An Efficient Method of Analysis of Co-Planar Dipole Array Antennas

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Abstract - **Generalized impedance formulas for non-uniform array configurations based on the *Improved Circuit Theory* (ICT) are presented for the first time in an English text. To further enhance the ICT as an accurate and fast method for evaluations, a new and more compact closed-form formula is derived to replace a function requiring time intensive numerical integration during the implementation of the ICT algorithm in co-planar dipole array antenna evaluations. The resulting ICT computational scheme reduces the required CPU time by a factor of two and has the same order of accuracy as the conventional MoM. The new ICT implementation would be of considerable use as a *Computer Aided Design* (CAD) tool of co-planar dipole array antennas.**

I. Introduction

The *Improved Circuit Theory* (ICT), which was first published in 1969 [1], is still an attractive analytical method for multielement dipole antenna evaluations. It was originally developed from the understanding that the classical EMF method, which had been used for decades, included several inconsistencies. Classical EMF theory assumes the current distribution is independent of the combinations of the driving voltages, that the self-impedances are not affected by the presence of other elements, and that the mutual impedances of elements are determined by the related two elements only [2]. The expression of the input impedance obtained by the EMF method coincides with that of the method of variations, which is the reason why the results are satisfactory for situations [3] where the antenna length is about a half wavelength for which the current distribution is well expressed as a simple sinusoidal function. The idea of the ICT was to introduce a second function for the current distribution in the variational method to improve the EMF method. The ICT method executed this by employing King's [4] two term current expression. But, it is surprising that only one additional current function in the EMF method made the ICT method an accurate analytical method compared to MoM. This later turned out to correspond to the Galerkin's method applied in the implementation of the MoM.

An important feature of the ICT method is that it excludes all the above inconsistencies of the EMF method, and achieves an accurate evaluation scheme for the impedance and other important antenna parameters like the gain and far-field radiation patterns.

The ICT method has considerable advantages over other conventional methods of evaluating antennas such as the MoM. It has been empirically known that the required CPU time is less than that in a standard MoM by a factor of more than ten. For certain applications, run time is important. Our task in this paper is to achieve this optimization using an improved ICT method in co-planar dipole array analysis.

The presentation of the rest of the paper follows. Section II discusses the essentials of the ICT method which includes the presentation of the generalized input impedance formulas for any arbitrary array configuration based on the ICT theory for the first time in English text. The main advantages of the ICT method compared to other conventional methods like the classical EMF and MoM methods are also discussed in this section. Section III discusses the application of the ICT method and points out the importance of an optimum running time in an antenna CAD tool. Section IV considers the optimization of the CPU time by replacing a function requiring time intensive numerical integration with an approximate but accurate closed-form equivalent. Section V demonstrates the accuracy and the reduction in the CPU time with case studies of array systems using the ICT method with the numerical integration and closed-formed schemes compared with the conventional MoM method. Section VI states the conclusion followed by an appendix in Section VII where details of the main formulas are given.

II. Essentials of ICT Method

Figure 1 shows a general antenna system which can be analyzed by the ICT. This include arrays with elements of different lengths, spacing and radii which

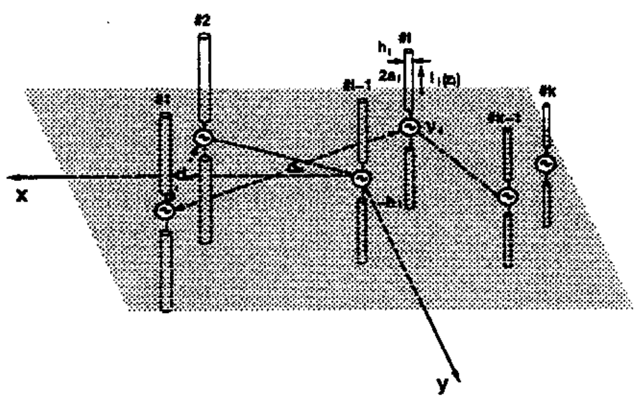

Figure 1: Dipole Antenna Array of Non-uniform Geometry.

are symmetrical with respect to the x-y plane with feeds on a common plane such as in Yagi-Uda and log-periodic dipole arrays.

Before the advent of modern computers and the possibility of versatile numerical methods like MoM, a common conventional method of analyzing such arrays for various antenna parameters was by the EMF method. The EMF method makes assumptions which lead to inaccuracies especially for non-resonant dipole antenna lengths [2]. However, the variational characteristics of the impedance of the EMF allows us to treat such a system as a variational problem. This is because the impedance expression in terms of current distribution by EMF coincides with the variational expression except for the definition of the inner product. The inner product takes the complex conjugate form in EMF while it takes the direct symmetric product form in the variational method [10].

For resonant dipoles the current functions are approximated by real functions, and the two forms coincide with each other. Thus, if the N element system in Fig. 1 is regarded as a variational problem, the expression of the current function $I_i(z_i)$ adopted from the Ritz method is [1]

$$I_i(z_i) = \sum_{l=1}^{M} I_i^l f_i^l(z_i), \quad f_i^l(0) = 1, \quad (i = 1, 2, \cdots, N). \tag{1}$$

The ICT uses the two-term current function represented as $M = 2$ given as

$$\begin{aligned} f_i^1(z_i) &= \sin k(h_i - |z_i|) \\ f_i^2(z_i) &= 1 - \cos k(h_i - |z_i|). \end{aligned} \tag{2}$$

Based on this current function, the ICT circuit equation for the system in Fig. 1 is given as [1]

$$\sum_{m=1}^{2} [Z^{lm}][I^m] = [V], \quad (l = 1, 2), \tag{3}$$

where the $N \times 1$ matrices $[V]$ and $[I^m]$ are the input voltage and current respectively and $\left[Z^{lm}\right]$ is the generalized impedance matrix of order $N \times N$ whose details are given in Section VII.A of the appendix. The ICT circuit equation as defined in Eq. (3) is a considerable improvement over the EMF method for the evaluation of multielement antennas.

It is important to note that in the ICT method, the use of the current functions in Eq. (2) makes it possible to express the far-field radiation patterns in closed-form as [8]

$$F(\theta, \phi) = \sum_{i=1}^{N} \sum_{m=1}^{2} I_i^m g_i^m(\theta, \phi) \tag{4}$$

where

$$\begin{aligned} g_i^1(\theta, \phi) &= \frac{\cos(kh_i \cos\theta) - \cos kh_i}{\sin\theta} \\ g_i^2(\theta, \phi) &= \frac{\sin(kh_i \cos\theta)\sin\theta}{\cos\theta} \\ &- \frac{sinkh_i - \sin(kh_i\cos\theta)\cos\theta}{\sin\theta}. \end{aligned} \tag{5}$$

This yields a considerable saving in time compared with the conventional MoM. In the latter, more than two current expansion terms per element are required to accurately evaluate the same quantities.

III. Applications of the ICT method

The ICT method has been popular among antenna designers in Japan. For instance, Inagaki et al. [5]

carried out design of dipole arrays with specified radiation patterns in the magnetic plane. The configuration of the elements in the array are non-uniform and the integral of the current distribution is expressed applying the Improved Circuit Theory. This was the first antenna synthesis theory which considered the change of current distributions to be dependent on the combination of the driving voltages. Oyama et al. applied the ICT method in finding an expression for the characteristics of Yagi-Uda antennas with a vector diagram. They also applied it in the optimum design of Yagi-Uda antennas with a minimum gain specified [6-7]. Kawakami et al. applied ICT in the analysis of log-periodic dipole antennas [8].

The ICT scheme has been installed in an expert system for linear antennas developed by ATR, Japan. Originally designed as a prototype *Computer Aided Engineering* (CAE) tool of wire antennas on PCs, this system now runs on an *Engineering Work Station* (EWS) and can be used to display graphics of analyzed results of different antenna geometries simultaneously [9]. An ICT algorithm has been found useful in this system because of the relatively quicker response required by such interactive and real time system.

IV. Faster ICT Implementation Scheme

For the antenna designer, the importance of having a fast method of evaluating an antenna system cannot be overemphasized. A faster method is especially desirable in an expert system discussed in the previous section.

In furtherance of the objective of achieving a faster antenna evaluating method, a recent study [11-12] shows the usefulness of replacing time consuming functions within the ICT algorithm. To achieve this in the *Faster Improved Circuit Theory* (FICT) [11-12], closed-form approximate formulas are derived to replace the most time consuming function in the algorithm. Part of resulting formula in FICT is bulky because it is based on polynomial functions derived from data banks using least square curve fitting techniques.

In this section, the same function [11-12] is analyzed again and a new and more compact closed-form formula is proposed which avoids time intensive numerical integration in the implementation of the ICT method. The new formula is shown to be capable of reproducing the results of the numerical integration scheme and reduces more than half the time required in implementing the conventional ICT method. It is applicable to the practical range of most co-planar dipole arrays of arbitrary configurations.

A. *Closed-Form Formula*

The current method of evaluating Eq. (27) of Section VII.A of the appendix which is restated here as

$$E_y(x) = 2\int_0^x \frac{\exp\left(-j\sqrt{t^2+y^2}\right)}{\sqrt{t^2+y^2}}dt. \qquad (6)$$

is by numerical integration, which is considerably time intensive. In Eq. (6) each variable ($x = kh, y = kd$), has been multiplied by k (the wavenumber) to achieve non-dimensionality. This function has been shown [11-12] to require more than fifty percent of the total CPU time during a typical analysis of a multielement system of linear arrays and has therefore been identified as the most time consuming function in the ICT algorithm. The objective here is to find a closed-form formula which avoids numerical integration. Two partial formulas are derived for large and small element spacing respectively, and then combined to replace the original equation (Eq. 6).

1) *Partial Formula for Large Inter-Element Spacing:* For the general case of large inter-element spacing between elements of a linear array of antennas the following formula can be derived to replace Eq. 6. For such situations, we make the substitution $t = y \sinh w$ in Eq. (6) and simplify to get

$$E_y(x) = 2\int_0^{\sinh^{-1}\frac{x}{y}} e^{-jy\cosh w}dw. \qquad (7)$$

Using Mathematica [13] the integrand of Eq. (7) is expanded in a power series with respect to w and then integrated over the limits given. The resulting approximate function is given as the two-variable function

$$E_y(x) \approx 2f(x,y)e^{-jy}. \qquad (8)$$

Details of $f(x, y)$ are given in Section VII.B of the appendix.

2) *Partial Formula for Small Inter-Element Spacing:* It will later be shown that Eq. (8) is accurate only for large values of y. For this reason we would seek for another closed-form solution of Eq. (6) which is valid for smaller values of y. This applies to evaluation of the self-term where the spacing is equivalent to the element radius and also for situations of electrically small spacing. For this case Eq. (6) is first expressed as

$$E_y(x) \quad = E_y(y) + F_y(x) \tag{9}$$

where $E_y(y)$ is as defined in Eq. (8) and

$$F_y(x) = 2\int_y^x \frac{\exp\left(-j\sqrt{t^2+y^2}\right)}{\sqrt{t^2+y^2}}dt. \tag{10}$$

To find an approximate function for Eq. (10) we make the substitution $w = t + \sqrt{t^2+y^2}$ to get

$$F_y(x) = 2\int_{(\sqrt{2}+1)y}^{x+\sqrt{x^2+y^2}} \frac{e^{-j\left(\frac{w}{2}+\frac{y^2}{2w}\right)}}{w}dw. \tag{11}$$

The following component of the function in the integrand of Eq. (11) which is difficult to treat analytically is expanded into a power series with respect to w

$$f(y) = e^{-j\frac{y^2}{2w}}. \tag{12}$$

Using Mathematica [13] it can be shown that in the interval $2 \leq y \leq 10$, the difference between a five term power series expansion of $f(y)$ of Eq. (12) and the original function is less than 10^{-4}.

Therefore using this five term series expansion of Eq. (12) we can express Eq. (11) approximately as

$$F_y(x) \approx 2\int_{(\sqrt{2}+1)y}^{x+\sqrt{x^2+y^2}} \left(1 - \frac{y^4}{8w^2} - j\frac{y^2}{2w}\right)\frac{e^{-j/2w}}{w}dw. \tag{13}$$

Equation (13) is then easily integrated using Mathematica [13] to give

$$F_y(x) = 2g(x,y) \tag{14}$$

where details of $g(x,y)$ is given in the Section VII.C of the appendix. *It should be noted that Eq. 9 is valid when Eq. 8 is also valid, but the converse is not necessarily so.*

3) *Valid Regions of Partial Formulas:* In this section we shall establish the valid regions of Eqs. 8 and 9. To achieve this we examine an array configuration of two elements of length $(2h/\lambda)$ in the interval $0.1 \leq 2h/\lambda \leq 2$ and inter-element spacing in the interval $10^{-4} \leq d/\lambda \leq 10$. This is then used to validate Eqs. (8) and (9). Most practical arrays of interest would normally fall within this region. The lower values of d/λ represent the radius of an element in the array. Relative error in amplitude for Eqs. (8) and (9) on one hand, and Eq. (6) on the other (all dimensionless complex quantities) are computed and compared. The one with least error would be best in representing Eq. (6) for any particular array configuration. Our aim is to have a function which gives minimum error at optimum time in order to reduce the time required to carry out the analysis of wire antennas using the ICT method.

As shown in Eq. (29) of Section VII.B of the appendix, Eqs. (8) and (9) are functions of the power expansions of b defined as

$$b = \ln\frac{x+\sqrt{x^2+y^2}}{y}. \tag{15}$$

Eq. (15) is such that, for certain combinations of x and y, higher order terms of it are vanishingly small and so some terms of Eq. (29) which are series expansion of this equation can be truncated, thereby optimizing the CPU time.

To determine the valid regions of the two equations we have computed and plotted various contour graphs of the percent relative amplitude errors between them and Eq. (6). A first heuristic choice is to fix a maximum permissible relative amplitude error of 1%. Fig. 2 (a) shows the 1% relative amplitude error contours for Eqs. (8) and (9). From Fig. 2(a) we can deduce that except for very small portions (shaded), at least one of the Eqs. (8) or (9) can be used in the computation of Eq. (6) for any particular array configuration. Increasing the maximum permissible error to 1.26% gives the contour plot in Fig. 2(b) which shows that at least one of the equations can be used to compute Eq. (6) with maximum error of 1.26% for the entire region.

4) *Combined Formula:* For this condition we have fitted a linear relationship between x and y as:

$$s(x,y) = x - 0.216831y - 0.0533064, \tag{16}$$

where Eqs. (8) and (9) are evaluated based on the conditions

$$E_y(x) \approx \begin{cases} 2f(x,y) & s(x,y) \geq 0 \quad \text{Eq. 8} \\ 2f(x,y) + 2g(x,y) & s(x,y) < 0 \quad \text{Eq. 9.} \end{cases} \tag{17}$$

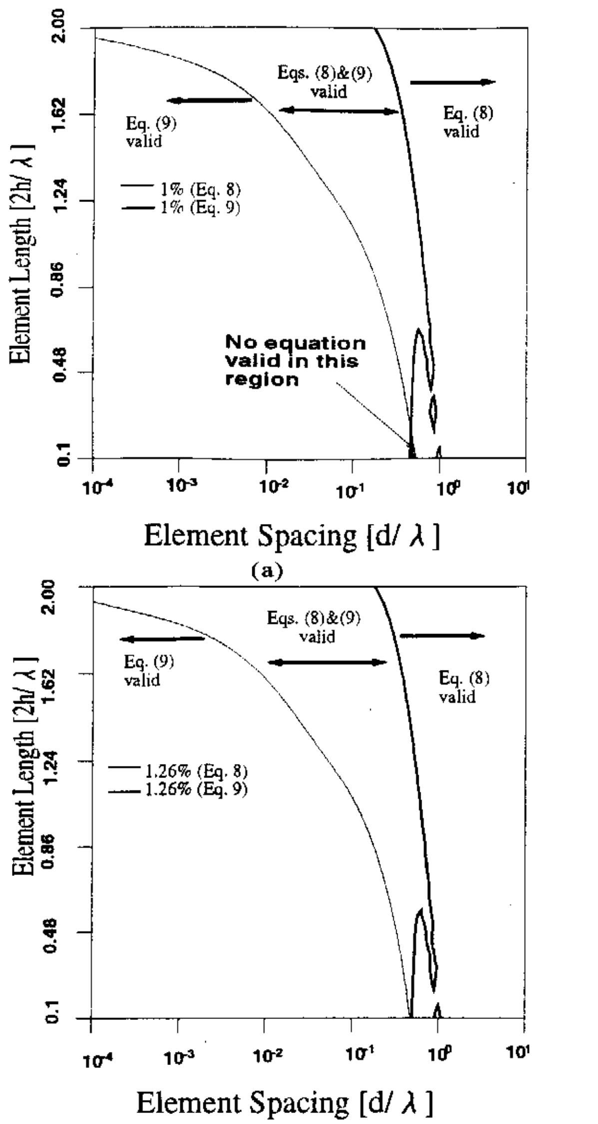

Figure 2: Contour plots of relative error in amplitude (a) 1%, (b) 1.26%, with valid formulas for each region indicated.

Further more detailed analysis shows that the following relations can be used to truncate unnecessary terms (Eq. 15) of Eq. (29) in the evaluation of Eq. (17):

$$\begin{aligned} s_1(x,y) &= 4.4612y - x - 0.2458 \\ s_2(x,y) &= 0.57312y - x + 0.811 \\ s_3(x,y) &= 0.11y - x + 0.33 \\ s_4(x,y) &= 32.475y - x - 3.4895. \end{aligned} \qquad (18)$$

Table 1: Conditions for detail evaluation of Eq. (17).

Solution of Eq. (6) with	Terms of Eq. (29) Required	Relation Between x and y
Eq. (8)	21	$s_1(x,y) < 0$
Eq. (8)	15	$s_1(x,y) \geq 0$, $s_2(x,y) < 0$
Eq. (8)	9	$s_3(x,y) < 0$
Eq. (8)	5	$s_3(x,y) \geq 0$
Eq. (9)	9	$s_4(x,y) \leq 0$
Eq. (9)	5	$s_4(x,y) > 0$

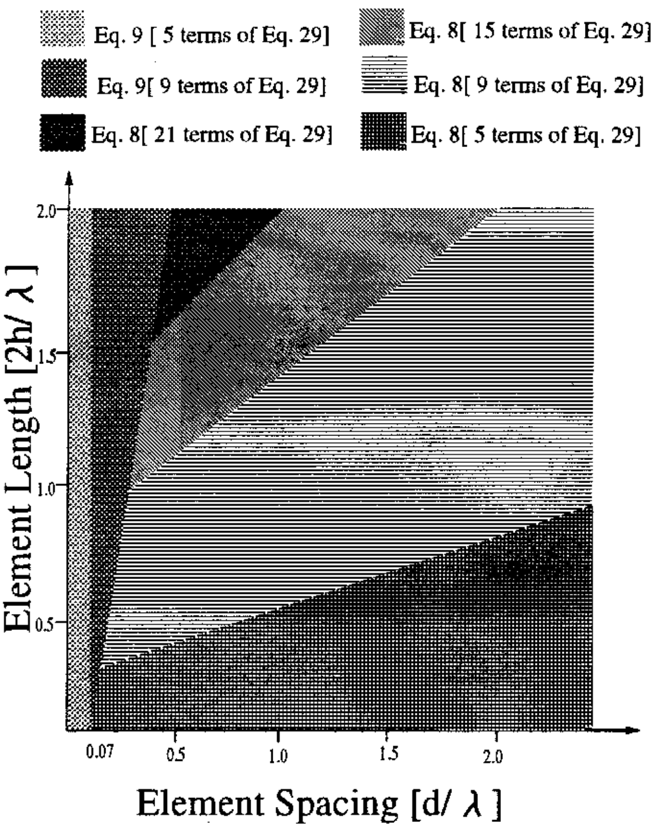

Figure 3: Demarcated regions.

Using the relations defined in Eq. (18) the following conditions in Table 1 are used for detailed evaluation of the closed-formed function defined in Eq. (17).

By way of explaining the rationale behind Table 1, the accurate evaluation of Eqs. 8 and 9 depends on the number of terms(defined by Eq. 15) of Eq. 29 used. Now for certain array configurations, some terms of Eq. 29 can be ignored since they are negligibly small but their very inclusion could only serve to increase the CPU time without improving on the numerical accuracy. The truncation of such terms using the relations in the term column of the Table 1 has been shown to improve considerably the computational efficiency of the algorithm.

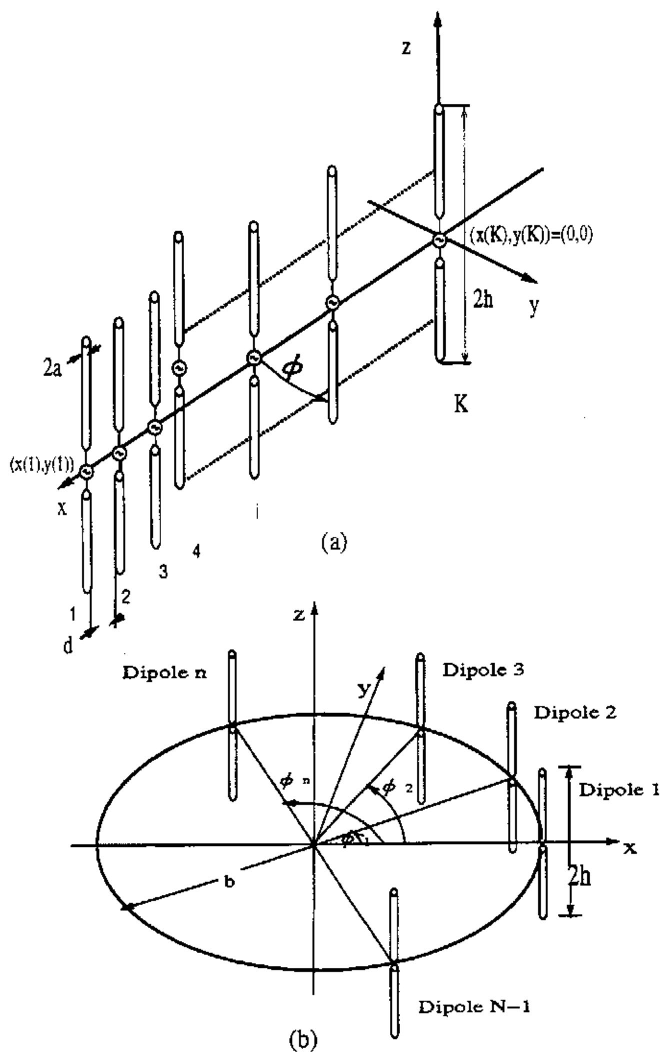

Figure 4: (a) Linear and (b) circular array systems.

The relations in Table 1 have been used to draw an approximate and more graphic demarcation for each function as shown in Fig. 3. These relations are then easily used as a closed-form replacement of Eq. 6. The valid region of Eq. 17 falls within the practical range of linear array systems.

V. Case Study Using Modified Algorithm

In this section we validate the new ICT algorithm by considering the analysis of various co-planar dipole arrays including tightly-coupled systems like Yagi-Uda array antennas. The results are compared with those of the conventional ICT method and MoM.

The computational times and storage required by these methods are also compared to demonstrate the efficiency of the new ICT implementation. The ICT algorithm using the closed-formed formula (Eq. 17) is designated as MICT.

A. *Linear and Circular Arrays of Equal Lengths and Spacing*

Analysis of typical linear and circular array systems of the form shown in Fig. 4 have been carried using ICT and MICT. The results are compared with analysis of a similar system by Thiele et al. [14]. The MoM procedure in this paper is based on seven current expansions a piecewise sinusoidal *Garlekin method.* These number of expansion functions have been found adequate in accuracy compared to the two ICT methods.

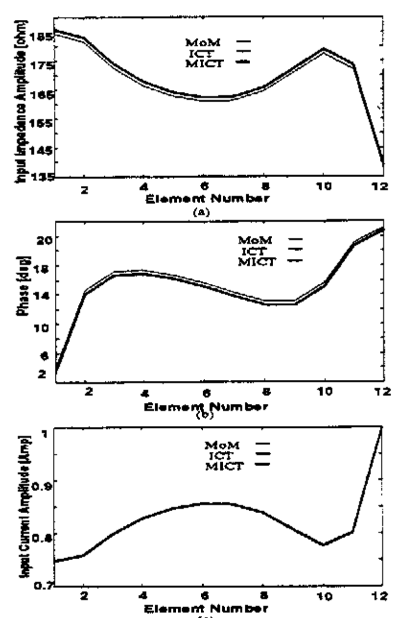

Figure 5: Twelve Element Linear Array (a) Impedance, (b) phase of impedance, and (c)current.

Following [14], we consider a typical linear array made of 12 elements each of equal length, radius and inter-element spacing of $\lambda/2$, 0.0001λ and $\lambda/2$ respectively. Each element in the array is considered to be supplied from co-phasal sources of uniform amplitude and with a series resistance of 72Ω. For this end-fire linear array, the main beam maximum is pointed at $\phi = 45°$ as shown in Fig. 4(a) [14]. Fig. 5 shows the results of an analysis with the different methods.

The circular array we consider is similar to the linear system described above except that the main beam direction is now pointed at $\phi = 0°$ shown in Fig. 4(b). The results of the three methods are given in Fig. 6. Fig. 7 shows the far-field patterns of linear and circular arrays described above. It is clear from these results that the three methods are very much in agreement, especially in the computation of the far-field radiation patterns.

In particular the results of the two ICT methods are so close that we can not easily distinguish between them, thus validating the accuracy of the closed-form formula.

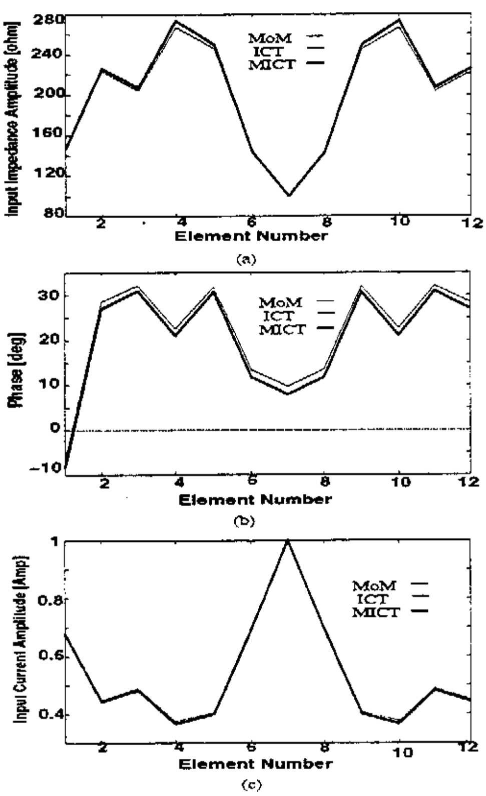

Figure 6: Twelve Element Circular Array (a) Impedance, (b) phase of impedance, and (c) current.

B. *Analysis of Yagi-Uda Arrays*

Using the generalized impedance formulas defined in Eqs. 19-20 and the new closed-form formula (Eq. 17), we have analyzed various Yagi-Uda Arrays of a typical form shown in Fig. 8.

The characteristic of a particular case of a six element Yagi-Uda Array ($L_R = 0.482\lambda$, $L = 0.456\lambda$, $L_D = 0.437\lambda$) of equal inter-element spacing of (0.2λ) and equal radii of 0.0025λ are shown in Table 2 for the different methods. Here L_R, L and L_D are the lengths of reflector, driver and directors respectively. The results of a similar structure in [14] also based on MoM are shown. The MoM used in this paper is based on [15] with 126 pulse expansion functions used to achieve the input impedance formulas as the other methods. All results are in reasonable agreement.

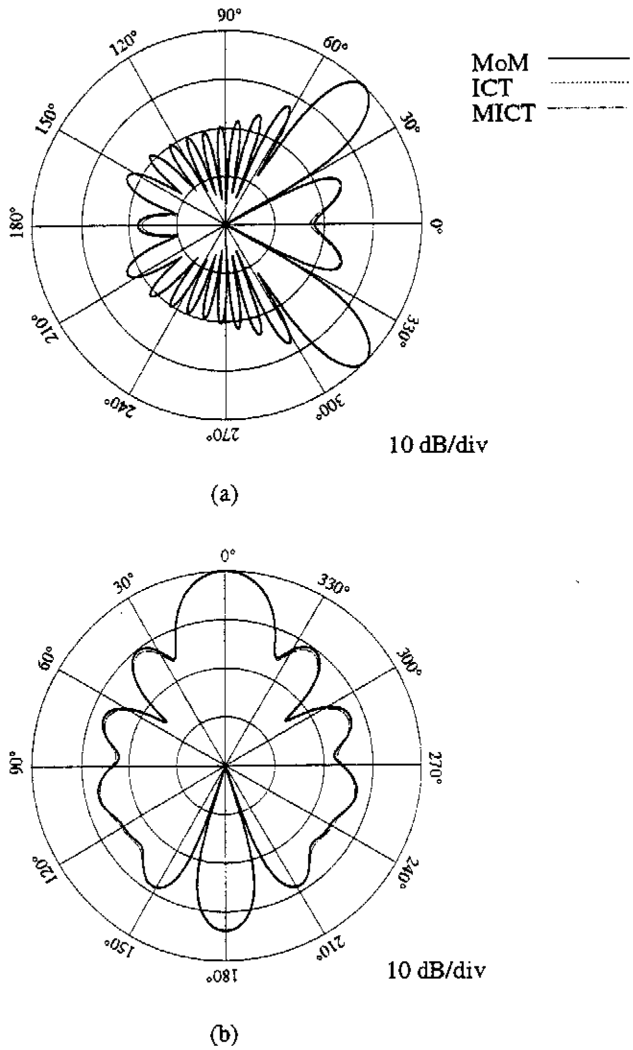

Figure 7: Patterns of 12 Element (a) linear with main beam direction steered to $\phi = 45°$, (b) circular array.

Figure 8: Configuration of a general Yagi-Uda Array.

Table 2: Characteristics of Equally Space Yagi-Uda Array.

Method	Gain [dB]	Input Impedance [Ω]
MoM [14]	11.2	$51.3 - j1.9$
MoM[15]	11.23	$50.6 - j4.03$
ICT	11.25	$50.3 - j2.48$
MICT	11.25	$50.3 - j2.54$

Table 3: Characteristics of a Yagi-Uda Array for TV Channel 15 operation.

Method	Gain [dB]	Input Impedance [Ω]
MoM [14]	11.50	$59.50 + j47.50$
MoM[15]	11.54	$59.45 + j44.61$
ICT	11.53	$59.26 + j43.70$
MICT	11.52	$59.17 + j43.50$

The characteristics and H-plane radiation pattern of a six element Yagi-Uda with the following geometry: $L_R = 0.5\lambda$, $L = 0.47\lambda$, $L_D = 0.43\lambda$ $S_R = 0.25\lambda$, $S_D = 0.30\lambda$ are shown in Table 3 and Fig. 9 respectively. Each element is of radius 0.0026λ. This configurations is a typical Yagi-Uda antenna for operation of midband frequency for Channel 15 [14]. Again, all the methods are in close agreement.

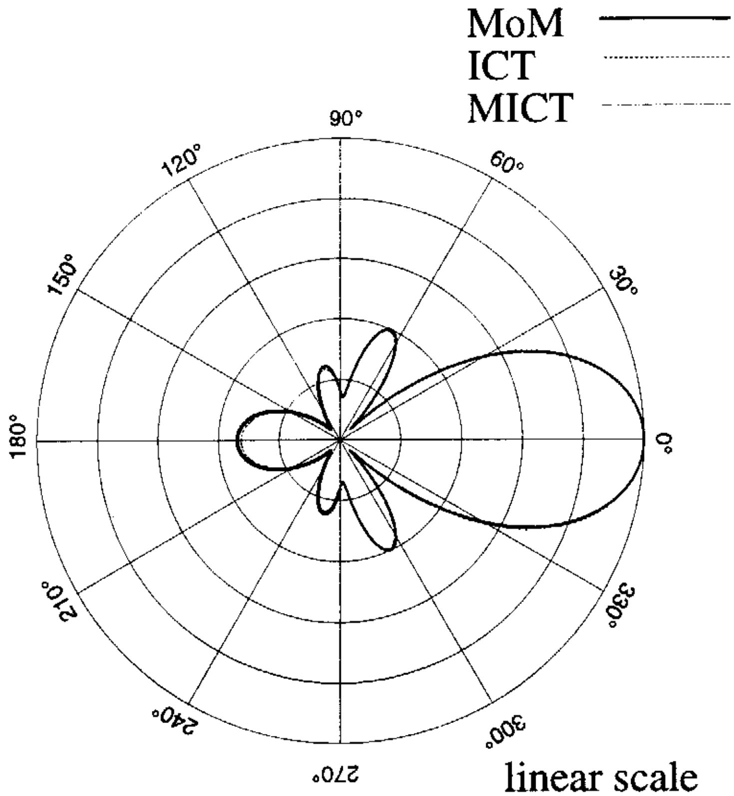

Figure 9: H-plane pattern of Six-element Yagi-Uda array for TV Channel 15 [14].

C. *CPU Time and Computer Storage*

Since the conventional workstation is time sharing by nature with variable loading not easily controllable by a particular user, it is quite difficult to establish similar conditions in order to accurately determine the CPU time requirements of each method described in the previous section. It has therefore be found more convenient to use an NEC PC-9801VX personal computer to carry out the CPU time statistics analysis.

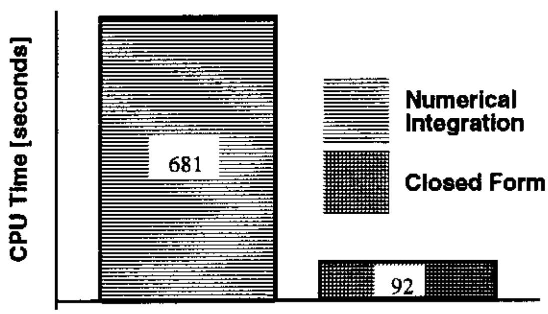

Figure 10: Closed form and numerical integration CPU times.

1) *CPU Time of Numerical and Closed-form Formulas:* The CPU time required in the evaluation using the closed-form and numerical integration is first carried out. To ensure that we cover each subregion of the closed-form formula according to the conditions defined in Table 1 we did the evaluation with 101×101 points in the intervals $0.1 \leq 2h/\lambda \leq 2$ and $10^{-4} \leq d/\lambda \leq 10$. The results are shown in the Barchart of Fig. 10.

It is clear from the results that the closed-form formula is capable of reproducing Eq. (6) accurately and at one seventh the CPU time on the average. For configurations requiring a lesser number of terms, as shown in Fig. 3, the CPU time could be shorter.

Using the same personal computer as before the end-fire linear array described in Section IV.A has been analyzed using the two ICT and MoM algorithms. The results are shown in the Barchart in

Fig. 11. The advantage of the ICT and MICT methods in CPU time required to analyze wire antennas is clearly demonstrated here.

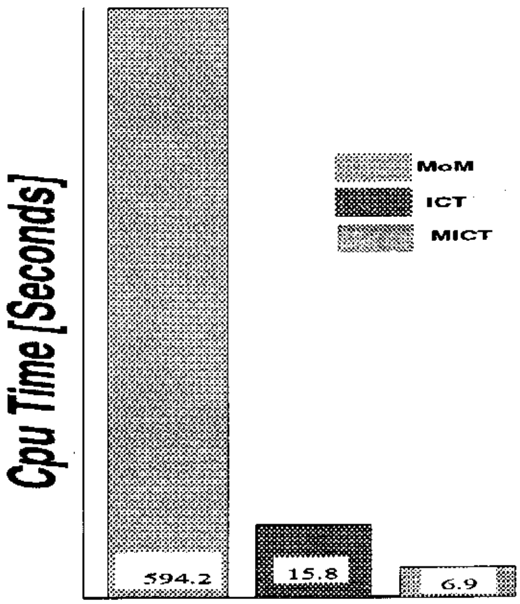

Figure 11: CPU time Statistics of 12 element linear Array analyzed with two ICT and MoM algorithms.

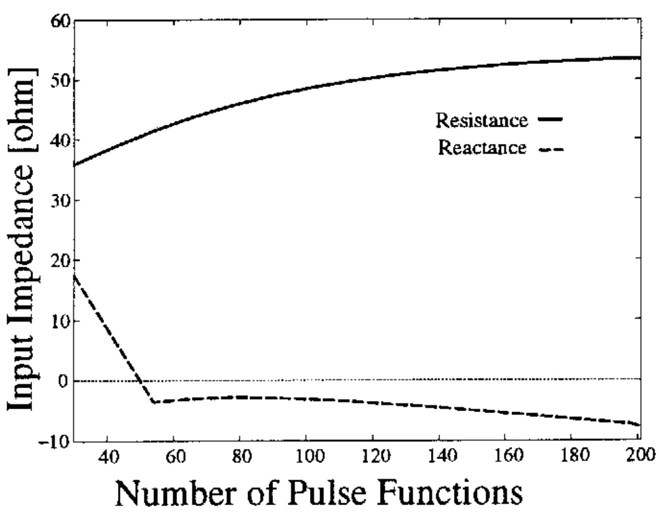

Figure 12: Curve showing convergence of input impedance as the number of pulse expansion functions with point matching is increased for six Element Yagi-Uda antenna.

2) *Computer Storage Limitation :* The moment method used for the Yagi-Uda Array is based on [15] and required about 126 pulse functions to achieve accurate impedance results comparable to the other methods. This required considerable storage and we are therefore not surprised that the computation can not be easily carried out on both NEC PC-9801VX and NEC PC-9801DA personal computers without storage problems.

On Sun Workstation (Sparcstation 10), the rough[1] computational times were 36 seconds for the MoM scheme, 47 milliseconds for ICT and 20 milliseconds for MICT.

Of course the relatively large computational requirement and storage of the MoM can be explained in this case by the fact that though the array is actually a discontinous structure in configuration, the evaluation has been carried out as if the array elements were connected togather and formed a continuous structure. Piecewise pulse functions are then used to express closed-form the formula for the impedance matrix. But this requires us to use a large number of expansion functions to achieve reasonably accurate results. This leads to large matrix to be inverted. Figure 12 shows convergence curves for the input impedance for a six Element Yagi-Uda antenna. The largest matrix to be inverted by the ICT method is $2N \times 2N$ where N is the number of elements in the array.

In MICT all formulas are expressed in closed-form, making it very efficient in terms of CPU time. The usefulness of MICT as an efficient CAD/CAE tool as described in Section III is clearly demonstrated.

VI. Conclusion

We have for the first time in English text presented the Improved Circuit Theory generalized input impedance formulas which can be useful in the analysis of multielement dipole antennas with arbitrary configurations. By deriving a new and closed-form formula which is valid for practical range of most multielement antenna systems to avoid time intensive numerical integration, it has been possible to reduce more than half the CPU time required to analyze multielement antennas using the ICT method. It has also been demonstrated that the ICT method offers a much faster method of analyzing multielement antennas compared with the conventional method of moment. This faster ICT implementation scheme further enhances the method as a useful CAD/CAE tool.

[1]Workstation was shared by other users during computation and so same conditions could not be guaranteed for each method unlike a PC which can ensure this.

However, the conventional ICT two current functions defined in Eq. 2 has been found to be inadequate for co-planar dipole element lengths greater that 1.85λ [16]. Even though this covers a large range of practical antennas, applications to much longer dipole antennas (e.g. large log-periodic dipole arrays required for wide bandwidth applications [17]) would necessitate the use of more appropriate trial function. A choice of better trial functions for expanded application of ICT is currently under study [16].

VII. Appendix

A. *Generalized Impedance for Non-uniform Array Configurations*

If the configurations of a multielement-element system are non-uniform, (the sizes, length and inter-element spacing vary), then the generalized impedance of the system is given as [8,18]

$$\begin{aligned} Z^{11}_{nm} &= \frac{j}{4\pi}\sqrt{\frac{\mu_0}{\varepsilon_0}}\,[-C^+\sin\theta^+ - C^-\sin\theta^- \\ &+ S^+\cos\theta^+ + S^-\cos\theta^-] \end{aligned} \tag{19}$$

$$\begin{aligned} Z^{12}_{nm} &= Z^{21}_{mn} = \frac{j}{4\pi}\sqrt{\frac{\mu_0}{\varepsilon_0}}\,[C^+\cos\theta^+ - C^-\cos\theta^- \\ &+ S^+\sin\theta^+ - S^-\sin\theta^- - 2E_{kd}(\theta_m)\cos\theta_n \\ &+ E_{kd}(\theta^+) - E_{kd}(\theta^-)] \end{aligned} \tag{20}$$

$$\begin{aligned} Z^{22}_{nm} &= \frac{j}{4\pi}\sqrt{\frac{\mu_0}{\varepsilon_0}}\,[C^+\sin\theta^+ - C^-\sin\theta^- \\ &- S^+\cos\theta^+ + S^-\cos\theta^- - 2E_{kd}(\theta_n)\sin\theta_m \\ &- 2E_{kd}(\theta_m)\sin\theta_n + \theta^+E_{kd}(\theta^+) - \theta^-E_{kd}(\theta^-) \\ &- j2\exp\left\{-j\sqrt{(kd)^2 + (\theta^+)^2}\right\} \\ &+ j2\exp\left\{-j\sqrt{(kd)^2 + (\theta^-)^2}\right\}] \end{aligned} \tag{21}$$

where

$$d = \begin{cases} a & n = m \\ d_{nm} & (n \neq m) \end{cases} \tag{22}$$

d_{nm} is the inter-spacing between elements n and m in the array

$$\theta^{\pm} = \theta_n \pm, \theta_m, \quad \theta_n = kh_n, \quad \theta_m = kh_m \tag{23}$$

$$\begin{aligned} C^{\pm} &= C_{kd}(\theta_n) \pm C_{kd}(\theta_m) - C_{kd}(\theta^{\pm}) \\ S^{\pm} &= S_{kd}(\theta_n) + S_{kd}(\theta_m) - S_{kd}(\theta^{\pm}) \end{aligned} \tag{24}$$

$$\begin{aligned} C_y(x) &= E_i\left\{-j\left(\sqrt{x^2+y^2}+x\right)\right\} \\ &- E_i\left\{-j\left(\sqrt{x^2+y^2}-x\right)\right\} \end{aligned} \tag{25}$$

$$\begin{aligned} S_y(x) &= jE_i\left\{-j\left(\sqrt{x^2+y^2}+x\right)\right\} \\ &- jE_i\left\{-j\left(\sqrt{x^2+y^2}-x\right)\right\} \\ &- j2E_i(-jy) \end{aligned} \tag{26}$$

$$E_y(x) = 2\int_0^x \frac{\exp\left(-j\sqrt{t^2+y^2}\right)}{\sqrt{t^2+y^2}}dt \tag{27}$$

$$E_i(-jt) = -\int_t^{\infty}\frac{\exp(-js)}{s}ds \tag{28}$$

B. *Details of $f(x,y)$*

The function $f(x,y)$ is simplified from the Mathematica [13] integration as

$$\begin{aligned} f(x,y) &= b - j\frac{y}{6}b^3 - y\left[\frac{j+3y}{120}\right]b^5 \\ &+ y\left[\frac{-j-15y+j15y^2}{5040}\right]b^7 \\ &+ y\left[\frac{-j-63y+210jy^2+105y^3}{362880}\right]b^9 \\ &+ 10^5\delta\left[-6+j50+80y^2+23.7y^3\right]b^{11} \\ &+ 10^5\delta y\left[j3+10.6y-8.35y^2-1.67y^3\right]b^{13} \\ &+ 10^4\delta y\left[j1.5+9.2y-j14.47y^2-7.234y^3\right. \\ &+ \left.j1.0334y^4\right]b^{15} + 100\delta y\left[j5+58y-163.4y^2\right. \\ &- \left.154.3y^3+j53.2y^4+5.7y^5\right]b^{17} \\ &+ 10\delta y^2\left[30-j135y-215y^2+j135y^3\right. \\ &+ \left.34y^4-j2.83y^5\right]b^{19} + \delta y^4\left[-200\right. \\ &+ \left.j200y+100y^2-j20y^3-1.3y^4\right]b^{21} \end{aligned} \tag{29}$$

where

$$b = \ln\frac{x+\sqrt{x^2+y^2}}{y} \quad \text{and} \quad \delta = 10^{-11}y^2. \tag{30}$$

C. *Details of* $g(x,y)$

The function $g(x,y)$ is simplified from the Mathematica [13] integration as

$$\begin{aligned} g(x,y) &= -j10^{-2}y\,[20.710678 - j1.0723305y \\ &- 1.2944174]\,e^{-jv} \\ &- j10^{-2}\left[j6.25 - 50y^{-2} + 3.125\right] y^4 u^{-2} \\ &- \left[E_i(v) - E_i\left(\frac{u}{2}\right)\right]\left(1 - \frac{y^2}{8}\right)^2 . \qquad (31) \end{aligned}$$

Here $v = -j1.207107y$, $u = -j(x + \sqrt{x^2+y^2})$ and $E_i(z)$ is as defined in Eq. (28).

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