0}^{a} \int_{u=0}^{1} f(x',y')\, Kx' \, \frac{e^{-jkR}}{R} dx'du \\ &\quad + \int_{y'=0}^{b} \int_{v=0}^{1} f(x',y')\, y' \, \frac{e^{-jkR}}{KR} dvdy'. \end{aligned} \tag{4}$$

K is the cell aspect ratio

$$K = \frac{b}{a}. \tag{5}$$

The change of variable

$$y' = Kx'\,u, \quad dy' = Kx'\,du \tag{6}$$

is used in the first integral and the substitution

$$x' = \frac{1}{K}y'\,v, \quad dx' = \frac{1}{K}y'\,dv \tag{7}$$

is used in the second. In the new first integral, at the test point, the integrand is now given by

$$\begin{aligned} f(x',y')\,Kx'\,\frac{e^{-jkR}}{R}\bigg|_{x'\to 0} &\cong \frac{f(x',y')\,Kx'}{\sqrt{(x')^2+(Kx')^2u^2}} \\ &= \frac{f(x',y')}{\sqrt{(1/K)^2+u^2}}. \end{aligned} \tag{8}$$

In the second, the integrand is

$$\begin{aligned} f(x',y')\,\frac{1}{K}y'\,\frac{e^{-jkR}}{R}\bigg|_{y'\to 0} &\cong \frac{f(x',y')\,y'/K}{\sqrt{(y'/K)^2v^2+(y')^2}} \\ &= \frac{f(x',y')}{\sqrt{v^2+K^2}}. \end{aligned} \tag{9}$$

These results are both nonsingular at the original test point, permitting the two integrals in (4) to be evaluated using standard numerical quadrature routines. As shown below, the cancellation of the singularity depends on the cell aspect ratio, K, and the SC approach can yield poor overall accuracy when K is very small or very large.

A third approach is an extension of the SE method described above, obtained by extracting a second term from the integrand [3]. The extended singularity extraction (ESE) approach may be implemented as

$$\begin{aligned} I(x,y) = \iint \left\{ \begin{array}{l} f(x',y')\dfrac{e^{-jkR}}{R} \\ -f(x,y)\dfrac{(1-k^2R^2/2)}{R} \end{array} \right\} dx'dy' \\ + f(x,y)\iint \frac{1}{R}dx'dy' - \frac{f(x,y)k^2}{2}\iint R\,dx'dy'. \end{aligned} \tag{10}$$

A closed-form expression for the final integral in (10) is described in the following section. The first integral, as in the SE method, is to be evaluated by quadrature.

In order to evaluate the effectiveness of the above approaches a method capable of providing high accuracy is needed. One such approach is based on a MacClurin series expansion of the Green's function [4], followed by the closed-form evaluation of the integrals of each term in the series. This series closed-form (SCF) approach is described in the following section.

II. FORMULATION OF THE SCF METHOD

For illustration, consider the evaluation of (1) for a rectangular cell $0 < x' < a$, $0 < y' < b$, $f(x',y') = 1$, and the test point (singularity) at $x = y = 0$. The Green's function may be expanded as

$$\frac{e^{-jkR}}{R} = S_1 - jS_2 \tag{11}$$

where

$$S_1 = \frac{1}{R} - \frac{k^2}{2!}R + \frac{k^4}{4!}R^3 - \frac{k^6}{6!}R^5 + \ldots, \tag{12}$$

$$S_2 = k - \frac{k^3}{3!}R^2 + \frac{k^5}{5!}R^4 - \frac{k^7}{7!}R^6 + \ldots. \tag{13}$$

Since the expansion in (13) is regular and causes no undue difficulty, we focus on (12) and the integral

$$I = \iint \left\{ \frac{1}{R} - \frac{k^2}{2!}R + \frac{k^4}{4!}R^3 - \frac{k^6}{6!}R^5 \ldots \right\} dx'dy'. \tag{14}$$

The SC approach can be applied to (14) to yield

$$I = I_0 + I_1 + I_2 + \ldots \tag{15}$$

where

$$\begin{aligned} I_0 &= \int_{x'=0}^{a}\int_{u=0}^{1} \frac{Kx'}{\sqrt{(x')^2+(Kx')^2u^2}}\,dudx' \\ &\quad + \int_{y'=0}^{b}\int_{v=0}^{1} \frac{y'}{K\sqrt{(y'/K)^2v^2+(y')^2}}\,dvdy' \\ &= \int_{x'=0}^{a}\int_{u=0}^{1} \frac{1}{\sqrt{(1/K)^2+u^2}}\,dudx' \\ &\quad + \int_{y'=0}^{b}\int_{v=0}^{1} \frac{1}{\sqrt{v^2+K^2}}\,dvdy' \ , \end{aligned} \tag{16}$$

$$\begin{aligned} I_1 &= -\frac{k^2}{2!}\left[\begin{aligned} &\int_{x'=0}^{a}\int_{u=0}^{1} Kx'\sqrt{(x')^2+(Kx')^2u^2}\ dudx' \\ &+\int_{y'=0}^{b}\int_{v=0}^{1} \frac{y'}{K}\sqrt{(y'/K)^2v^2+(y')^2}\ dvdy' \end{aligned}\right] \\ &= -\frac{k^2}{2!}\left[\begin{aligned} &K^2\int_{x'=0}^{a}\int_{u=0}^{1} (x')^2\sqrt{1/K^2+u^2}\ dudx' \\ &+\frac{1}{K^2}\int_{y'=0}^{b}\int_{v=0}^{1} (y')^2\sqrt{v^2+K^2}\ dvdy' \end{aligned}\right], \end{aligned} \tag{17}$$

and the *n-th* term can be expressed as

$$\begin{aligned} I_n &= (-1)^n \frac{k^{2n}}{(2n)!} \\ &\times\left[\begin{aligned} &K^{2n}\int_{x'=0}^{a}\int_{u=0}^{1} (x')^{2n}\left[\sqrt{1/K^2+u^2}\right]^{2n-1} dudx' \\ &+\frac{1}{K^{2n}}\int_{y'=0}^{b}\int_{v=0}^{1} (y')^{2n}\left[\sqrt{v^2+K^2}\right]^{2n-1} dvdy' \end{aligned}\right]. \end{aligned} \tag{18}$$

The problem reduces to finding an analytical evaluation of

$$I = \int_0^1 \left(\sqrt{\delta^2+z^2}\right)^{2n-1} dz \ . \tag{19}$$

With the aid of the transformation $z = \delta\tan u$, $dz = \delta\sec^2 u du$, $u_1 = \tan^{-1}\left|1/\delta\right|$, equation (19) can be written as

$$\begin{aligned} I &= \int_0^{u_1}\left(\sqrt{\delta^2+\delta^2\tan^2 u}\right)^{2n-1}\delta\sec^2 udu \\ &= \delta^{2n}\int_0^{u_1}\left(\sqrt{1+\tan^2 u}\right)^{2n-1}\sec^2 udu \\ &= \delta^{2n}\int_0^{u_1}\frac{1}{\cos^{2n+1}u}du \\ &= \delta^{2n}\left\{\begin{aligned} &\frac{\sin z}{2n}\left[\begin{aligned}&\sec^{2n} z + \\ &\sum_{k=1}^{n-1}\frac{(2n-1)(2n-3)\cdots(2n-2k+1)}{2^k(n-1)(n-2)\cdots(n-k)}\sec^{2n-2k} z\end{aligned}\right] \\ &+\frac{(2n-1)!!}{2^n n!}\ln\sqrt{\frac{1+\sin z}{1-\sin z}} \end{aligned}\right\}. \end{aligned} \tag{20}$$

The identity used in the final statement of (20) is found in [5; 2.519.2 & 2.526.9]. In practice the number of terms required for the evaluation of (15) to full quad precision is approximately 15, although it was always evaluated to machine precision using as many terms as necessary, per (23).

The formulation can be expanded to include polynomials such as

$$f(x,y) = x^p y^q \tag{21}$$

or by any function that can be represented by combinations of such polynomials. The associated integrals have the form

$$\begin{aligned} I_n &= \int_0^a x^p dx\int_0^b y^q R^{2n-1}dy \\ &= \frac{a^{2n+p+q+1}}{2n+p+q+1}K^{2n+q}\int_0^1 u^q\left(\sqrt{\left[\frac{1}{K}\right]^2+u^2}\right)^{2n-1} du \\ &\quad +\frac{b^{2n+p+q+1}}{2n+p+q+1}\frac{1}{K^{2n+p}}\int_0^1 v^p\left(\sqrt{v^2+K^2}\right)^{2n-1} dv. \end{aligned} \tag{22}$$

Evaluation of these integrals proceeds in a manner similar to the earlier method.

III. METHODOLOGY

The present study investigates the numerical accuracy obtained from the preceding methods, and the relative computational efficiency (run times) required for each method to produce a specified level of accuracy. The use of single, double, and quad precision for some or all of the calculations is considered. The objectives of the testing were:

- To examine the effect of machine precision on the accuracy of the SCF method.
- To investigate the accuracy of the SE and SC methods.

The location of the test point is rarely at the exact corner of a patch and so, in practice, the domain is divided into four rectangular sub-patches each with a corner at the test point. These sub-patches will frequently have aspect ratios significantly different from $K = 1.0$. As the location of the test point may well be the result of using a quadrature rule, it is instructive to examine the location of test points required by various quadrature formulae. In particular, one is interested in the smallest dimension involved in an application. Examples are shown in Table I. The third rule, "Linlog+Sqrt singularity" possesses the capability to integrate a log singularity and a square root singularity at the same end point.

Table I. Locations of the first test point for various Gaussian quadrature rules, $0 \le x \le 1$

# of nodes in the quadrature rule	Type of quadrature rule		
	Gauss-Legendre	Linlog [6]	Linlog + Sqrt singularity [7]
16	5.30E-3	8.28E-5	4.99E-6
32	1.37E-3	5.69E-6	9.86E-8
48	6.14E-4	1.15E-6	9.35E-9
64	3.47E-4	3.73E-7	1.73E-9

From Table I, one can see that in some instances the location of the test point may result in rectangles with aspect ratios of $K \approx 10^{-9}$. Therefore the range over which tests were conducted was $1.0 \times 10^{-10} \le K \le 1.0$. Two tests were designed. In the first, the location of the test point is at the corner of a patch that has one side dimension of 0.1λ and the other a dimension of 10^{-n}, where $1 \le n \le 11$. The second test uses a constant patch size of $0.1\lambda \times 0.1\lambda$. The test point is located on the line stretching from the center of the patch, at $(0.05, 0.05)$ to the corner of the patch, at $(0.0, 0.0)$, in steps of 10.0^{-n} where $0 \le n \le 10$. The integration over the patch is achieved by dividing it into four sub-patches each with a corner at the test point. The purpose of the second test is to evaluate the impact of the high aspect ratio sub-cell on the overall integral.

As a baseline for comparison, a reference result for the series (15) was evaluated in Multi-Precision, MP, arithmetic [8] using an epsilon value of 10.0^{-400} and reported out in quad (REAL*16) precision. Such precision may seem extreme. However, the comparative accuracy of the other results was based on these reference values.

The effect of machine precision was investigated not only for the present new formulation but also for the SE and SC methods. The integrals requiring the use of quadrature rules were evaluated with an adaptive Gauss-Kronrod-Patterson, GKP, procedure using tabulations derived in MP from an algorithm published by Patterson [9]. These integrals were evaluated so that:

$$\frac{|I_n - I_{n-1}|}{|I_n|} \le 2\varepsilon \tag{23}$$

where I_n is the value of the integral after the n^{th} evaluation. *Epsilon*, ε, is defined as the difference between 1.0 and the smallest number which is greater than 1.0, that can be represented by the compiler. Two other compiler parameters needed to be considered - *tiny* and *huge* - which are the smallest and largest positive numbers respectively that can be represented by the compiler. These are shown in Table II for single, double and quad precision, as well as for the level of precision used in the MP calculations performed in this study.

Relative error was used to evaluate the different schemes using:

$$Error = \log_{10}\left|\frac{I - I_{ref}}{I_{ref}}\right| . \tag{24}$$

Table II. Compiler specific parameters for various levels of precision.

precision	*epsilon*	Log10(*epsilon*)	*tiny*	*huge*
single	1.19E-07	-6.92360	1.18E-38	3.40E+38
double	2.22E-16	-15.6536	2.22E-308	1.80E+308
quad	1.93E-34	-33.7154	3.36E-4932	1.19E+4932
MP	1.00E-400	-400.000	6.19E-14449439	6.19E+14449439

Here, I and I_{ref} are the values of the relevant integral, evaluated in the stated machine precision, and the reference value respectively.

IV. NUMERICAL RESULTS

The results for the SE method, the SC method and the SCF method when evaluated in quad precision are shown in Figure 1.

The susceptibility of the SC method to aspect ratio has already been mentioned in the literature, [4, 10], and investigating this phenomenon was an early motivation for this study. Such suspicions appear to be confirmed as that approach essentially fails for $K < 10^{-4}$. The criticism of the SE method is that although the obvious singularity has been removed, the first integral in the right-hand side of (3) is still not "smooth" in a mathematical sense due to the derivatives of the integral being unbounded at one of the integration limits. Nevertheless, the results over the test range are accurate to better than double precision. The results for the SCF method show that, even in quad precision, there is degradation for the more extreme aspect ratios.

When the same study was carried out in single precision, none of the methods provided acceptable results. It is doubtful that one would encounter such extreme aspect ratios as $K = 1.0\times10^{-10}$ when using single precision. Nevertheless the underlying causes for these failures were examined as an aspect ratio of $K = 1.0\times10^{-6}$ could arise in single precision work.

Two factors were determined to play a role in the failures – the value of *epsilon* and the values of *tiny/huge* in the Fortran complier. Examination of the calculations in the SCF method of (20) indicated the need to carry numbers with a wide range of values – wider than is available with single precision. Routines were written that accepted single precision input and returned single precision output, but within the routines the working precision was either double or quad precision. The results for the SCF method are reported in Table III.

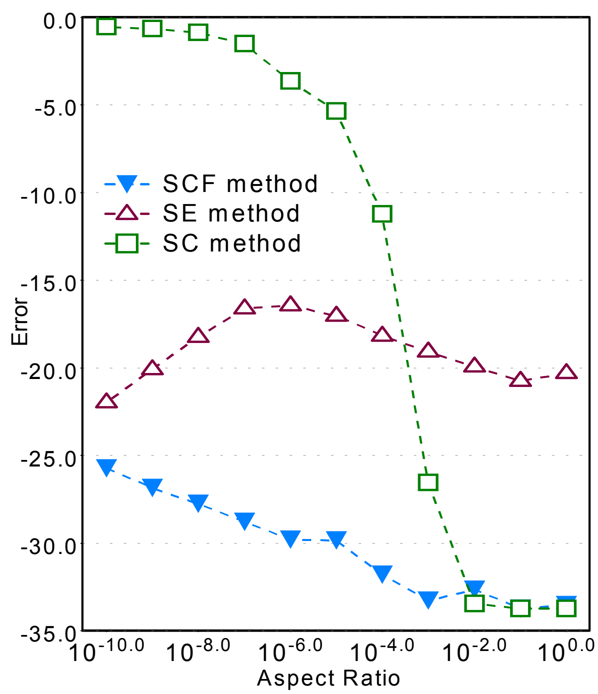

Fig. 1. Plots of results for the SCF, SE, and SC methods when using quad precision for all calculations.

In Table III, the first column reports the aspect ratio, AR, for the calculations. The column headed "single" reports the results of simply using single precision. The column headed "tiny" again uses single precision throughout but guards against $\cos^{2n}(z)$ being less than the tiny value in Table II. When this potential violation is detected, the routine exits with the last value calculated prior to the detection. The columns headed "double" and "quad" indicate use of the special routines mentioned earlier. Use of double precision prevented total failure of the SCF method. However, quad precision was needed to provide

Table III. Effect of machine precision on error (24) of the SCF solution. (d.b.z. is divide-by-zero fault)

	Mode			
AR	single	tiny	double	quad
1	-6.92369	-6.92369	-6.92369	-6.92369
0.1	-6.92369	-6.92369	-6.92369	-6.92369
1.00E-02	-5.97828	-5.97828	-6.92369	-6.92369
1.00E-03	-5.64612	-5.64612	-6.92369	-6.92369
1.00E-04	-5.05785	-5.05785	-6.92369	-6.92369
1.00E-05	d.b.z.	-4.06992	-6.92369	-6.92369
1.00E-06	d.b.z.	0.00E+00	-6.92369	-6.92369
1.00E-07	d.b.z.	0.00E+00	-6.92369	-6.92369
1.00E-08	d.b.z.	0.00E+00	-6.75083	-6.92369
1.00E-09	d.b.z.	0.00E+00	-6.00224	-6.92369
1.00E-10	d.b.z.	0.00E+00	-5.05588	-6.92369

success over the entire range. The reason that double precision does not provide success over the entire range has to do with the calculation of $R=\sqrt{c^2+x^2}$. Precision is lost whenever the ratio of the two numbers, c^2 and x^2, or its reciprocal, is less than the relevant value of ε . This can be seen quite clearly in the "double" results, where precision is lost when the aspect ratio exceeds 1.0E-07.

When the SE approach was examined with single precision it was found that the integration of the "non-singular" part was performing satisfactorily, but evaluation of the singular part was not good, as seen in the "single" column of Table IV. Once this was performed in double precision, the errors were at their lower limit until the aspect ratio reached a value of $K=1.0\times10^{-9}$. To cover the entire range, it was necessary to use quad precision. This is an important finding, as the SCF method has not yet been developed for more general situations and hence one may still need to resort to SE and/or SC.

The problems in the SC method can be understood when one considers the effect of K on the evaluation of the inner integrals of (4). When K takes on an extreme value one of the inner integrals is essentially independent of the variable of integration whereas the other inner integral approaches $O\left(1/R\right)$. This latter effect makes the Gauss-Legendre integration perform very poorly.

Table IV. Effect of machine precision on error (24) of the SE solution.

AR	Single	Double	Quad
1	-6.92369	-6.92369	-6.92369
0.1	-6.41765	-6.92369	-6.92369
1.00E-02	-5.94832	-6.92369	-6.92369
1.00E-03	-5.07121	-6.92369	-6.92369
1.00E-04	-3.90065	-6.92369	-6.92369
1.00E-05	-2.42700	-6.92369	-6.92369
1.00E-06	2.73E-03	-6.92369	-6.92369
1.00E-07	2.37E-03	-6.92369	-6.92369
1.00E-08	2.10E-03	-6.92369	-6.92369
1.00E-09	1.89E-03	-6.92369	-6.92369
1.00E-10	1.71E-03	-6.10287	-6.92369

The solution is to use the SE method in conjunction with the SC method. That is, use (3) in each of the integrals of (4). The results are reported in Table V. Again the importance of evaluating the extracted component in double precision is to be noted. If the extracted component is only evaluated in single precision there is a degradation of the accuracy for intermediate values of the aspect ratio.

Similar findings were made when using double precision as the underlying machine precision. It was necessary to write routines that accepted double precision input and returned double precision output with the internal calculations performed in quad precision. The relevant results are shown in Tables VI. The heading "Double + quad ESE" in this table means that the main quadrature routines used double precision while the evaluation of the extracted components was performed in quad precision. Note that it is necessary to extract two terms in the SE method that is use ESE, as is also the case in the SC method.

Table V. Effect of machine precision on error (24) of the SC solution.

AR	Single only	Single + single SE	Single + double SE
1	-6.92369	-6.92369	-6.92369
0.1	-6.92369	-6.92369	-6.92369
1.00E-02	-6.92369	-5.97828	-6.92369
1.00E-03	-6.92369	-5.55185	-6.92369
1.00E-04	-4.09225	-4.97039	-6.92369
1.00E-05	-2.08886	-3.98617	-6.92369
1.00E-06	-1.1023	-2.52202	-6.92369
1.00E-07	-0.70402	-1.96481	-6.92369
1.00E-08	-0.53765	-1.30139	-6.92369
1.00E-09	-0.43983	-1.34868	-6.92369
1.00E-10	-0.3739	-1.39132	-6.92369

are close to, if not identical, to the results for the SCF method.

So far, the reported results were concerned with accuracy. The timing results for quad precision are shown in Table VII. These show that the SCF method is clearly superior to the other two methods. It is important to note that all of the calculations are performed in the same precision. When one runs similar timing tests in double and single precision the SE method is superior. The reason for the shift is that the SCF method is still largely performed in quad precision, whereas the quadrature routines run mainly in double and single precision respectively.

In the second test series the test point was not located at a corner of the patch. Instead it was moved on a diagonal extending from the center to close to the corner. The patch was then sub-divided into four sub-patches each with a corner at the test point. Two of the sub-patches are square and two have potentially extreme aspect ratios. The test was conducted for each of the three methods, incorporating the information learned in

Table VI. Error (24) for double precision on the three methods of integration.

	SCF		SE		SC	
AR	Double	Quad internal	Double	Double + quad ESE	Double	Double + quad ESE
1	-13.8785	-15.6536	-15.6536	-15.464	-15.6536	-15.6536
0.1	-13.2513	-15.6536	-15.6536	-15.6536	-15.6536	-15.6536
1.00E-02	-7.91709	-15.6536	-14.1219	-15.6536	-15.6536	-15.6536
1.00E-03	-5.77831	-15.6536	-13.5385	-15.6536	-15.6536	-15.6536
1.00E-04	-3.82704	-15.6536	-12.2983	-15.6536	-11.2123	-15.6536
1.00E-05	-3.91093	-15.6536	-11.7914	-15.6536	-5.33437	-15.6536
1.00E-06	-3.98120	-15.6536	-10.1758	-15.6536	-3.62397	-15.6536
1.00E-07	-4.04162	-15.6536	-9.18065	-15.6536	-1.50125	-15.6536
1.00E-08	-4.09416	-15.6536	-8.38368	-15.6536	-0.85638	-15.6536
1.00E-09	-4.13613	-15.6536	-8.43096	-15.6536	-0.64213	-15.6536
1.00E-10	-4.12927	-15.6536	-6.14382	-15.6536	-0.52264	-15.6536

Returning to the use of quad precision as the underlying, and only, precision level, the SE method is modified to incorporate the extraction of two terms. The SC method is modified to incorporate the incorporation of one term and then two terms. The results appear in Figure 2, which has the same scaling as Figure 1 – for direct comparison. It is clear that when two terms are extracted in both the SE and SC methods the results

the first test series. Thus it was unsurprising that the integrations all performed well. An exception occurred in the quad precision studies where it was found that, in the case of the SE method, better results were obtained when the integration area was divided into four subsections, each with a corner at the location of the (extracted) singularity – just as was necessarily done for the SCF and the SC methods. The two results for the SE method are shown in Table VIII. In the cases of single precision and double precision, integration over the entire cell gave results at the limit of precision.

Table VII. Results for relative times for the three different methods when set up for greatest accuracy, using quad precision.

AR	SCF	SE	SC
1	4.59E-02	4.48	2.20
0.1	3.91E-02	2.31	4.15
1.00E-02	4.00E-02	1.46	6.18
1.00E-03	4.10E-02	1.12	6.18
1.00E-04	4.00E-02	1.12	6.15
1.00E-05	4.00E-02	0.86	6.01
1.00E-06	4.00E-02	0.69	6.12
1.00E-07	4.10E-02	0.73	6.13
1.00E-08	3.91E-02	0.17	2.64
1.00E-09	4.00E-02	0.15	2.14
1.00E-10	4.00E-02	0.16	2.76

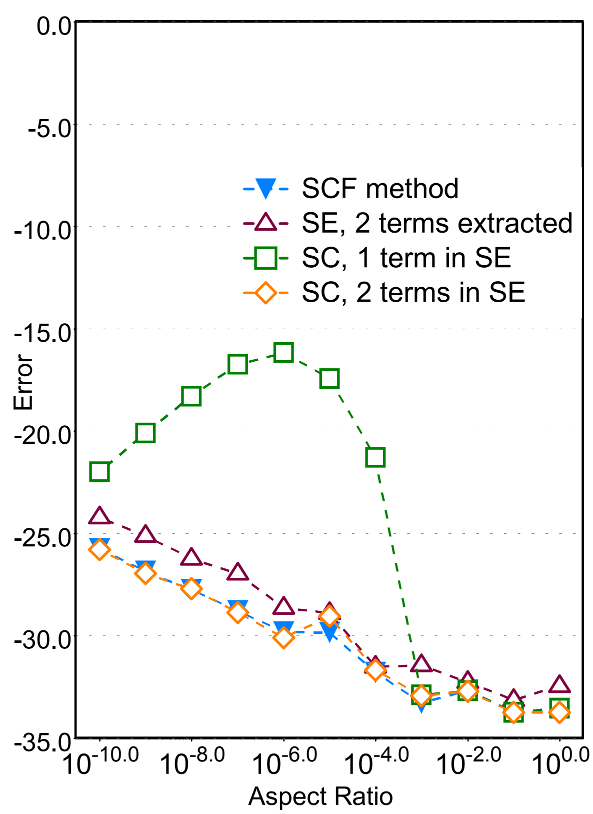

Fig. 2. Plots of results for the SCF, SE, and SC methods when using quad precision for all calculations and incorporating two term extraction.

Table VIII. Results for error (24) for the SE method, using quad precision.

AR	Entire cell	4 sub-cells
1	-16.7542	-33.5543
0.1	-18.3499	-32.6836
1.00E-02	-21.8931	-32.3831
1.00E-03	-23.9701	-32.3669
1.00E-04	-27.0118	-32.4292
1.00E-05	-30.7153	-32.4119
1.00E-06	-32.1547	-32.2953
1.00E-07	-32.4289	-32.4651
1.00E-08	-32.6807	-32.6807
1.00E-09	-32.3217	-32.3083
1.00E-10	-32.3644	-32.3797

V. CONCLUSIONS

The SCF method was developed as a fast, accurate method to evaluate the integral of the free-space Green's function. This method was also used to investigate the effect of machine precision on other approaches to this same evaluation. It was found that in order to span the range of aspect ratios investigated here:

- The SCF method needed to be evaluated in quad precision regardless of the default precision of the compiler.
- The analytical term(s) extracted in the SE method needed to be evaluated in a level of precision higher than the default precision.
- The SC method was successful only when singularity extraction was applied to the inner integrals, and only if those extracted terms were evaluated in a higher level of precision than the default level.
- When extracted terms were evaluated in quad precision it was also important to extract a second term and evaluate it in quad precision.

When dealing specifically with planar patches and polynomial basis functions, the SCF method is the fastest approach of the three considered when using quad precision. Otherwise, the SE approach is faster.

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