A NUMERICAL EXAMPLE OF A 2-D SCATTERING PROBLEM USING A SUBGRID*

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ABSTRACT

In this paper we present a detailed application of a subgridding scheme for the finite difference time domain (FDTD) numerical solution to Maxwell's equations. The subgridding scheme will be necessary for greater detail and for localized calculations when other methods for the subcell modifications of the regular FDTD are not applicable. We have made comparative calculations, as a function of mesh size, of the reflection coefficient and shunt capacitance associated with two infinite parallel plates with a finite discontinuity in plate separation.

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1. Statement of the Problem

Shown in Figure 1 is a cross-section of infinite parallel plane conductors. The heights of the two sections are such that only the TEM wave will propagate for a chosen frequency. The plates should be considered to extend to infinity in the x and z directions, even though the boundaries for our calculations are at $X_L = 2\lambda$ to the left of the discontinuity, and $X_R = 11\lambda$ to the right. We will show that our calculational algorithm that uses grids of different sizes in different regions can give meaningful results. These results will be compared to calculations using a uniform coarse grid and a uniform fine grid throughout.

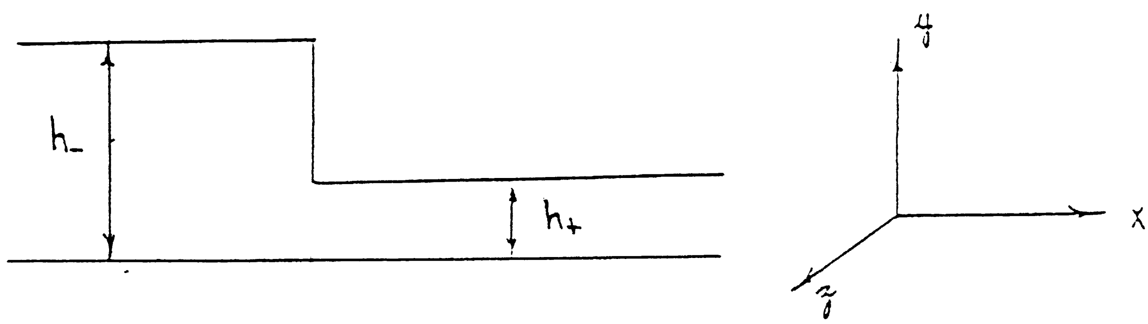


Fig. 1. The cross-section of infinite parallel plane conductors.

Our calculational tool, as sketched in Fig. 2, is the finite difference time domain algorithm[1] with various grid sizes and time steps in various regions.[2]

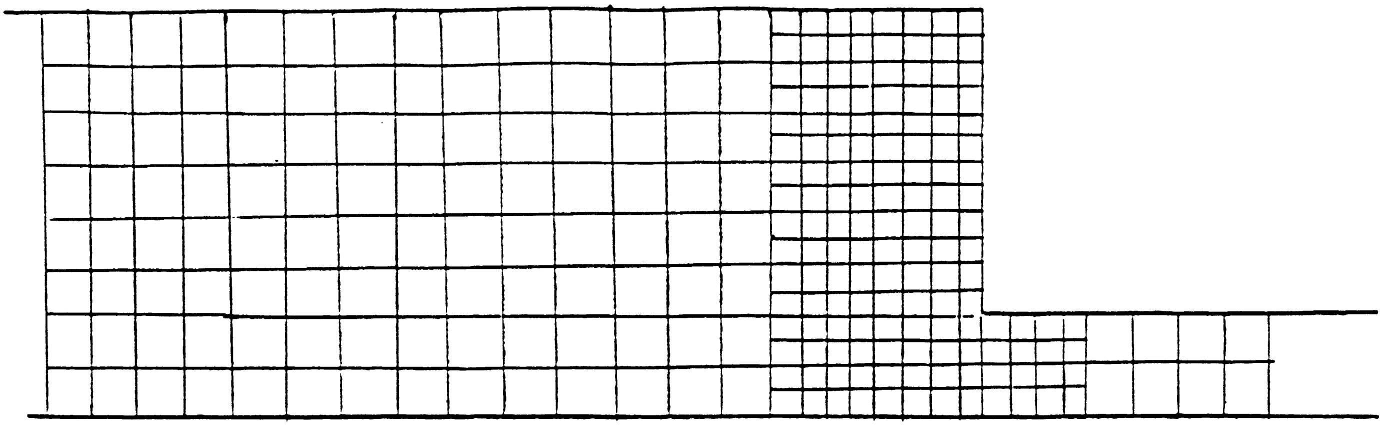


Fig. 2. A possible zoning for the FDTD calculation.

When a sinusoidal TEM wave traveling from the left encounters a step discontinuity, higher order TM modes will be generated in order to satisfy the boundary conditions. The frequency and the heights will be so chosen that only the TEM mode will propagate. Regions of various spatial and time divisions are shown in Fig. 3.

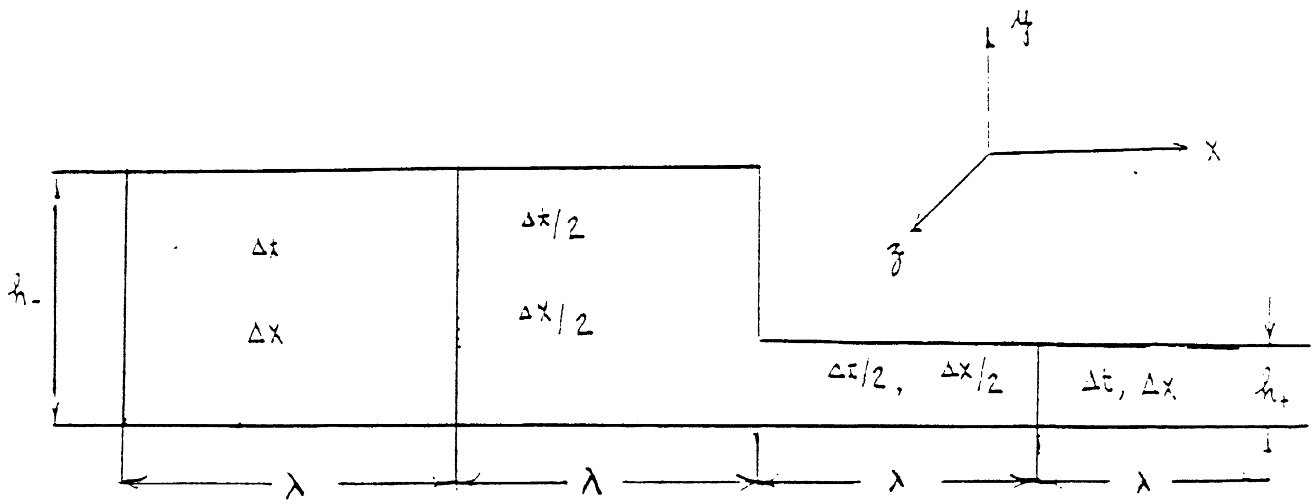


Fig. 3. Regions of different spatial and time subdivisions.

2. Input Information

We chose a sinusoidal wave with $\vec{E}^i = \hat{y}\, E^i$; $\vec{H}^i = \hat{z}\, H^i$. E^i and H^i do not depend on z.

$$E^i(x,t) = \mathrm{Re}\,\{A\, e^{j\omega t}\, e^{j\theta}\, e^{-jkx}\}\ ;\ \text{where } k = \omega\sqrt{\mu\varepsilon}\quad . \tag{1}$$

$$f = \frac{3}{4} \times 10^{10}\ \text{Hertz} \tag{1a}$$

Therefore, in free space

$$\lambda = 3 \times 10^8 \Big/ \frac{3}{4} \times 10^{10} = .04\ \text{m} = 4\ \text{cm}\ ;\ \theta = -\frac{\pi}{2} \qquad (1b)$$

choose $\theta = -\frac{\pi}{2}$ so that the source will be a sine function. Also let

$$h_- = 1\ \text{cm}, \qquad (2a)$$

$$h_+ = \alpha h_-,\ \text{where}\ 0 \le \alpha \le 1. \qquad (2b)$$

It is physically plausible that the boundary conditions (namely, the vanishing of the tangential components of the electric field on the conducting planes) can be satisfied with TM modes and the TEM mode. The lowest TM mode is the TM_{10} mode (half sinusoidal in y and no z dependence). If we choose $\alpha = 0.25$, so that $h_+ = 0.25$ cm, then

$\lambda_c^+ = 2\,(.25) = .5$ cm, the cutoff wavelength to the right. (3a)

$\lambda_c^- = 2\,(1.00) = 2$ cm, the cutoff wavelength to the left. (3b)

but $\lambda = 4$ cm $> \lambda_c^+$ and $> \lambda_c^-$; therefore, the TM_{10} mode cannot propagate on either half of the guide. Choose $\Delta x = h_-/8$ (this will allow 32 zones per wavelength, a very fine zoning).

3. Boundary Condition on the Left

Far away to the left of the discontinuity, we have

$$\vec{E}(x,t) = \hat{y}\,(E^i(x,t) + E^r(x,t)) \qquad (4)$$

where $E^r(x,t)$ is a reflected TEM wave. If x_L is the left boundary,

$$
\begin{aligned}
E^r(x_L,t+\Delta t) &= E^r(x_L+ct+c\Delta t) \\
&= E^r((x_L+c\Delta t) + ct)) \\
&= E^r(x_L+c\Delta t,t)
\end{aligned}
$$

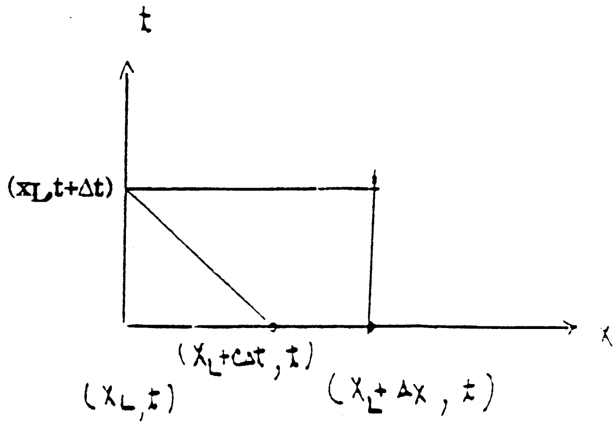


Fig. 4. Interpolation to get $E^r(x_L,t+\Delta t)$.

The value of E^r at $(x_L+c\Delta t,t)$ will be obtained by interpolation.

Thus,

$$E^r(x_L,t+\Delta t) = \frac{\Delta x - c\,\Delta t}{\Delta x} E^r(x_L,t) + \frac{c\,\Delta t}{\Delta x} E^r(x_L+\Delta x,t) \qquad (4a)$$

It is to be observed that $c\Delta t \le \Delta x$ is due to stability considerations. We also note that $E^i(x_L,t+\Delta t) = E^i(x_L-c(t+\Delta t))$ and this is given.

$$E(x_L,t+\Delta t) = E^i(x_L,t+\Delta t) + E^r(x_L,t+\Delta t) \qquad (5)$$

$$= E^i(x_L,t+\Delta t) + \frac{\Delta x - c\Delta t}{\Delta x}(E(x_L,t) - E^i(x_L,t))$$

$$+ \frac{c\,\Delta t}{\Delta x}(E(x_L,+\Delta x,t) - E^i(x_L+\Delta x,t))$$

$$E^i(x,t) = \mathrm{Re}\left\{ \exp\left(j\omega t + jkx_L - jkx - j\frac{\pi}{2}\right)\right\}$$

$$= \sin(\omega t + kx_L - kx) \quad \text{if} \quad \omega t + kx_L - kx > 0$$

$$E^i(x,t) = 0 \text{ if } \omega t + kx_L - kx < 0.$$

Or

$$E^i(x_L,t) = \begin{cases} \sin \omega t \ , & \text{if } t \geq 0 \ ; \\ 0 & \text{if } t < 0 \ . \end{cases}$$

$$E^i(x_L+\Delta x,t) = \begin{cases} \sin(\omega t - k\,\Delta x) \ . & \text{if } \omega t - k\,\Delta x \geq 0 \ ; \\ 0 \ , & \text{if } \omega t - k\,\Delta x < 0 \ . \end{cases} \tag{5a}$$

$$\omega = 2\pi f = \frac{3\pi}{2} \times 10^{10} \ ; k = \omega/3 \times 10^{8} \ . \tag{5b}$$

4. Boundary Condition on the Right

Far away to the right of the discontinuity, we have only a transmitted wave traveling to the right. If we let x_R be the right boundary, we have

$$E(x_R, t + \Delta t) = E(x_R - c(t+\Delta t)) = E((x_R - c\Delta t) - ct)$$

$$= E(x_R - c\Delta t, t)$$

Or

$$E(x_R, t+\Delta t) = \frac{\Delta x - c\Delta t}{\Delta x} E(x_R,t) + \frac{c\Delta t}{\Delta x} E(x_R - \Delta x,t) \tag{6}$$

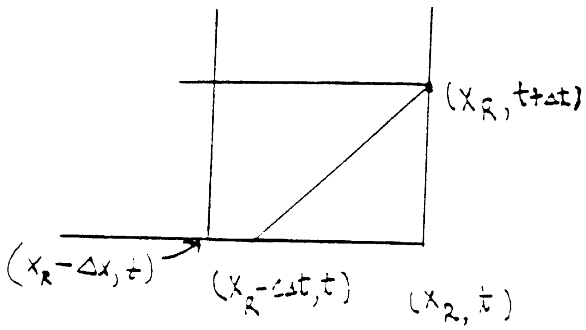


Fig. 5. Interpolation to get $E(x_R, t+\Delta t)$.

5. The Transmission Line Approximation

We can obtain a transmission line approximation of our problem by assuming that the electromagnetic field is that given by 1-D TEM Maxwell's equations.

$$\varepsilon \frac{\partial E_y}{\partial t} = - \frac{\partial H_z}{\partial x} \tag{7a}$$

$$\mu \frac{\partial H_z}{\partial t} = - \frac{\partial E_y}{\partial x} \tag{7b}$$

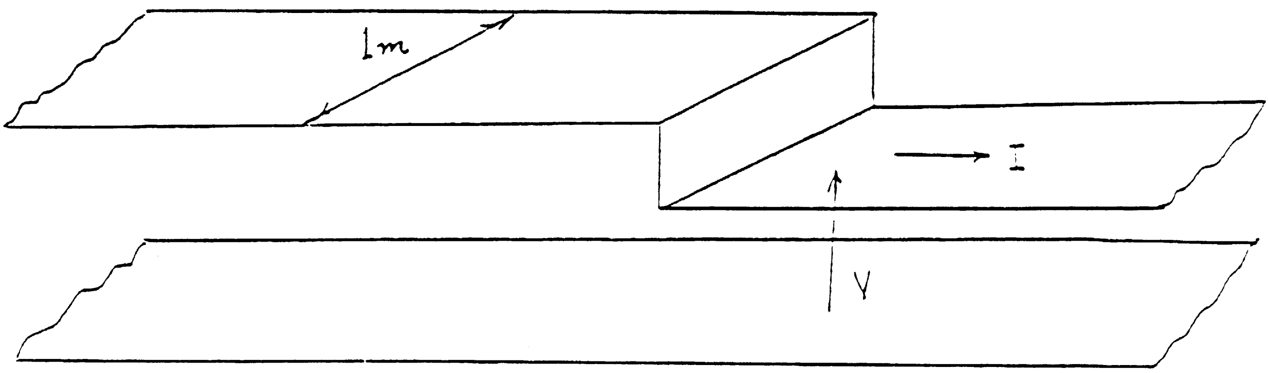


Fig. 6. A unit width two-line transmission line.

Referring to Figure 1 for the coordinates and Figure 6 for the direction of the voltage and current, we let

$$V = -h\, E_y$$

$$I = -H_z \bullet 1\text{ m}$$

where h is the separation between the parallel planes.

(7b) and (7a) can be written as

$$\frac{\partial V}{\partial x} = - h\mu \frac{\partial I}{\partial t} \tag{8a}$$

$$\frac{\partial I}{\partial x} = - \frac{\varepsilon}{h} \frac{\partial V}{\partial t} \tag{8b}$$

(8a) and (8b) are the familiar transmission-line equations with

$$L = h\,\mu \qquad \text{henrys/m} \tag{9a}$$

$$C = \frac{\varepsilon}{h} \qquad \text{farad/m} \tag{9b}$$

The wave velocity is

$$\upsilon = \left(\sqrt{LC}\right)^{-1} = \frac{1}{\sqrt{\mu\varepsilon}}$$

and the characteristic impedance is

$$Z_o = \sqrt{\frac{L}{C}} = \sqrt{\frac{\mu}{\varepsilon}}\, h = \eta h \; ; \; \eta = \sqrt{\frac{\mu}{\varepsilon}}$$

where η is the "characteristic" impedance of a plane TEM wave. For the purpose of analysis, we take x=0 at the discontinuity as shown in Figure 7.

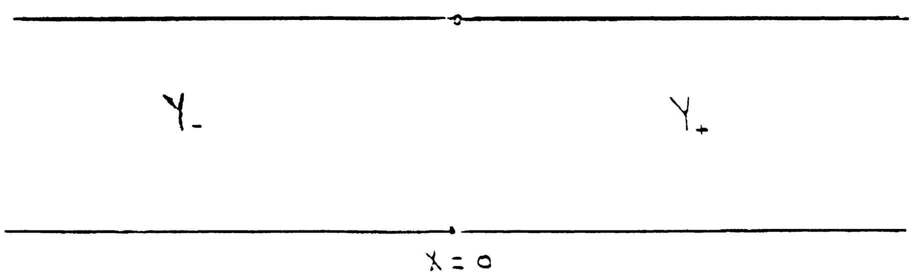


Fig. 7. Two different transmission lines connected at x=0.

Our crude approximation is equivalent to the wave propagation in two different transmission lines connected at x=0. The symbols in Figure 7 are

$$Z_- = Y_-^{-1} \text{ , the characteristic impedance for } x < 0,$$
$$Z_+ = Y_+^{-1} \text{ , the characteristic impedance for } x > 0,$$

If we use the superscript + to denote a positive traveling wave and the superscript - to denote a negatively traveling wave, we get at x=0.

$$V = V^+ + V^-$$
$$I = I^+ + I^- = (1/Z_-)(V^+ - V^-)$$

For $x=0_+$, we have

$$V = Z_+ I$$

as there is only an outgoing wave to the right. Combining

$$(V^+ + V^-) \Big|_{x=0_-} = V \Big|_{x=0_+} = Z_+ I \Big|_{x=0_+}$$

$$= Z_+ I \Big|_{x=0_-} = \frac{Z_+}{Z_-} (V^+ - V^-) \Big|_{x=0_-}$$

yields

$$V^- = V^+ \frac{Z_+ - Z_-}{Z_+ + Z_-} = \mathcal{R}\, V^+ \qquad \text{(10a)}$$

At $x=0_+$, we have

$$V^+ \Big|_{x=0_+} = (Z_+ I^+) \Big|_{x=0_+} = (V^+ + V^-) \Big|_{x=0_-}$$

$$V^+ \Big|_{x=0_+} = (1 + \mathcal{R}) V^+ \Big|_{x=0_-} = \mathcal{T} V^+ \Big|_{x=0_-} \tag{10b}$$

where $\mathcal{R}$ and $\mathcal{T}$ are the reflection and transmission coefficients, respectively, at x=0.

$$\mathcal{R} = \frac{Z_+ - Z_-}{Z_+ + Z_-} \tag{11a}$$

$$\mathcal{T} = 1 + \mathcal{R} = \frac{2Y_-}{Y_- + Y_+} \tag{11b}$$

If we calculate (or measure) the reflection coefficient at a position $n\lambda$ to the left of x=0, we get the same value as given in (11a). And, if we calculate (or measure) the transmission coefficient at a position $n\lambda$ to the right of x=0, we would get the same $\mathcal{T}$.

6. The Transmission Line Approximation with Approximate Account for Fringing at the Discontinuity

A more refined approximation would be to use the transmission line approximation for the dominant TEM mode with a shunt capacitor at x=0 to account for the fringing of the electric field at the discontinuity (x=0). Figure 8 shows such an admittance.[2]

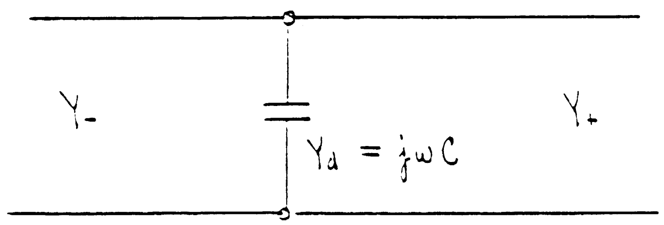


Fig. 8. Shunt admittance at the discontinuity.

For the two semi-infinite transmission lines connected together as shown in Figure 8, the boundary conditions at x=0 are

$$V(0_-,t) = V^+(0_-,t) + V^-(0_-,t) = V(0_+,t)$$

$$I(0_-,t) = Y_-(V^+(0_-,t) - V^-(0_-,t)) = (Y_d + Y_+)\, V(0_+,t)$$

with

$$V^-(0_-,t) = \mathcal{R}\, V^+(0_-,t) \;;\; V(0_+,t) = \mathcal{T}\, V^+(0_-,t) \;,$$

We find that

$$\mathcal{R} = \frac{Y_- - Y_+ - Y_d}{Y_- + Y_+ + Y_d} \;; \qquad (11c)$$

$$\mathcal{T} = 1 + \mathcal{R} = \frac{2Y_-}{Y_- + Y_+ + Y_d} \;. \qquad (11d)$$

Also , if

$$V^+(x,t) = \mathrm{Re}\,(e^{j\omega t}\, e^{-jkx}\, e^{+jkx_L}\, e^{-j\pi/2}) \quad \text{with} \quad x_L = n\lambda$$

then

$$V^-(x,t) = \mathrm{Re}\,(\mathcal{R}\, e^{j\omega t}\, e^{+jkx_L}\, e^{jkx}\, e^{j\pi/2}) \,.$$

For $\mathcal{R} = |\mathcal{R}|\, e^{j\theta_r}$ we get

$$V^-(x_L,t) = |\mathcal{R}| \sin(\omega t + \theta_r)$$

Both $|\mathcal{R}|$ and θ_r can be obtained from the time history of

$$E^r(x_L,t) = E(x_L,t) - E^i(x_L,t)$$

We can then go back to (11d)

$$Y_d + Y_- + Y_+ = \frac{2Y_-}{1 + \mathcal{R}}$$

or

$$Y_d = \frac{2Y_-}{1 + \mathcal{R}} - Y_- - Y_+ \qquad (12)$$

From (12) we can get C as required in Figure 8.

In Appendix A we give more details to compute Y_d and hence C from our numerical scheme.

7. Calculational Results

We take $h_- = 1$ cm and $\Delta x = \Delta y = 1/8$ cm. This would allow 32 zones per wavelength. The left calculational boundary is 2λ from the discontinuity and the right calculational boundary is 11λ from the discontinuity. The left boundary condition is imposed at $x = -2\lambda$ and the right boundary condition is imposed at $x = 11\lambda$. For the time interval of calculation, the effect of the right boundary is not felt at the left boundary, as we only wish to test the effect of the change of grid sizes. At t=0 we set the electric and magnetic fields equal to zero. This will give us the initial condition. For $t \geq 0$ we set

$$E_y^i \Big|_{x_L = -2\lambda} = \sin \omega t$$

In Figure 9a-c we show

$$E_y^r \Big|_{x_L = -2\lambda} \qquad \text{for } t \geq 0 .$$

Using Tables 1a-c we can calculate the time difference, Δt, between the cross-over points of the incident and reflected E_y for different values of h_+/h_-. Since ω is known, we can then calculate the phase angle between these E_y from $\theta_r = \omega\Delta t$. We then derive the shunt capacitance as shown in Appendix A.

In the last column of Table 2 we give the reflection coefficient based on equation (A.4). The last column of Table 3 is based on the exact static formula [equation (13) below] due to the fact that the electric field lines at the neighborhood of the discontinuity are curved (fringing).

A static approximation for C shown in Figure 8 can be obtained. It is the excess capacitance over what occurs when the field lines are uniformly distributed and straight across. The formula is[3]

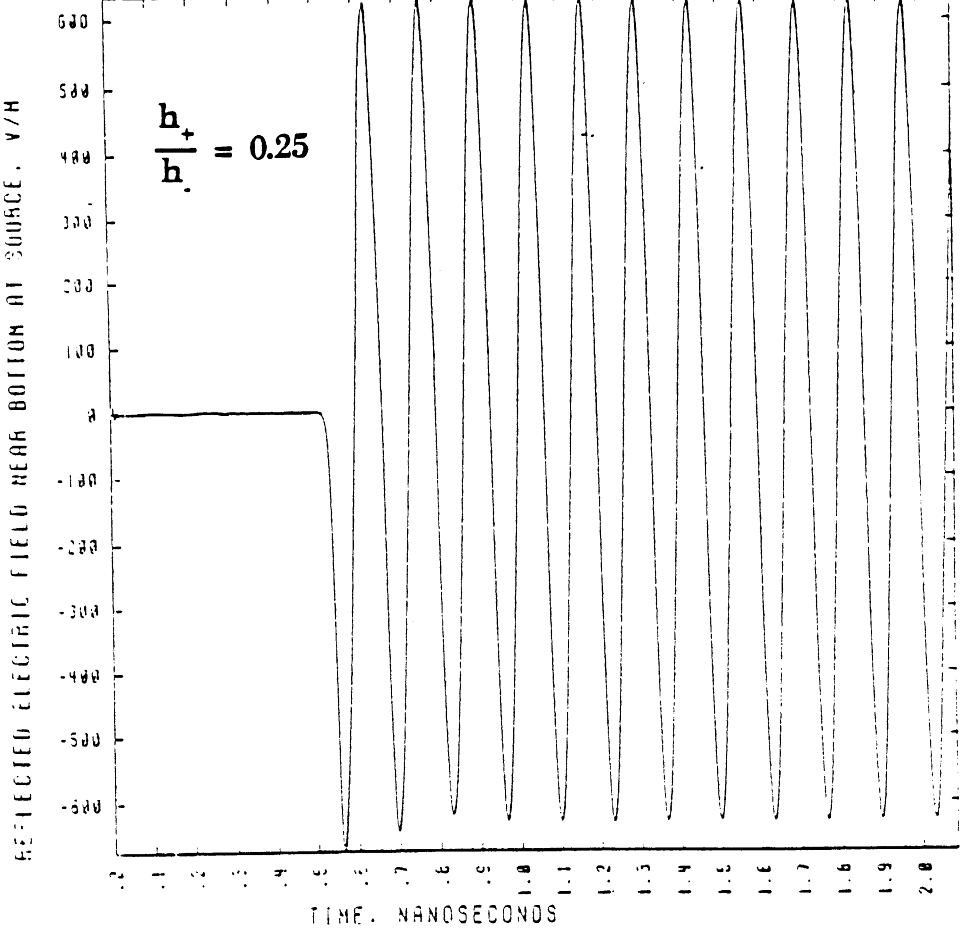
$$C = \frac{\varepsilon}{\pi} \left\{ \frac{(\alpha^2 + 1)}{\alpha} \ln \frac{(1 + \alpha)}{(1 - \alpha)} - 2 \ln \left(\frac{4\alpha}{1 - \alpha^2} \right) \right\} \text{ f/ meter width} \tag{13}$$

where

$$\alpha = h_+ / h_-$$

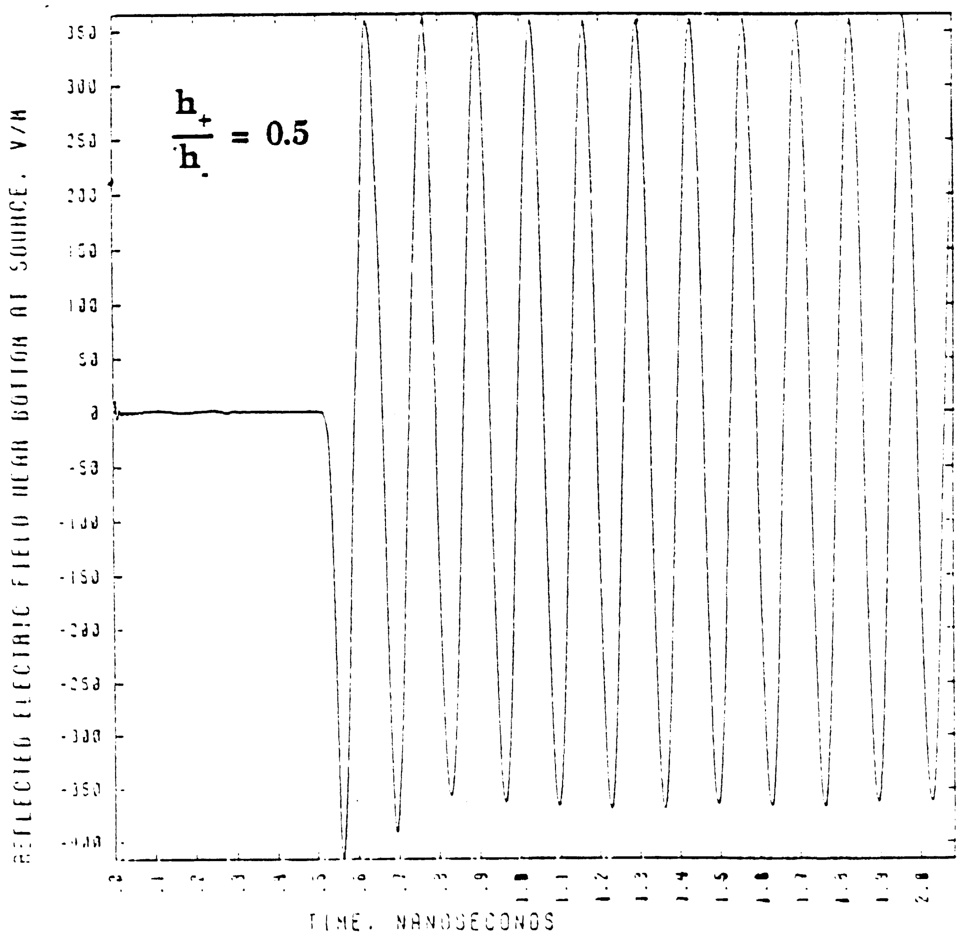
8. Conclusions

The sets of calculations of the reflection coefficients and capacitances for the infinite parallel plates with a finite discontinuity show that using grids of different sizes in different regions gives meaningful results. This approach will save considerable running time and require less memory than a finer grid throughout would need. In addition, it will enable smaller objects to be modeled more accurately.



REFLECTED ELECTRIC FIELD NEAR BOTTOM AT SOURCE, V/M
TIME, NANOSECONDS
$\frac{h_+}{h_-} = 0.25$

Fig. 9a



REFLECTED ELECTRIC FIELD NEAR BOTTOM AT SOURCE, V/M
TIME, NANOSECONDS
$\frac{h_+}{h_-} = 0.5$

Fig. 9b

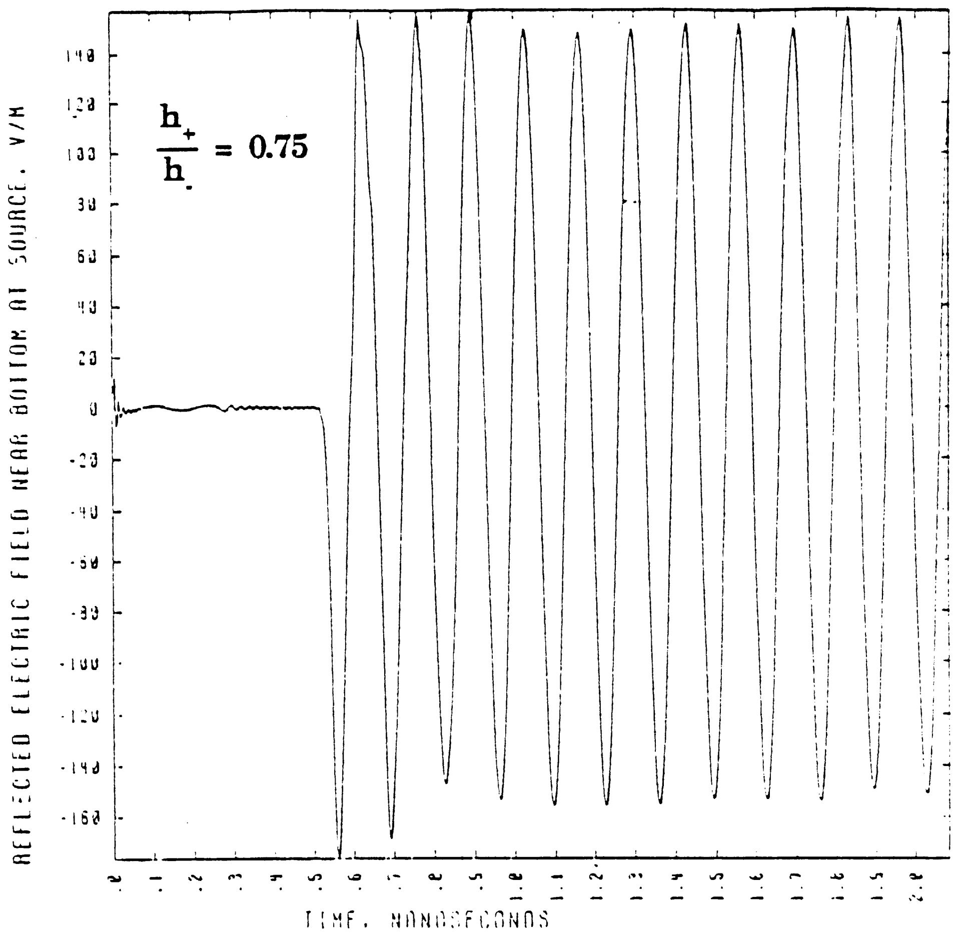


Fig. 9c

Figure 9. Reflected y-component of the electric field 2λ away from the discontinuity.

Fig. 9a:	$h_+/h_- = .25$
Fig. 9b:	$h_+/h_- = .50$
Fig. 9c:	$h_+/h_- = .75$

The incident electric field at $x = -2\lambda$ is $E^i(-2\lambda,t) = 1000 \sin \omega t$ for $t \geq 0$.

Time (ns)	$E_y{}^i$, Incident	$E_y{}^r$, Reflected
1.9957333	-1.9509e+02	4.8245e+01
1.99791667	-9.8016e+01	-1.3807e+01
2.00000	+1.3923e-03	-7.5750e+01

Table 1a: $h_+/h_- = .25$

1.99375000	-2.9028e+02	1.0602e+01
1.99583333	-1.9509e+02	-2.4998e+01
1.999791667	-9.8016e+01	
2.000000	1.392923e-03	

Table 1b: $h_+/h_- = .50$

1.99375000	-2.9028e+02	1.6992e+00
1.99583333	-1.9509e+02	-1.2897e+01
1.999791667	-9.8016e+01	
2.00000	1.392923e-03	

Table 1c: $h_+/h_- = .75$

Table 1a,b,c. The incident and reflected E_y at $x= -2\lambda$. The left boundary of the calculational grid is at $x = -2\lambda$ and the right boundary of the calculational grid is at $x = +11\lambda$.

h_+/h_-	Calculated $\|\mathfrak{R}\|$			Transmission-Line Approx. $\|\mathfrak{R}\|$
	Uniform Coarse Grid	Mixed Grid	Uniform Fine Grid	
.25	.629	.627	.627	(4-1)/(4+1)
.50	.363	.362	.362	(2-1)/(2+1)
.75	.151	.150	.151	(4-3)/(4+3)

Table 2. Calculated and transmission-line-approximation reflection coefficients.

h_+/h_-	Calculated $C \times 10^{12}$ f/m			From Equation (13)
	Uniform Coarse Grid	(Mixed Grid)	Uniform Fine Grid	
.25	5.20	5.1	5.22	5.75
.50	2.31	2.27	2.29	2.21
.75	.59	.566	.570	5.73

Table 3. Calculated shunt capacitance and the static approximation based on Equation (13).

ACKNOWLEDGMENT

We appreciate Dr. Hans Kruger's encouragement.

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APPENDIX A
Calculation of the Shunt Capacitance from the Numerical Output

From the text we have the incident wave at $x_L = n\lambda$

$$E^i(x_L,t) = \sin \omega t \qquad t > 0 \qquad \text{(A.1)}$$

[Equation (5a)]

The reflected wave is

$$E^r(x,t) = \mathrm{Re}\,\{\mathcal{R}\, e^{jkx - j\pi/2 + j\omega t}\}$$

with

$$\mathcal{R} = |\mathcal{R}|\, e^{j\theta_r} \text{ and } x = -n\lambda$$

$$E^r(x,t) = |\mathcal{R}| \cos(\omega t + \theta_r - \pi/2) = |\mathcal{R}| \sin(\omega t + \theta_r)$$

$$= -|\mathcal{R}| \sin(\omega t + \Delta_r) \qquad \text{(A.2)}$$

where

$$\Delta_r = \theta_r - \pi \;(\text{or } \theta_r = \pi + \Delta_r)$$

In our numerical examples, Δ_r is a small number. Δ_r can be obtained directly from the E^r vs. ωt plot. From equation (12)

$$Y_d = \frac{2\,Y_-}{1 + \mathcal{R}} - Y_- - Y_+ \qquad \text{(A.3)}$$

Let

$$\mathcal{R} = |\mathcal{R}| \cos\theta_r + j\,|\mathcal{R}| \sin\theta_r = -|\mathcal{R}| \cos\Delta_r - j\,|\mathcal{R}| \sin\Delta_r$$

$$Y_d = \frac{2Y_-(1+\mathcal{R}^*)}{(1+\mathcal{R})(1+\mathcal{R}^*)} - Y_- - Y_+$$

$$= \left\{ \frac{2Y_-(1-|\mathcal{R}|\cos\Delta_r)}{(1-|\mathcal{R}|\cos\Delta_r)^2+|\mathcal{R}|^2\sin^2\Delta_r} - Y_- - Y_+ \right\}$$

$$+ \frac{j\,2Y_-|\mathcal{R}|\sin\Delta_r}{(1-|\mathcal{R}|\cos\Delta_r)^2+|\mathcal{R}|^2\sin^2\Delta_r}$$

The value $|\mathcal{R}|$ can be read off directly from the computer output, and Δ_r can be obtained from a numerical interpolation. We also recall that

$$Y_- = \frac{1}{\sqrt{\frac{\mu}{\varepsilon}}\,h_-} \approx \frac{1}{377}\frac{1}{h_-} \qquad \text{for free space.}$$

For the transmission line approximation,

$$\mathcal{R} = \frac{Z_+ - Z_-}{Z_+ + Z_-} = \frac{h_+ - h_-}{h_+ + h_-} \tag{A.4}$$

APPENDIX B
Symbols and "Pseudo" FORTRAN Flow Chart

In this appendix, we show the names of some variables and a "pseudo" FORTRAN flow chart. The electric field components in zone 1 will be E1x, E1y; the horizontal zone boundaries will be I1L, I1R, etc. The vertical boundaries will be J1B, J1T, etc. A pseudo flow chart outlining the steps of calculations, interpolations, etc. is shown in steps (1)-(13).

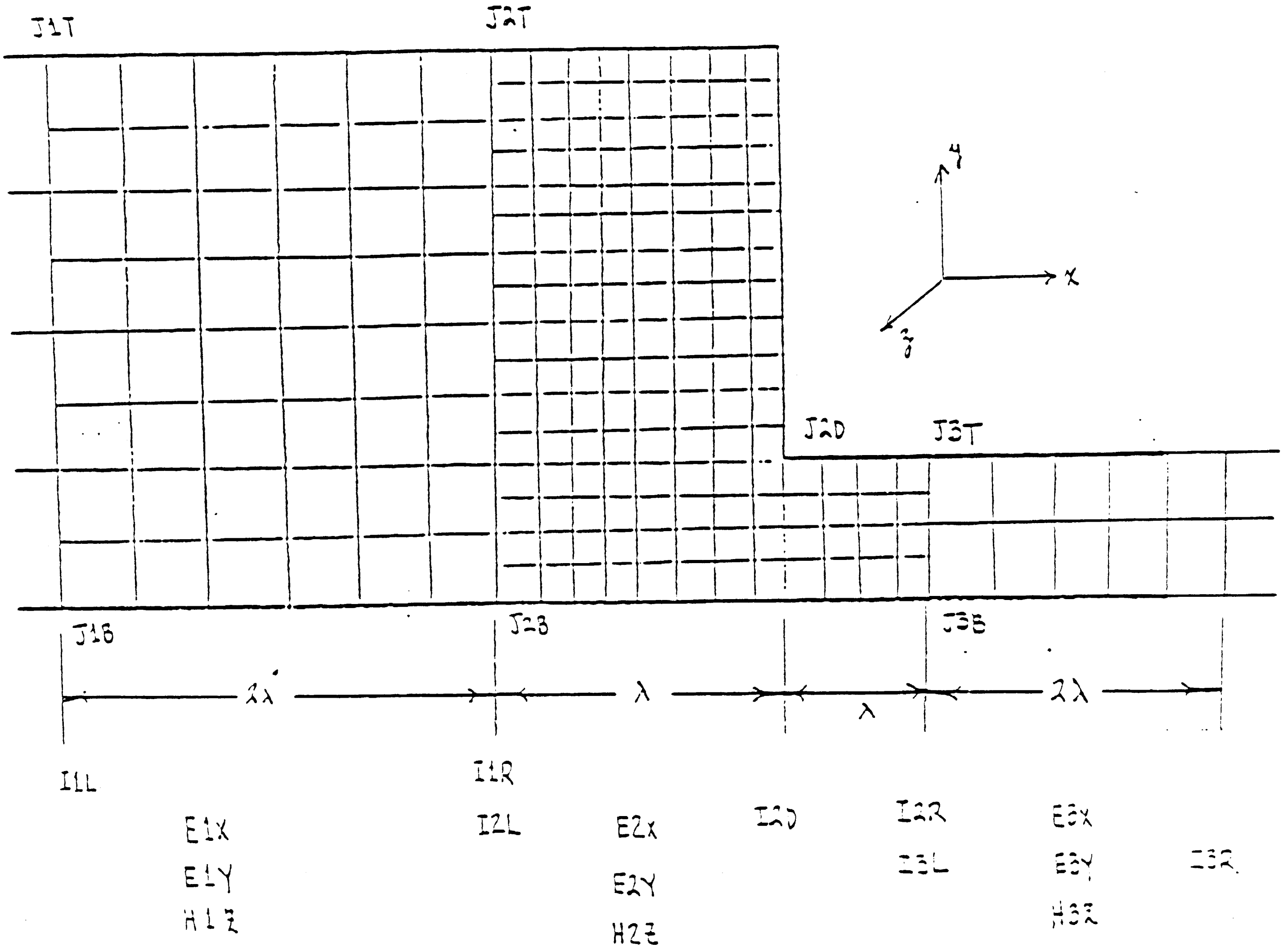

Fig. B.1. The variable zones, indices and variables.

Back from (13)

(1) Use ΔT1, advance to $t = \frac{\Delta T1}{2}$

```
I1 = I1L, I1R
    J1 = J1B, J1T - 1
    H1Z(I1,J1) = H1Z(I1,J1) + ....
Continue
I3 = I3L - 1, I3R - 1
    J3 = J3B, J3T - 1
    H3Z(I3,J3) = H3Z(I3,J3) + ....
Continue
```

(2) Use ΔT1, advance to t = ΔT1

```
I1 = I1L, I1R - 1
    J1 = 2, J1T - 1
    E1x(I1,J1) = E1x(I1,J1) + ....
Continue
I3 = I3L, I3R-1
    J3 = 2, J3T - 1
    E3x(I3,J3) = E3x(I3,J3)
Continue
```

Fix to t = ΔT1

```
Fix the boundary condition for E1Y(I1L,J1)
Fix the boundary condition for E3Y(I3R,J3)
```

(3) At the interface
Saving the old values for interpolation

```
J1 = J1B, J1T - 1
        E1YT(J1) = E1Y(I1R,J1)
Continue
J3 = J3B, J3T - 1
        E3YT(J1) = E3Y(I3L,J3)
Continue
                                        to t = 0
```

(4) At the interface
Interpolate and extrapolate **to t = 0**

```
        E2Y(I2L,1) = E1Y(I1R,1)
        E2Y(I2L,J2T-1) = E1Y(I1R,J1T-1)
        E2Y(I2R,1) = E3Y(I3L,1)
E2Y(I2R,J2D-1) = E3Y(I3L,J3T-1)

        K = 2, J1T-1
E1Y(I2L,2*K-1)
        = .75*E1Y(I1R,K)
          +.25*E1Y(I1R,K-1)

E2Y(I2L,2*(K-1)) = .25*E1Y(I1R,K)
        + .75*E1Y(I1R,K-1))
Continue

        K = 2, J3T-1
E2Y(I2R,2*K-1) = .75*E3Y(I3L,K) + .25*E3Y(I3L,K-1)
E2Y(I2R,2*(K-1)) = .25*E3Y(I3L,K) + .75*E3Y(I3L,K-1)
Continue
```

(5) Use ΔT2, advance H2Z to $t = \frac{\Delta t2}{2} = \frac{\Delta t1}{4}$

```
I2 = I2L, I2D-1
    J2 = J2B, J2T-1
H2Z(I2,J2) = H2Z(I2,J2) + ...
Continue

I2 = I2D, I2R-1
    J2 = J2B, J2D-1
H2Z(I2,J2) = H2Z(I2,J2) + ...
Continue
```

(6) Use Δt1, advance E1Y and E3Y to t = Δt1

```
I1 = I1L+1, I1R
    J1 = J1B, J1T-1
        E1Y(I1,J1) = E1Y(I1,J1) + ...
Continue

I3 = I3L, I3R-1
    J3 = J3B, J3T-1
        E3Y(I3,J3) = E3Y(I3,J3) + ...
Continue
```

(7) Time average values for E1Y and E3Y at the interface to $t = \Delta t_2 = \frac{\Delta t1}{2}$

```
J1 = J1B, J1T-1
    E1YT(J1) = .5*E1YT(J1) + .5*E1Y(I1R,J1)
Continue

J3 = J3B, J3T-1
    E3YT(J3) = .5*E3YT(J3) + .5*E3Y(I3L,J3)
Continue
```

(8) Spatial average to get E2Y at the interface to $t = \Delta t_2$

```
E2Y(I2L,1) = E1YT(1)
E2Y(I2L,J2T-1) = E1YT(J1T-1)
E2Y(I2R,1) = E3YT(1)
E2Y(I2R,J2D-1) = E3YT(J3T-1)
    K = 2, J1T-1
E2Y(I2L, 2*K-1) = .75*E1YT(K) + .25*E1YT(K-1)
E2Y(I2L, 2*(K-1)) = .25*E1YT(K) + .75*E1YT(K-1)
Continue

    K = 2, J3T - 1
E2Y(I2R, 2*K-1) = .75*E3YT(K) + .25*E3YT(K-1)
E2Y(I2R, 2*(K-1)) = .25*E3YT(K) + .75*E3YT(K-1)
Continue
```

(8a)

Advance the interior E2x, E2Y to $t = \Delta t_2$

(9) Replacing the calculated values in (1) by spatial average (tempor.) to $t = \frac{\Delta t_2}{2}$

```
J1 = 1, J1T - 1
H1Z(I1R,J1) = .25*(H2Z(I2L,1+2*(J1-1))
              + H2Z(I2L+1,1+2*(J1-1))
              + H2Z(I2L,2*J1) + H2Z(I2L+1,2*J1))
Continue

J3 = 1, J3T - 1
H3Z(I3L-1, J3) = .25*(H2Z(I2R-1,1+2*(J3-1))
              + H2Z(I2R-2,1+2*(J3-1))
              + H2Z(I2R-1,2*J3)
              + H2Z(I2R-2,2*J3)
Continue
```

(10) Use Δt2, advance H2Z to $t = \frac{3}{2}\Delta t_2$

```
I2 = I2L, I2D-1
      J2 = J2B, J2T-1
H2Z(I2,J2) = H2Z(I2,J2) + ...
Continue

I2 = I2D, I2R-1
      J2 = J2B, J2D-1
H2Z(I2,J2) = H2Z(I2,J2) + ...
Continue
```

to (11)

(11) Time average of (9) and (10). This replaces the calculated boundary values by (1) for $t = \frac{1}{2}\Delta t_1$. These values are then advanced by (1) to a new time $t = \frac{3}{2}\Delta t_1$

```
┌─J1 = 1, J1T-1
│        H1Z(I1R,J1) = .5*H1Z(I1R,J1)
│                + .5*.25*(H2Z(I2L, 1+2*(J1-1)
│                + H2Z(I2L+1, 1+2*(J1-1))
│                + H2Z(I2L, 2*J1) + H2Z(I2L+1, 2*J1))
└─Continue

┌─J3 = 1, J3T-1
│        H3Z(I3L-1, J3) = .5*H3Z(I3L-1, J3)
│                + .5*.25*(H2Z(I2R-1, 1+2*(J3-1))
│                + H2Z(I2R-2, 1+2*(J3-1))
│                + H2Z(I2R-1, 2*J3)
│                + H2Z(I2R-2, 2*J3))
└─Continue
```

to (12)

(12) Spatial average to get E2Y at the interface to t = $\Delta t_1 = 2\Delta t_2$

```
E2Y(I2L,1) = E1Y(I1R,1)
E2Y(I2L,J2T-1) = E1Y(I1R,J1T-1)
E2Y(I2R,1) = E3Y(I3L,1)
E2Y(I2R,J2D-1) = E3Y(I3L,J3T-1)

    K = 2, J1T-1
E2Y(I2L,2*K-1) = .75*E1Y(I1R,K) + .25*E1Y(I1R,K-1)
E2Y(I2L,2*(K-1)) = .25*E1Y(I1R,K) + .75*E1Y(I1R,K-1)
    Continue

    K = 2, J3T-1
E2Y(I2R,2*K-1) = .75*E3Y(I3L,K) + .25*E3Y(I3L,K-1)
E2Y(I2R,2*(K-1)) = .25*E3Y(I3L,K) + .75*E3Y(I3L,K-1)
    Continue
```

(13)

Advance the interior E2Y, E2x to t = $2\Delta t2 = \Delta t_1$

Back to (1)

APPENDIX C
The Finite Difference Equations

The finite difference equations used in this report are derived from the integral forms of Maxwell's equations, which are

$$\oint \vec{E} \bullet d\vec{\ell} = -\frac{\partial}{\partial t} \iint \mu \vec{H} \bullet d\vec{S} \qquad (C.1)$$

$$\oint \vec{H} \bullet d\vec{\ell} = \frac{\partial}{\partial t} \iint \varepsilon \vec{E} \bullet d\vec{S} \qquad (C.2)$$

For the TM wave appropriate to our problem, we have

$$\vec{E}(x,y,z,t) = \hat{x} E_x(x,y,t) + \hat{y} E_y(x,y,t) \qquad (C.3)$$

$$\vec{H}(x,y,z,t) = \hat{z} H_z(x,y,t) \qquad (C.4)$$

and there is no z dependence.

Using (C.1), we have

$$-\mu \frac{H_z^{n+1/2}(i+1/2, j+1/2) - H_z^{n-1/2}(i+1/2, j+1/2)}{\Delta t} \Delta x \, \Delta y$$

$$= \Delta x \left(E_x^n(i+1/2, j) - E_x^n(i+1/2, j+1)\right)$$

$$+ \Delta y \left(E_y^n(i+1, j+1/2) - E_y^n(i, j+1/2)\right)$$

resulting in

$$H_z^{n+1/2}(i+1/2, j+1/2) = H_z^{n-1/2}(i+1/2, j+1/2) - \frac{\Delta t}{\mu \Delta y}\left(E_x^n(i+1/2, j) - E_x^n(i+1/2, j+1)\right) - \frac{\Delta t}{\mu \Delta x}\left(E_y^n(i+1, j+1/2) - E_y^n(i, j+1/2)\right)$$

Using (C.2), we have

$$\varepsilon \frac{E_x^{n+1}(i+1/2,j) - E_x^{n}(i+1/2,j)}{\Delta t} \Delta y \bullet 1 =$$

$$1 \bullet \left(H_z^{n+1/2}(i+1/2,j+1/2) - H_z^{n+1/2}(i+1/2,j-1/2)\right)$$

$$\varepsilon \frac{E_y^{n+1}(i,j+1/2) - E_y^{n}(i,j+1/2)}{\Delta t} \Delta x \bullet 1 =$$

$$1 \bullet \left(H_z^{n+1/2}(i-1/2,j+1/2) - H_z^{n+1/2}(i+1/2,j+1/2)\right)$$

resulting in

$$E_x^{n+1}(i+1/2,j) = E_x^{n}(i+1/2,j) + \frac{\Delta t}{\varepsilon \Delta y}\left(H_z^{n+1/2}(i+1/2,j+1/2) - H_z^{n+1/2}(i+1/2,j-1/2)\right)$$

$$E_y^{n+1}(i,j+1/2) = E_y^{n}(i,j+1/2) + \frac{\Delta t}{\varepsilon \Delta x}\left(H_z^{n+1/2}(i-1/2,j+1/2) - H_z^{n+1/2}(i+1/2,j+1/2)\right)$$