A Comparative Study of Expansion Functions Using the Boundary Residual Method on a Linear Dipole - Part I: Entire-Domain Functions

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Abstract. The Boundary Residual Method, which is a specialization of the Least Squares Method, is described. A significant benefit of the approach is that error in the residual satisfaction of the boundary condition is explicitly reported. Use of error values facilitates better monitoring of solution convergence as expansion functions are added to the underlying model. Furthermore, better discrimination between competing models is possible when errors are known. These concepts are explored and applied to dipoles of various lengths with key findings reported.

Introduction.

One of the earliest models, if not the earliest model, of the current on a linear dipole was that of a sinusoidally distributed current, an entire domain function. This model still appears in textbooks produced today but was already under critical review in the early 1950s. The shortcomings in this simple model motivated King [1] in 1959 to develop a two-term formula for the current. Subsequently, in 1967 King [2] proposed his "three-term theory" that provided a reasonable prediction of the current distribution on monopoles up to lengths in excess of one and one quarter wave lengths. According to a review by Duncan and Hinchey [3] in 1960, Storm [4] was the first to introduce the idea of expanding the dipole current in a Fourier Series. Others [5] subsequently proposed the use of polynomial expansions. However, it is the Fourier series that appears to have been adopted by most workers using entire-domain expansions. In a paper in 1965, Richmond [6] examined not only the performance of the Fourier series, but also the series of Maclaurin, Chebyshev of the first kind, Hermite and Legendre. He examined them up to the first five terms. His comments were "it would be advantageous to expand the current in a series of functions which converges more rapidly than the Fourier series……the Chebyshev and Legendre series appear most promising, but this matter requires further investigation". No record of this further investigation, or similar ones, can be found. The purpose of this current work is to report the results of such an investigation – one that takes advantage of the enhanced tools made available by the passage of time.

Review of the Methodology.

The Method of Weighted Residuals, MWR, has been discussed and researched in many forms [7]. One form of the MWR utilizes least squares minimization in order to arrive at a solution to whatever problem is under consideration. The Boundary Residual Method, BRM, first introduced by Davies in 1973 [8], appears to be the first application of the least squares method to computational electromagnetics (an earlier application in the area of acoustics had been reported [9]). In 1990, Bunch and Grow [10] [11] developed a formalism for implementing the BRM in a manner that appears much like point matching, but without the associated short-comings. The method of Bunch and Grow utilizes nodes and weights from widely used integration formulae and the resulting equations are solved using available procedures for solving rectangular matrices. In this work, Singular Value Decomposition, SVD [12], is used. Use of SVD results not only in the solution of the design matrix, but also provides both the value of the condition number of the operator matrix and an estimate of the square of the residual error. It will be shown in this paper that information on these two measures proves to be of great help in the understanding, and evaluation, of the results.

The ideas of Bunch and Grow are reviewed next. Assume a boundary-value problem in which the electric field tangential to a boundary can be expressed as:

$$L\sum_{1}^{n}\alpha_i f_i\left(s\right) \cong E(s) \tag{1}$$

where s is the boundary over which the tangential field is matched, and $E(s)$ is the value of the tangential incident field at this boundary. The sum of the functions, in conjunction with the operator, L, is assumed to approximately represent the field on the boundary. Weighting functions, W_j, $1 \le j \le m, m \ge n$, can be multiplied on both sides of equation (1) and integrated over the boundary, s, to produce a matrix equation:

$$MA \cong B \tag{2a}$$

$$M_{ji} = \int_s W_j(s) Lf_i(s) ds \tag{2b}$$

$$B_j = \int_s W_j(s) E(s) ds \tag{2c}$$

Defining the residual along the boundary as:

$$R(s) = MA - B \tag{3}$$

and minimizing the integral of the residual magnitude over the boundary in the least squares sense produces the basic equation, or *normal* equation, of the Boundary Residual Method, namely:

$$M^t MA = M^t B \tag{4}$$

The condition number of $M^t M$ is the square of the condition number of M alone and hence the solution of (4) will be unstable at an earlier stage than if only M is considered. For this reason, a solution of (4) is not attempted. Instead Bunch and Grow restate (2) as:

$$\vec{M}A = \vec{B} \tag{5}$$

$$\vec{M} = \begin{bmatrix} \sqrt{q_1} Lf_1(s_1) & \dots & \sqrt{q_1} Lf_n(s_1) \\ \vdots & \vdots & \vdots \\ \sqrt{q_m} Lf_1(s_m) & \cdots & \sqrt{q_m} Lf_n(s_m) \end{bmatrix}$$

$$\vec{B} = \begin{bmatrix} \sqrt{q_1} E(s_1) \\ \vdots \\ \sqrt{q_m} E(s_m) \end{bmatrix}$$

and solve this set of equations directly. q_i and s_i are, respectively, the weights and locations associated with any conventional integration method. An approach similar to this was discussed, in a general mathematical sense, by Hildebrand [13] as early as 1952.

The matrix $\vec{M}$ is generally not square, with the number of rows, m, intentionally greater than the number of columns, n. Equation (5) can be solved using, among others, the Singular Value Decomposition, SVD, method [12] [14]. This approach decomposes the matrix into:

$$\vec{M} = U \sigma V^t \tag{6}$$

where U and V are orthogonal matrices and σ is a diagonal matrix of singular values. The ratio of the largest to the smallest singular values, σ_1 / σ_n, is the 'condition number' for the matrix and indicates the number of significant digits needed in manipulating $\vec{M}$. The sought for solution is then:

$$A = \vec{M}^{-1} B = V \sigma^{-1} U^t B \tag{7}$$

The squared residual norm, ρ^2, is given by:

$$\rho_{LS}^2 = \| e_k \|^2 = \| MA^{(k)} - B \|^2 = \sum_{k+1}^{m} U^t B \tag{8}$$

where k is the rank of $\vec{M}$ [14, p242]. The ready availability of the value of this norm provides a tool for assessing the suitability of the expansion functions used in the calculation of A. ρ_{LS}^2 appears as a function of m and its value depends on the degree of match between the two sides of equation (1). Typically, the right-hand side of (1) is smooth and when the expansion functions used on the left-hand side are also smooth then $m \ge n + 10$ is found to be more than sufficient to achieve a stable value of ρ_{LS}^2. When the expansion functions are sub-domain then the relationship between m and n needs further attention. In order to avoid the need for concern with this issue and because the objective of this study was accuracy, all results were computed using m=128. In subsequent sections of this paper, both condition number and the squared residual norm; ρ_{LS}^2, will be used in the evaluation of the fit of the expansion functions. In particular, only results where the condition number is well within a limit suitable for the precision (double) of the calculations are presented.

In order to evaluate the suitability of an expansion function series, two measures of convergence are defined.

1) ***Global***; for purposes of observing convergence over the entire domain, De Boor [15, p23] proposed a decay exponent that anticipates that, as a function of n, $\| e_n \|$ decreases to zero like βn^{α} for some constant β and some (negative) constant, alpha, α. If

$\| e_n \| \sim \beta n^{\alpha}$ then $\| e_n \| / \| e_m \| = (n/m)^{\alpha}$, and we can estimate the decay exponent from

$$\alpha = (\log(\| e_n \|) - \log(\| e_m \|)) / \log(n/m) \quad (9)$$

Ideally, initially α would be a large, negative number and then, as n increases, at some point quickly approach zero. This ideal state is most likely to occur when the model is an excellent representation of the problem.

2) ***Local***; frequently, the value of a specific variable at a specific location is both required and used to monitor convergence – the current at the center of a dipole for example. Consistent with the above model, we assume

$$X(n) = X_{\infty} + A.n^{D} \quad (10a)$$

which leads to:

$$X(n) = X_{\infty} + (\partial X / \partial n) n / D \quad (10b)$$

Use of (10b) permits both a qualitative assessment of the convergence process as well as a quantitative estimate of X_{∞} when the data supports it. In this study, at least, the data do not support this form of analysis. However, the equation provides the basis for an approach described here as scatter diagrams. In these diagrams one plots $(X(n) - X_{\infty})$ versus $n(\partial X / dn)$ to obtain the qualitative assessment. As the value of X_{∞} is not known, the value at n=64 is used as a surrogate. Unfortunately, equations (10a) and (10b) are unsuitable for use in evaluating global convergence as the values of ρ_{LS}^{2} vary over orders of magnitude, hence the use of α for that purpose.

These two concepts, global and local convergence, were used to define the following three types of convergence:

Type I. The strongest type of convergence is manifest when α in equation (9) behaves in the desired manner – initially being a large negative number and then at some decisive point assuming a value of, or close to, zero as n is increased. It is assumed that global convergence produces local convergence.

Type II. .This is an intermediate level between Type I and Type III (to be defined next). α behaves well initially but may not stay close to zero as n increases; however Type III is clearly observed.

Type III. α may behave poorly. However, when the data associated with $(X(n) - X_{\infty})$ and $n(\partial X / dn)$ is plotted, the scatter plot clearly shows a point of convergence.

Numerical Procedures

The numerical algorithms and procedures used in this study are summarized next.

1) All calculations were made using double complex precision.
2) The SVD routines are those provided in LAPACK, release 3.0, and the associated BLAS [16]. The routines for various Bessel functions were those provided in ACM TOMS 644 [17].
3) The number of variables, n, represents the number of terms in each of the series. There is obviously a need for a further variable in connection with the constant in Hallen's equation when it is used, but this is not included in the reported value of n.
4) The condition number is reported as Cn = $\log_{10}(\sigma_1/\sigma_n)$ and the error-squared is reported as Err-sq = $\log_{10}(\rho_{LS}^{2})$.
5) The smallest singular value permitted was 10.0^{-10}, or Cn = 10.0.
6) The integrals were evaluated using multiple applications of a 42-point quadrature rule using a procedure specifically designed to accommodate logarithmic singularities [18].
7) The 128 nodes and weights used in m instances were the positive nodes of a 256 point Gauss-Legendre quadrature formula to reduce the computational time by taking advantage of symmetry.

Numerical Findings.

Infinite Cylinder. Primarily for purposes of illustrating a Type I convergence, the case of a TM wave, $\phi_{inc} = 180^{0}$, being scattered from an infinitely long circular cylinder of circumference 1.0λ was examined. The formulation for this problem is in [19, p37]. The current on the surface of the cylinder was modeled as the sum of a surface-conforming Fourier series, namely:

$$I = \sum_{n=0}^{N} a_n \cos(n\phi).$$

Figure 1a clearly shows the desired behavior of alpha and the associated Err-sq. When n>8 the values of alpha are essentially zero and global convergence has been achieved. The components of the surface current at $\phi = 180^0$, shown in Figure 1b, also appear to possess excellent convergence and stability. Finally, for values of n greater than 7 a value of 6.2366 is found for the magnitude of the current which agrees well with the value reported in [19, p43].

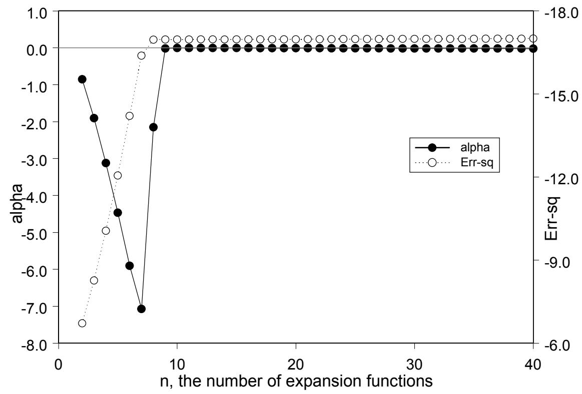

Figure 1a. Err-sq and alpha plotted as a function of the number of expansion functions.

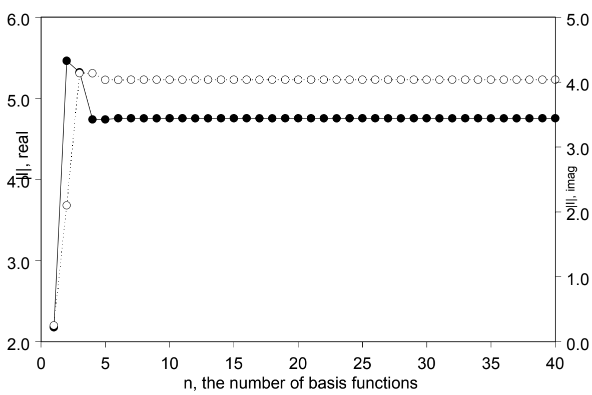

Figure 1b. Plots of the components of the current on a perfectly conducting cylinder illuminated by a TM wave.

Linear Dipole. In what follows, the operator is that associated with the Hallen solution for a linear dipole [20]. The relevant equation is: $\int_{-h}^{+h} I(z')G(z,z')dz' + C\cos(kz) = \int_{0}^{z} E_i(z')\sin(k(z-z'))dz'$

where $G(z,z') = \dfrac{1}{4\pi}\displaystyle\int_{\phi'=0}^{2\pi} \frac{e^{-jkR}}{R} d\phi'$

and $R = \sqrt{(z-z')^2 + (2a\sin(\frac{\phi'}{2}))^2}$ where I is the desired current, G is the Green's function, C is a constant to be determined and E_i is the incident excitation. The dipole, which is an open circular cylinder with an infinitesimally thin wall thickness, has a radius, a, equal to 0.007 λ in this work.

Two excitations were employed in the study. The first was a plane wave approaching the dipole from a direction at 90 degrees to the longitudinal axis of symmetry with its polarization parallel to that same axis. The second was a magnetic frill [21] located at the center of the dipole with b= 2.3a, where b is the outer radius of the frill. Four dipole lengths, 2h, were considered: 2h = 0.5, 1.0, 1.5 and 2.0 λ .

Various entire-domain expansion functions were examined. These are summarized in Table I below. In all cases, each term of the expansion function was multiplied by $\sqrt{(1-x^2)}$ (as x is defined in Table I.) as proposed by Richmond [22] to account for the edge mode.

When the ten current models shown in Table I were used in numerical calculations several important findings emerged.

1) Three of the series were found to be unsuitable for use. Both FS2 and Jn performed poorly, both globally and locally, for dipole lengths of 2h=1.50 and 2h=2.00. Pol was associated with high condition numbers which exceeded the threshold of Cn = 10.0 for n>6 for all dipole lengths. Consequently, these three series are not included in subsequent discussion and are not recommended for use in this type of application.
2) CH1, CH2 and LEG were found to give identical results, except in their values of Cn. This is not surprising as they are each an orthogonal representation of a simple power series.
3) FS1 and J0 gave identical results except that FS1 typically showed a lower value of Cn.
4) FS3 and Jmax also provided identical results except that FS3 provided slightly lower values of Cn.

The only difference within each of these latter three groups was in the values of Cn. For these reasons, the reported results are limited to the series that shows the lowest value of Cn for each group. These were CH2, FS1 and FS3. Results from the application of each of these series to each of the four dipole lengths are the basis for the following discussion.

The linear dipole with plane wave excitation.
Figure 2a shows the values of alpha for the twelve test cases. For each of the four lengths studied CH2 exhibits behavior indicative of Type II categorization. Neither FS1 nor FS3 exhibit Type II, or better, behavior. Although FS1 and FS3 perform well for specific lengths, only CH2 performs reasonably well for all four lengths examined.

With the preceding observations in mind, it is concluded that, for a plane wave excitation of a linear dipole, the best series of those examined is the Chebyshev series of the second kind which provides Type II convergence. The Chebyshev series of the first kind and the Legendre series have identical performance except, as indicated earlier, that the resulting condition numbers are somewhat higher.

Table I. Definitions of the entire-domain functions examined in this study.

Name	Type	Description $-h \le z \le +h;\ \ x = z/h$
Ch1	Chebyshev series of the first kind	$I(x) = a_1 + \sum_{i=2}^{n} a_i T_{2(i-1)}(x)$
Ch2	Chebyshev series of the second kind	$I(x) = a_1 + \sum_{i=2}^{n} a_i U_{2(i-1)}(x)$
FS1	Fourier series	$I(x) = a_1 + \sum_{i=2}^{n} a_i \cos((2i-3)\frac{\pi}{2} x)$
FS2	Fourier series	$I(x) = a_1 + \sum_{i=2}^{n} a_i \cos((i-1)kz)$
FS3	Fourier series	$I(x) = a_1 + \sum_{i=2}^{n} a_i \cos((i-1)\pi x)$
J0	Bessel series	$I(x) = a_1 + \sum_{i=2}^{n} a_i J_0(j_{i-1} x)$ where $j_1, j_2, j_3 ..$ denote the positive zeros of $J_0(y)$ arranged in ascending order of magnitude.
Jmax	Bessel series	$I(x) = a_1 + \sum_{i=2}^{n} a_i J_0(j_{i-1} x)$ where $j_1, j_2, j_3 ..$ denote the location of the maximums between the positive zeros of $J_0(z)$ arranged in ascending order of magnitude.
Jn	Bessel series	$I(x) = a_1 + \sum_{i=2}^{n} a_i J_0((i-1)x)$
Leg	Legendre series of the first kind	$I(x) = a_1 + \sum_{i=2}^{n} a_i P_{2(i-1)}(x)$
Pol	Polynomial expansion	$I(x) = a_1 + \sum_{i=2}^{n} a_i (kz)^{2(i-1)}$

The classical approach to displaying convergence in electromagnetic studies of convergence is provided in Figure 2b. As can be seen, it is difficult to visually distinguish between the convergence of the various plots. It is here that the scatter plots of (|I| - Iref) versus n.dI/dn, which correspond to the model of (10b), are useful. Such plots are shown here in Figure 2c. Only terms corresponding to values of n>10 are plotted.

The linear dipole with a magnetic frill excitation.
When excitation by a magnetic frill was examined it was found that none of the series described in Table I was able to adequately represent the

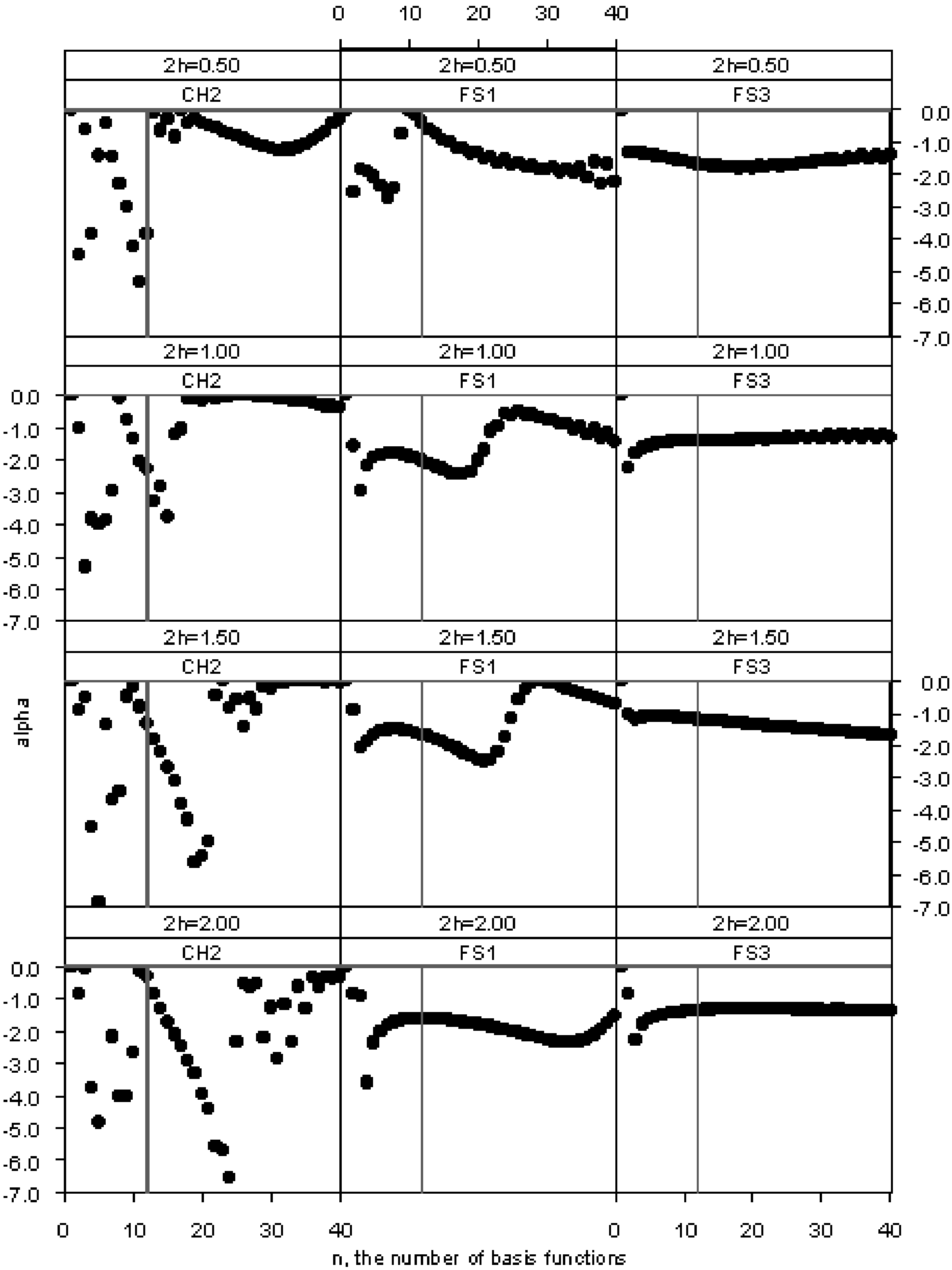

Figure 2a. Plots of alpha, for plane wave excitation, using the three main basis functions.

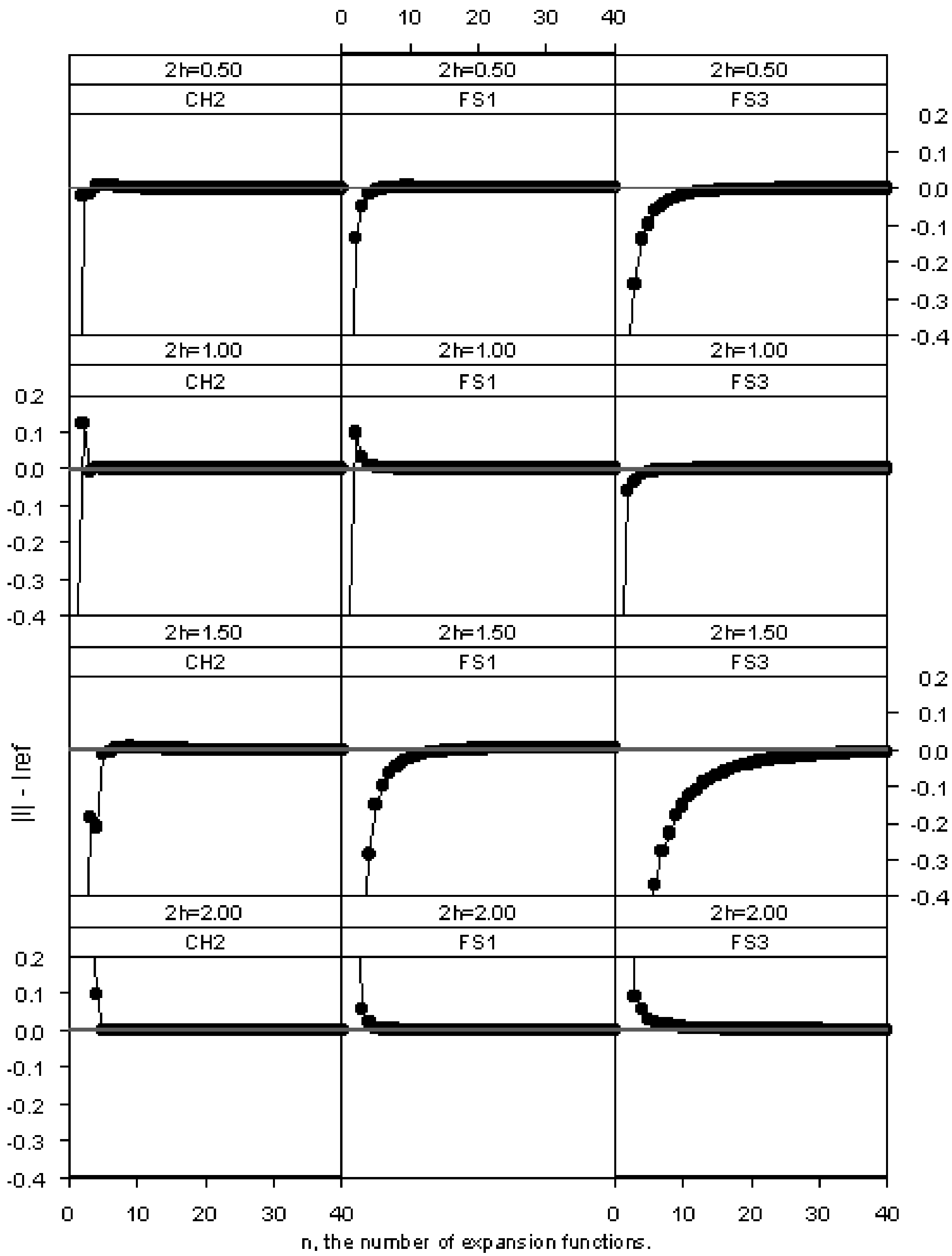

Figure 2b. Plots of (|I| - Iref) at the center of each dipole excited by a plane wave for each of the three main expansion functions.

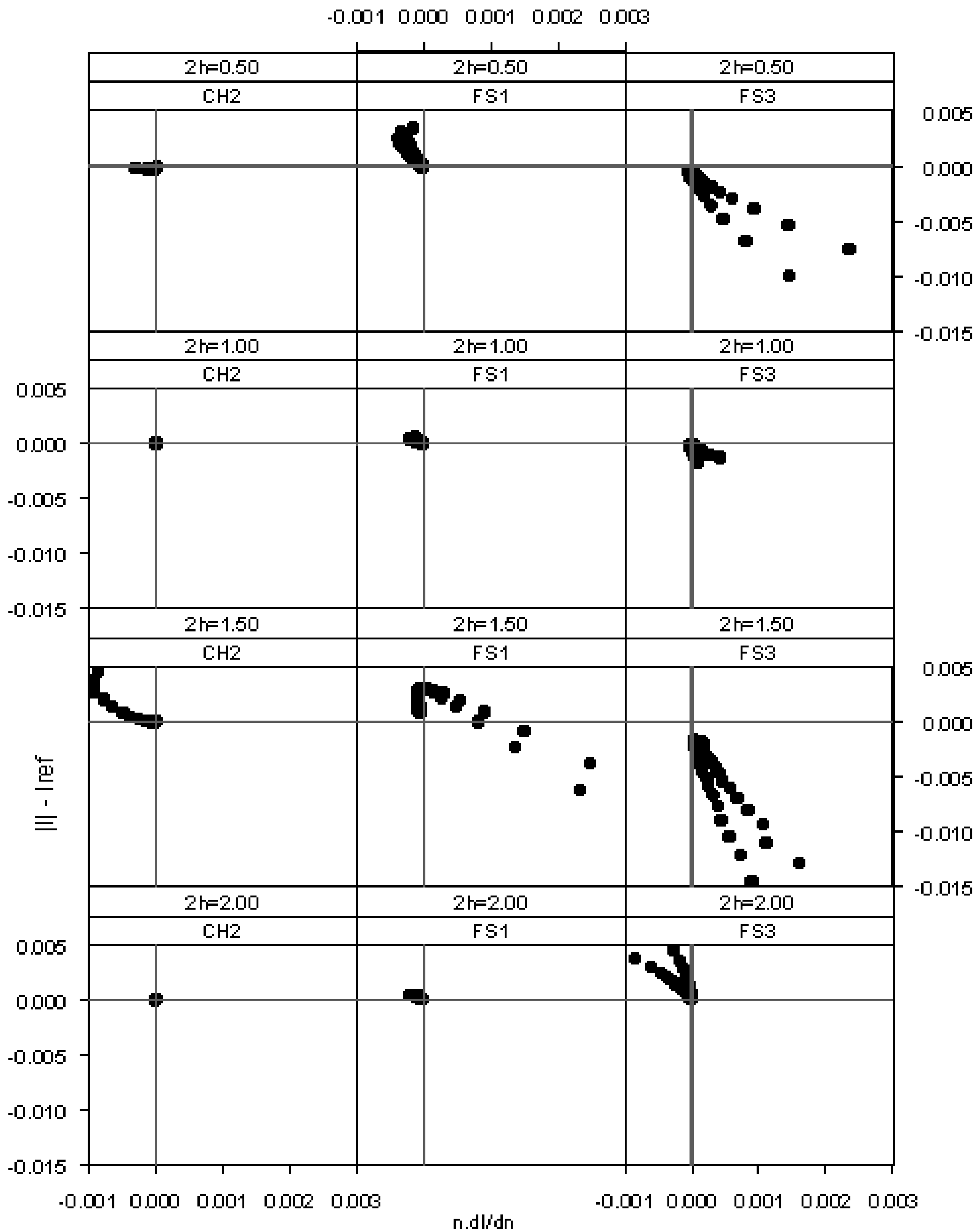

Figure 2c. Scatter plots, for plane wave excitation, using the three main basis functions. n>10.

current on the dipole in a global sense. When the scatter plots associated with (10b) were examined it was observed that CH2 performed very poorly. The best that can be said for the series in this part of the study is that FS1 and FS3 provide a weak form of Type III convergence.

In an attempt to improve the convergence properties of the series under study, each of them was supplemented with a term intended to model the out-going wave from a frill generator. Application of this concept was originally proposed by Richmond [22]. The form of this term is:

$$I(z) = \frac{2jkV_{in}}{\eta \ln(b/a)} \int_0^{\infty} \frac{[K_0(\gamma a) - K_0(\gamma b)]}{\gamma^2 K_0(\gamma a)} \cos(gz)dg$$

where V_{in} denotes the generator voltage, $\gamma^2 = g^2 - k^2, k = \omega\sqrt{\mu\varepsilon}, \eta = \sqrt{\frac{\mu}{\varepsilon}}$ and K_0 is the modified Bessel function. In this section of the study, the additional term is the second in any one series – the first term always being the α_1 term. Again, none of the series exhibited desirable alpha results. When the scatter plots were examined CH2 performed better than when the special term was not present but still at unacceptable levels. The scatter plots for FS1 and FS3 are shown in Figure 3a and Figure 3b – without and with the special term to account for the outgoing wave. Examination of these results clearly shows the benefits of incorporating the special term with either FS1 or FS3, with FS1 being the better performer. In fact, it is essential to incorporate the special term to assure the performance of FS1 over the four dipole lengths.

Discussion.

The work reported above was performed with great attention to detail. In particular, a precision of 15 digits was used, and the condition number rarely exceeded 6 digits except when the out-going wave from a frill generator was included, in which case it rose to 8 digits. Unfortunately, no estimate of the matrix-coefficient accuracy was available. The results identify two distinct families of expansion functions. In the first group are the two kinds of Chebyshev polynomials and the Legendre polynomials. In the second group we find two types of Fourier series and two types of Bessel Series. Whenever entire domain expansions are encountered in the literature, it is invariably the Fourier series. As no references can be found that explore this issue, it is not clear why this is. It is possible that familiarity and mathematical convenience play a role. As stated in the introduction, in a paper in 1965 Richmond [6] examined not only the performance of the Fourier series, but also the series of Maclaurin (in this paper POL), Chebyshev of the first kind, Hermite and Legendre. He examined them up to the first five terms. His comments were “it would be advantageous to expand the current in a series of functions which converges more rapidly than the Fourier series……the Chebyshev and Legendre series appear most promising, but this matter requires further investigation”. No record of this further investigation can be found. The present work suggests that five terms is an insufficient number from which to draw any conclusions. To increase the number beyond five would have been very expensive (computer wise) in 1965. Furthermore, the combination of the first two terms causes the condition number, Cn, to exceed a value greater than 4.0 which then increases monotonically up to a value in excess of 8.0 when n=64. This observation puts the approach outside the realm of single-precision calculations, probably in use in 1965, except for small values of n.

Accuracy/convergence in electromagnetic problems is frequently discussed in terms of $O(\Delta^p)$ where Δ is a cell size and p is some exponent – either to be determined or expected [19, p197]. Such an approach presumes a sub-domain model for the expansion functions, functions not used in this study. Nevertheless, there is a strong parallel between the use of $O(\Delta^p)$ and the use of $\| e_n \| \sim \beta n^{\alpha}$. Both approaches expect the errors to decrease as the number of expansion functions is increased. However, as shown in this study because the error term can be calculated explicitly it is possible to calculate α as each expansion function is added. When α becomes small there is then no point in adding further terms. In this way it is a strong indicator on when to terminate the addition of new expansion functions. In the case of the infinite cylinder, this termination point is very clear-cut. In the case of the dipole excited by a plane wave, there is some oscillation initially but the termination point is subsequently clear for each of the four dipole lengths. Use of the model,

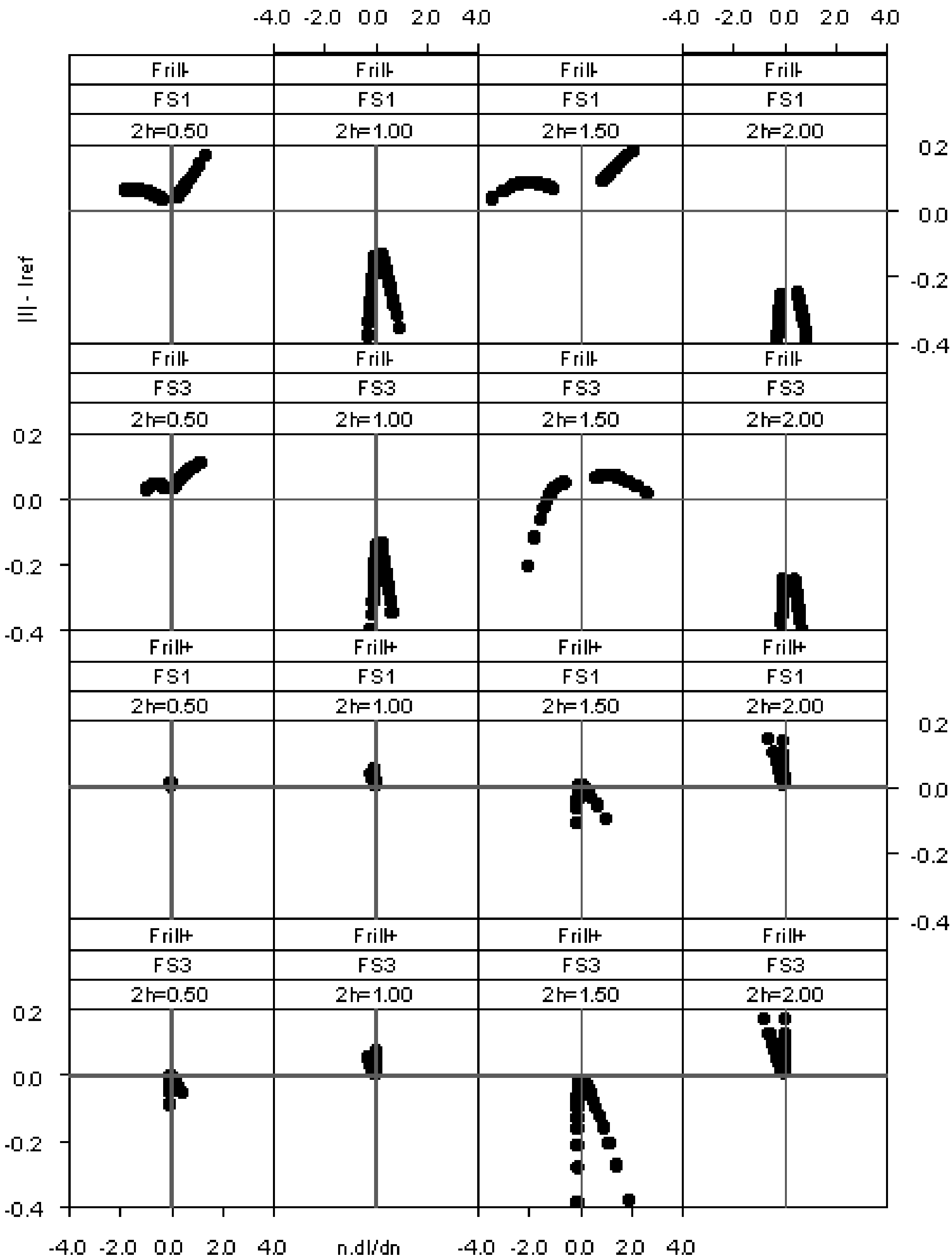

Figure 3. Scatter plots for dipoles of various lengths when excited by a magnetic frill with, Frill+, and without, Frill-, a special term representing the outgoing wave.

$\| e_n \| \sim \beta n^{\alpha}$, leads to the development of (10b). This latter equation was successfully applied, using backward differences to estimate the first derivative, to better discriminate between the performance of each of the series in place of the classical method of visually examining the magnitude of the current as a function of the number of expansion functions.

The preceding findings produce the results reported in Table II.

	Plane wave	**Frill** (b/a=2.3)
Converg-ence	**Type II**	**Type III**
Series	**CH2**	**FS1***
Dipole length		
2h=0.50	2.8247 -j1.9275	8.317 -j3.703
2h=1.00	0.5442 -1.3483	0.989 +j1.655
2h=1.50	-1.7095 -0.2114	6.707 -j1.980
2h=2.00	0.1289 -j0.0	1.201 +j1.733

* series includes the term for an outgoing wave due to the frill excitation.

Table II. Values of admittances, in mmhos, for dipoles of various lengths for two forms of excitation.

Conclusions.

- When a problem can be represented by an excellent model, it is possible to get both global and local convergence with relatively few terms. This was illustrated with the aid of an infinite cylinder. Such a statement is intuitively obvious but the truth of the statement is dramatically illustrated in this study.
- The best model for a dipole excited by a plane wave uses an orthogonal polynomial series which gives acceptable global convergence and excellent local convergence. In the case of the CH2 model, the number of terms required to reach convergence is 12, 17, 22 and 25 for dipole lengths of 2h=0.4, 1.0, 1.5 and 2.5 respectively.
- The best model for a dipole excited by a magnetic frill uses a Fourier series but this provides no global convergence and only acceptable local convergence with a relatively large number of terms in the series, including the special representation of the out-going wave.

As already mentioned, Richmond [22] commented that "it would be advantageous to expand the current in a series of functions which converges more rapidly than the Fourier series…". Richmond was referring to a dipole excited by a plane wave arriving at an oblique angle. For a plane wave normally incident on a dipole the current work suggests that a better series has been found. However, for a magnetic frill excitation the search must go on. This will be explored in future studies.

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