A TECHNIQUE FOR DETERMINING NON-INTEGER EIGENVALUES FOR SOLUTIONS OF ORDINARY DIFFERENTIAL EQUATIONS

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ABSTRACT: *A study of the determination of non-integer eigenvalues for ordinary differential equations with transcendental solutions is presented. An algorithm based on expansion in terms of Chebyshev polynomials and collocation is presented. The method is applied to the problem of computing the electric field external to a biconical radiating structure. Eigenfunction solutions for Legendre's differential equation satisfying the boundary conditions of the problem considered are presented.*

1. INTRODUCTION

Canonical analysis of boundary value problems commonly produce series solutions of transcendental functions, [1-3] where the index of summation is a set of eigenvalues (ν) determined by the boundary conditions of the problem. When considering perfectly conducting boundaries, the eigenvalues are those values such that the transcendental function of order ν, or its derivative, is equal to zero at the boundaries. Unfortunately, the determination of the eigenvalues that provide such results is usually limited to special cases, often when ν is an integer. Difficulty in determining the correct values for ν, when ν is not an integer, has limited the use of canonical analysis for these problems.

This paper presents a generalized numerical approach for determining the eigenvalues of transcendental functions subject to the Dirichlet and Neumann boundary conditions. The method proposed is based upon expanding the unknown function in a series of Chebyshev polynomials [4] and using the method of collocation [3] to obtain a well conditioned system of linear homogeneous equations. The eigenvalues (ν) are then found by using a bracketing and bisection technique [5].

This paper is organized as follows. Section 2 describes the mathematical formulation of the problem. Section 3 presents the method for numerically computing the eigenvalues. Section 4 presents sample results for Legendre's differential equation, which arises in the canonical analysis of spherical based problems.

2. MATHEMATICAL FORMULATION

Assume that the physical problem under consideration is described by the function $y_\nu(x)$, which is defined on the closed region [a,b], and satisfies a second order ordinary differential equation (ODE) of the following form:

$$A(x) \frac{d^2 y_\nu(x)}{dx^2} + B(x) \frac{dy_\nu(x)}{dx} + C(x,\nu)\, y_\nu(x) = 0 \quad . \tag{2.1}$$

The functions A(x), B(x), and C(x,ν) are taken to be continuous on the open region (a,b). The sought solutions are subject to boundary conditions which can be separated into different types according to the physical problem under consideration. Typical boundary conditions are:

$$y_\nu(a) = y_\nu(b) = 0 \tag{2.2}$$
$$y_\nu'(a) = y_\nu'(b) = 0 \tag{2.3}$$
$$y_\nu(a) = y_\nu'(b) = 0 \tag{2.4}$$
$$y_\nu'(a) = y_\nu(b) = 0 \tag{2.5}$$

where $y_\nu'(\xi)$ denotes the derivative of $y_\nu(x)$ with respect to x, evaluated at $x = \xi$.

Let $y_\nu(x)$ be expressed by the following Chebyshev polynomial expansion:

$$y_\nu(x) = \sum_{n=0}^{\infty} \alpha_n^\nu \, \mathrm{T}_n[\ell(x)] \qquad a \le x \le b \tag{2.6}$$

where $\mathbf{T}_n(z)$ is the nth Chebyshev polynomial of the first kind, and $\ell(x)$ is a linear mapping which maps the interval [a,b] to [-1,1]

$$\mathrm{T}_n(z) = \cos\left(n \cos^{-1}(z)\right) \qquad -1 \le z \le 1 \tag{2.7}$$

$$\ell(x) = \left[\frac{2}{(b-a)}\right] x - \frac{2a}{(b-a)} - 1 \qquad a \le x \le b \quad . \tag{2.8}$$

Since the set of Chebyshev polynomials are continuous on [-1,1], the first and second derivatives of $y_\nu(x)$ may be expressed as follows:

$$\frac{dy_\nu(x)}{dx} = \sum_{n=0}^{\infty} \alpha_n^\nu \, T_n' \, [\ell(x)] \, \frac{d\ell(x)}{dx} \tag{2.9}$$

$$\frac{d^2 y_\nu(x)}{dx^2} = \sum_{n=0}^{\infty} \alpha_n^\nu \, T_n'' \, [\ell(x)] \left[\frac{d\ell(x)}{dx}\right]^2 \quad . \tag{2.10}$$

$T_n'[\ell(x)]$ and $T_n''[\ell(x)]$ are the first and second derivatives of $T_n[\ell(x)]$ with respect to $\ell(x)$, and are given by:

$$T_n'(z) = \frac{1}{(1-z^2)} \left[-nz \, T_n(z) + n \, T_{n-1}(z) \right] \tag{2.11}$$

$$T_n''(z) = \frac{1}{(1-z^2)^2} \Big[\, T_n(z) \, \left((nz)^2 - nz^2 - n\right)$$

$$+ \, T_{n-1}(z)(-2n^2 z + 3nz) \, + \, T_{n-2}(z)(n^2 - n) \, \Big] \quad . \tag{2.12}$$

Substituting Eqs. (2.9) and (2.10) in Eq. (2.1) and approximating the series expansion for $y_\nu(x)$ by the first N terms, yields the following linear homogeneous equation for the expansion coefficients (α_n^ν):

$$\sum_{n=0}^{N-1} M_n^\nu(x) \, \alpha_n^\nu = 0 \tag{2.13}$$

where,

$$M_n^\nu(x) = A(x) T_n''[\ell(x)] \left[\frac{d\ell(x)}{dx}\right]^2 + B(x) T_n'[\ell(x)] \, \frac{d\ell(x)}{dx} + C(x,\nu) T_n[\ell(x)] \quad . \tag{2.14}$$

Enforcing Eq. (2.14) at N points, $\{x_i, i=1, N\}$, on the interval [a,b] leads to a system of N linear homogeneous equations, which may be written as the following matrix equation:

$$\begin{bmatrix} M_0^{\nu}(x_1) & M_1^{\nu}(x_1) & \dots & M_{N-1}^{\nu}(x_1) \\ M_0^{\nu}(x_2) & M_1^{\nu}(x_2) & \dots & M_{N-1}^{\nu}(x_2) \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ M_0^{\nu}(x_N) & M_1^{\nu}(x_N) & \dots & M_{N-1}^{\nu}(x_N) \end{bmatrix} \begin{bmatrix} \alpha_0^{\nu} \\ \alpha_1^{\nu} \\ \cdot \\ \cdot \\ \cdot \\ \alpha_{N-1}^{\nu} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \cdot \\ \cdot \\ \cdot \\ 0 \end{bmatrix} \tag{2.15}$$

Note that the matrix $\boldsymbol{M}$ is solely a function of ν, the desired eigenvalues of the given ODE. Since Eq. 2.13 is a homogeneous equation, it will have non-trivial solutions if and only if the determinant of the matrix $\boldsymbol{M}$ is zero. However, taking the determinant of the matrix $\boldsymbol{M}$ in its present form yields zero for any value of ν. This is because solutions exist for any given value of ν due to the fact that the boundary conditions have not yet been imposed. Hence, it is necessary to first impose the boundary conditions in order to obtain the desired values for ν.

Boundary conditions are imposed by replacing the first and last rows of the matrix $\boldsymbol{M}$ with the series representation for the boundary conditions. Thus if, $y_\nu(a) = y_\nu(b) = 0$ the first row of the matrix $\boldsymbol{M}$ is replaced by

$$\sum_{n=0}^{N-1} \alpha_n^{\nu}\ T_n[\ell(a)] = 0 \tag{2.16}$$

and the last row is replaced by

$$\sum_{n=0}^{N-1} \alpha_n^{\nu}\ T_n[\ell(b)] = 0 \quad . \tag{2.17}$$

The new matrix obtained will be denoted by $\tilde{\boldsymbol{M}}$. Because the boundary conditions require the function, or the derivative of the function, to be zero at the boundaries, the matrix equation remains homogeneous; thus, non-trivial solutions still exist if and only if the determinant of the matrix $\tilde{\boldsymbol{M}}$ is zero. Hence, the permissible values of ν, subject to the given boundary conditions, are obtained by requiring $\det(\tilde{\boldsymbol{M}}) = 0$.

3. COMPUTER IMPLEMENTATION

Numerical estimation of the eigenvalues ν_p $(p=1,\ldots,P)$, is based on the fact that the $\det(\tilde{M})$ is an oscillatory function of ν, with the $\det(\tilde{M}^{\nu_p}) = 0$ $(p = 1,\ldots,P)$. Hence, over any interval containing a single eigenvalue, $\det(\tilde{M})$ will change sign. This behavior allows the eigenvalues (ν_p) to be determined using a bracketing and bisection technique [5]. Scanning $\det(\tilde{M})$ for changes in sign over a given interval on the ν axis, provides the bracketing intervals for the eigenvalues. Care must be taken in selecting a maximum scan distance which is less than the minimum distance between any two adjacent roots. Scan distances which are too large may cause roots to be missed. Once the roots are bracketed the bisection technique may be implemented to compute the particular eigenvalue to the desired degree of accuracy. Since the bisection method requires only the computation of the sign of the determinant, the common problem of numerical over-flow, associated with determinant calculations, is avoided.

4. ANALYSIS OF A BICONICAL RADIATING STRUCTURE

The technique developed in section 2. is now applied to Legendre's differential equation which describes the electromagnetic field external to a biconical radiating structure (shown in figure 4.1). It can be shown that, under radiation conditions, the electric fields external to the structure have series solutions of the following form:

$$E_r = \sum_{\nu} c_\nu \, H^{(2)}_{\nu+1/2}(kr) \, L_\nu(\cos\theta) \tag{4.1}$$

where $H^{(2)}_{\nu+1/2}(\mathrm{kr})$ is a modified Hankel function of the second kind, and $L_\nu(\cos\theta)$ is an odd Legendre polynomial [6]. Boundary conditions require that $L_\nu(\cos\theta_1) = L_\nu(\cos\theta_2) = 0$. Thus, it is necessary to determine the eigenvalues whose eigenfunctions satisfy the given boundary conditions. For Legendre's differential equation, $A(x) = 1-x^2$, $B(x) = -2x$ and $C(x,\nu) = \nu(\nu+1)$.

Two different biconical radiating structures were chosen for analysis: Structure 1, $\theta_1 = 30°$ and $\theta_2 = 150°$; Structure 2, $\theta_1 = 10°$ and $\theta_2 = 170°$. Tables I and II provide the first four eigenvalues for each structure and compares the calculated eigenvalues with the eigenvalues estimated by Grimes using an asymptotic expansion technique [7]. Graphs of the corresponding eigenfunctions are shown in figures 4.2 and 4.3. For both radiating structures under study, coverage of the calculated eigenvalues occurred for matrix sizes on the order of N = 50 to 60.

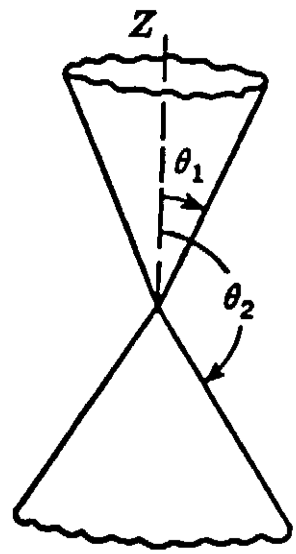

Figure 4.1 Geometry of the biconical radiating structure

TABLE I
Structure 1 $\theta_1 = 30°$, $\theta_2 = 150°$

Eigenvalue	Calculated Value	Grimes' Result
nu-1	2.439211	2.439211
nu-2	5.466996	5.466996
nu-3	8.477510	8.477309
nu-4	11.482784	-

TABLE II
Structure 2 $\theta_1 = 10°$, $\theta_2 = 170°$

Eigenvalue	Calculated Value	Grimes' Result
nu-1	1.621407	1.620624
nu-2	3.916836	3.915488
nu-3	6.188799	6.187171
nu-4	8.451585	8.450112

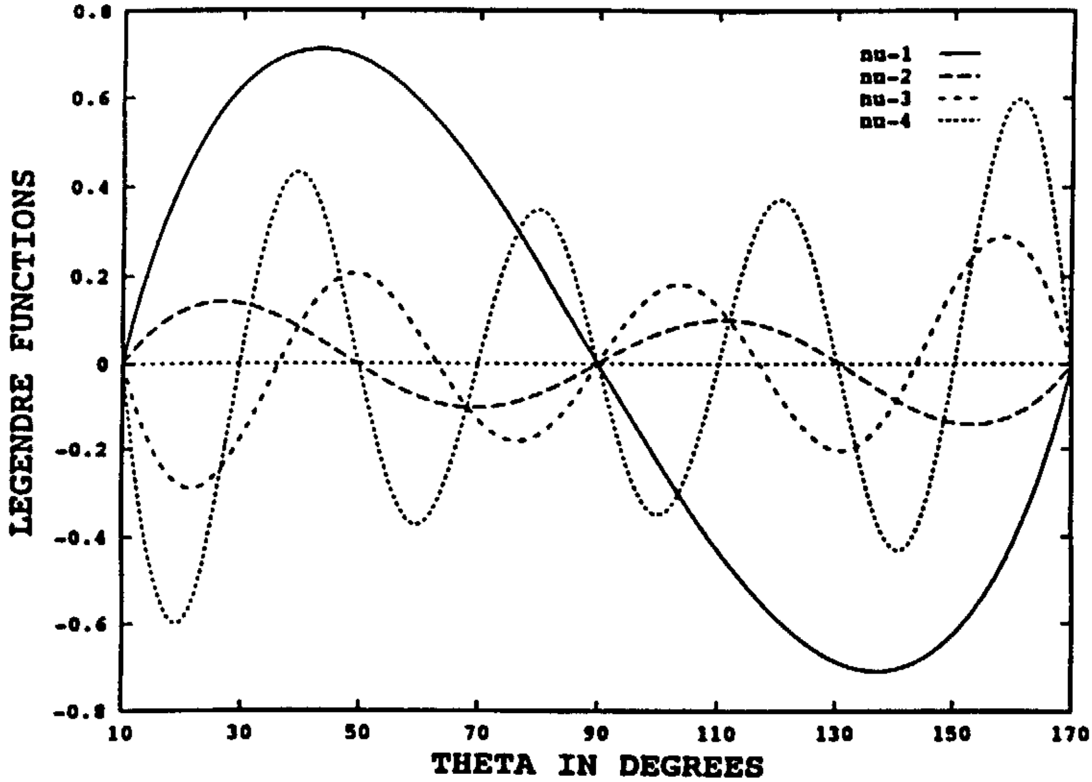

Figure 4.2 First four Legendre functions (eigenfunctions) for 10° biconical structure ($\theta_1 = 10°$, $\theta_2 = 170°$)

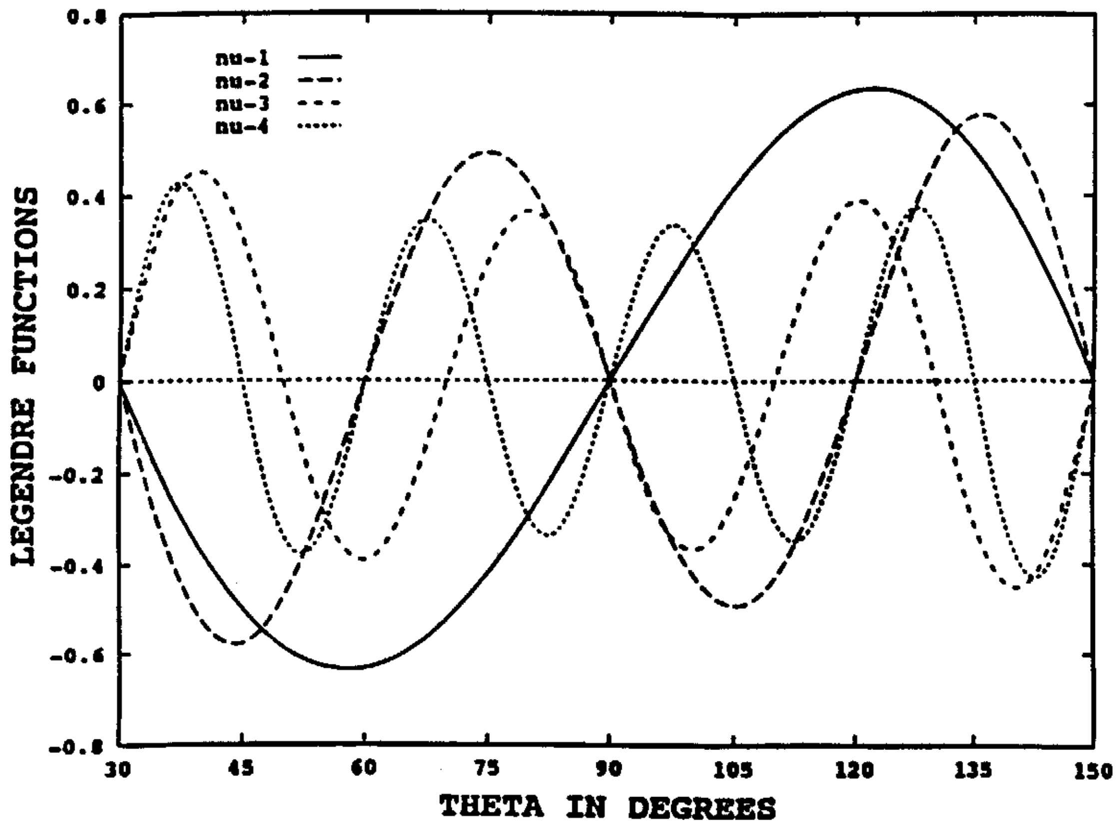

Figure 4.3 First four Legendre functions (eigenfunctions) for 30° biconical structure ($\theta_1 = 30°$, $\theta_2 = 150°$)

5. CONCLUSIONS

A generalized technique for determining non-integer eigenvalues for ordinary differential equations with transcendental solutions has been presented. To verify the technique, the method was applied to a biconical radiating structure and the computed eigenvalues were compared with results obtained by Grimes using an asymptotic method. Excellent agreement was established between the two methods. The method proposed here is completely general and has the advantages that only the sign of the determinant needs to be computed and that the matrix size needed for convergence is small.

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