A Posteriori Error Estimates for Two-Dimensional Electromagnetic Field Computations: Boundary Elements and Finite Elements

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Abstract

A brief summary of the variational boundary-value problem formulation of the 2D finite element/boundary element (FE/BE) method is presented. From this *a posteriori* error estimates and error indicators for the FE/BE method are developed and applied to electromagnetic scattering and radiation problems. The results obtained indicate that these error estimates and indicators can be obtained within negligible computational times and can be used successfully to obtain valuable *a posteriori* accuracy and convergence information regarding the reliability of the FE/BE method solutions.

1 Introduction

The 2D finite element/boundary element (FE/BE) method[1] has been used extensively over the past few years for solving electromagnetic problems numerically [1, 2, 3, 4, 5]. The method has proved to be highly successful and applicable to specifically scattering and radiation problems concerning inhomogeneous, arbitrarily shaped objects. However, the solution time and memory requirements of the method become impractical when electromagnetically large problems are considered. These limitations of the FE/BE method are obviously dependent on the computational hardware available.

A FE/BE method solution results in approximated field values in the regions under consideration. The accuracy of these solutions is dependent on the specific problem at hand as well as the approximation functions used with the FE/BE method. *A priori* knowledge of the accuracy and reliability of the FE/BE method is obviously an important consideration which has been investigated by a number of researchers [6, 1, 7]. Another important consideration would be an *a posteriori* error estimate of FEM or FE/BE method solutions. Reliable *a posteriori* error estimates could serve as criteria for required accuracies as well as convergence checks for the FE/BE method solutions obtained. *A posteriori* error estimates for the FEM as well as the BEM have been under investigation for the past few years and have been applied successfully to a number of general engineering problems [8, 9, 10]. However, very little has been published on this topic in the computational electromagnetic literature and only recently has work started appearing, for example [11, 12]. A recent special issue of this journal contained several papers on error estimation, none specifically on FEM error estimates although the paper by Hsiao and Kleinman considers boundary integral error control [13]. Some commercial FEM programs include error estimates and adaptive meshes, but the algorithms are often proprietary.

In this paper, a brief summary of the FE/BE method formulation will be presented (section 2). A number of *a posteriori* error estimates and error indicators for FE/BE method solutions of electromagnetic problems will be formulated in section 3. These include local (in each finite element) and global Element Residual Method (ERM) error estimates [8, 9, 10], L^2-norm boundary field and boundary field derivative error estimates [14, 15], a L^2-norm Neumann boundary condition error indicator and a radar width error indicator. It will be shown that these are highly efficient error estimates with negligible computational times compared to the solution times of FE/BE method solutions. The *a posteriori* error estimates and error indicators developed will be applied to a number of FE/BE method solutions of electromagnetic scattering problems. The results obtained will be used to investigate the accuracy, reliability and applicability of the different error estimates and error indicators when applied to FE/BE method solutions of 2D electromagnetic problems.

In section 4 general conclusions on the work presented in this paper will be drawn and further research that needs to be done will be discussed.

[1] The technique is also referred to as the finite element/moment method or finite element/integral equation method in the literature; there are sometimes differences in detail but the basic concepts are the same.

2 The 2D FE/BE Method

In this section the 2D FE/BE method formulation will be presented[2]. The development of this formulation is not new, but the specific notation used is crucial for the development of the FE/BE method error estimates. A thorough understanding of the mathematics behind the FE/BE method is essential for the development of any kind of error analysis associated with the method. A rather *informal* discussion of the functional analysis formulation of the FE/BE method will be presented in this paper. This discussion is sufficient to prepare the reader for the development of the error estimates. (Details of the formal functional analysis formulation of the FE/BE method can be found in reference [16].)

The formulation will be presented for 2D Transverse Magnetic (TM_z) polarized scattering problems.[3] The TM_z scattering formulation is a special case of the more general 2D scalar Helmholtz equation formulation. This equation serves as the governing equation for a variety of 2D electromagnetic problems, including static electric and magnetic problems[4], as well as closed and open boundary electromagnetic problems for TM_z and TE_z polarization. The FE/BE method and the *a posteriori* error estimate methods considered in this paper are applicable to all the above mentioned electromagnetic problems.

2.1 Variational boundary-value problem formulation of the FEM

Consider figure 1. For 2D scattering problems involving a TM_z polarization incident field, the 2D electromagnetic field equation can be written as a boundary value problem [6, pp.185] [1, pp.72-73]. (Throughout this paper the electric field component associated with the TM_z polarized field, E_z, will be denoted by E):

$$\nabla \frac{1}{\mu_r} \nabla E + \epsilon_r k_0^2 E = 0 \quad in \quad \Omega \tag{1}$$

with Dirichlet boundary condition on a perfectly conducting region

$$E = 0 \quad on \quad \Gamma^{D_0} \tag{2}$$

and Neumann boundary condition on the fictitious boundary

$$\frac{\partial E}{\partial n} = g_1 \quad on \quad \Gamma^{N_1} \tag{3}$$

where g_1 are the prescribed values for $\frac{\partial E}{\partial n}$ on Γ^{N_1}. The boundary-value problem of equations 1 to 3 can be written as a variational boundary-value problem [6, pp.219]:

$$\int_\Omega \left(\nabla E \cdot \frac{1}{\mu_r} \nabla v - \epsilon_r k_0^2 E v\right) d\Omega = - \int_{\Gamma^{N_1}} g_1 v \, d\Gamma^{N_1} \tag{4}$$

with v an arbitrary weighting or testing function on Ω. Using conventional finite element procedures [17][16, pp.14], equation 4 can be written in matrix form:

$$[S][E] + [T][\frac{\partial E}{\partial n}] = 0 \tag{5}$$

with $[S]$ the FEM system matrix, $[E]$ the unknown field coefficient column matrix, $[T]$ the FEM boundary condition system matrix and $[\frac{\partial E}{\partial n}]$ the field derivative column matrix. Equation 5 is the FEM matrix equation for TM_z polarized, 2D electromagnetic problems. The solution of the FEM matrix equation 5 yields the approximated field solution, E, if g_1 of equation 3 is known and used to construct the column matrix $[\frac{\partial E}{\partial n}]$.

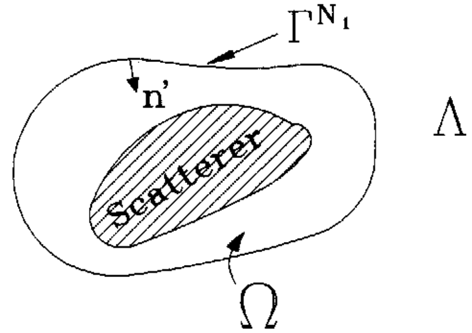

Figure 1: **Finite region, Ω, in which the FEM will be applied, enclosed by a boundary, Γ^{N_1}. Λ is an exterior free-space region extending to infinity.**

2.2 The boundary element method

Many practical electromagnetic field problems, such as scattering and radiation problems, are open boundary problems. The domain of these problems extends to infinity, making the finite element method unsuitable for

[2] Only two-dimensional problems will be considered in this paper. Variation in the 2D $x - y$ plane will be assumed (for all fields and structures) with no variation in the z-direction (extending to infinity).

[3] The convention for polarization in a 2D plane which will be used throughout this paper is as follows: For TM_z polarization the magnetic field vector has only x and y components, thus $\vec{H}$ is always transverse to the z plane. In this case $\vec{E}$ has only a z component. For TE_z polarization the electric field vector has only x and y components, thus $\vec{E}$ is always transverse to the z plane. In this case $\vec{H}$ has only a z component.

[4] In these cases, the 2D scalar Helmholtz equation reduces to the 2D Laplace or Poisons equations.

the solution of such problems. One way to overcome this problem is to couple the finite element method with an unbounded region method such as the boundary element method. In this way, the advantages of both the FEM and BEM are exploited [1][5].

A fictitious boundary denoted by Γ^{N_1} can be defined around the structure under consideration dividing the solution region into an interior region, Ω, and an exterior region, Λ (see figure 1). A BEM solution of the normal field derivatives on Γ^{N_1} can be used as Neumann boundary condition for the boundary value field problem in Ω. The FEM can thus be applied to the field problem in Ω with appropriate boundary conditions (obtained with the BEM solution) on Γ^{N_1}.

Consider the exterior region Λ bounded by Γ^{N_1} and infinity. For electromagnetic field problems, the homogeneous free-space Helmholtz equation must be satisfied in this region. The variational form of the boundary element method integral equation [1, pp.299][14, pp.41][16, pp.17] can be written in the BEM matrix equation form [16, pp.19]:

$$[H][\tilde{E}] - [G]\left[\frac{\partial \tilde{E}}{\partial n'}\right] = [F^{inc}] \qquad (6)$$

The matrix elements of $[H]$, $[G]$ and $[F^{inc}]$ can be calculated using the boundary element integral equation [16, pp.17] and the matrix elements of $[\tilde{E}]$ and $\left[\frac{\partial \tilde{E}}{\partial n'}\right]$ are unknown field coefficient values which can be obtained with the BEM solution. The elements of matrix $[F^{inc}]$ contain external source information. In the rest of this paper, E represents the FEM and $\tilde{E}$ the BEM computed solution.

2.3 Coupling the FEM and BEM

For a closed region, Ω, a FEM solution is obtainable if certain boundary conditions are known. These could be Dirichlet and/or Neumann boundary conditions. For open boundary problems one can use the fictitious boundary (discussed in the previous sections) as closure of the finite, interior region, Ω, with inhomogeneous Neumann boundary conditions specified on the fictitious boundary Γ^{N_1}. By using $\frac{\partial \tilde{E}}{\partial n'}$ of equation 6 as the Neumann boundary condition on Γ^{N_1}, a solution of the FEM in Ω is obtainable. That is, g_1 of equation 3 can be set equal to $\frac{\partial \tilde{E}}{\partial n'}$ of equation 6 on Γ^{N_1}. (The minus sign is due to the direction of the normal n' — see figure 1):

$$g_1 = -\frac{\partial \tilde{E}}{\partial n'} \quad on \quad \Gamma^{N_1} \qquad (7)$$

Using the above equation together with equations 4 and 5 leads to a modified FEM matrix equation

$$[S][E] - [T]\left[\frac{\partial \tilde{E}}{\partial n'}\right] = 0 \qquad (8)$$

Setting E of equation 5 equal to $\tilde{E}$ of equation 6 on Γ^{N_1} and coupling the two matrix equations 6 and 8 (each containing *two* unknown coefficient column matrices) is thus equivalent to solving the FEM matrix equation of equation 5 with the Neumann boundary condition of equation 7 on Γ^{N_1}.

3 *A Posterior* Error Estimates

A reliable *a posteriori* error estimate for a FE/BE method solution would enable one to obtain valuable convergence information without having to solve the same problem, using a *larger* number of unknowns. The error estimate can thus be used as a convergence check for practical electromagnetic problems for which no analytical solution exists. This would be especially advantageous when electromagnetically large problems, reaching the practical solution time and memory limits of the computer at hand, are considered. This is true regardless of the computational power available.

Local and global finite element method error estimates have been under investigation for the past decade [8, 9, 10]. Of these, the Element Residual error estimate Method (ERM) seems to be the most reliable and the easiest implementable method. Applications of this FEM error estimate method were directed mainly at mechanical and fluid-dynamical problems although the method is applicable to a wide variety of *positive definite* FEM type problems [18]. The governing equation for static electric field problems (Laplace's or Poisson's equation) is positive definite, and the successful application of the ERM to such problems has recently been published [19]. The ERM applied to wave equation type problems, of which electromagnetic scattering and radiation problems are examples, will be considered in this section. Although the FEM system equations for such problems are not necessarily positive definite, an element residual type error estimate method will be introduced, which is applicable to such problems.

A L^2-norm boundary **field** error estimate, which has been applied to linear acoustic problems [14, 15], and a L^2-norm boundary **field derivative** error estimate have been investigated and will be modified for application to the BEM part of the FE/BE method solution. A L^2-norm **Neumann boundary condition** error indicator for the FE/BE method will also be introduced. The L^2-norm error estimates and error indicator will be used to develop far-field error indicators for FE/BE method solutions of electromagnetic problems. These far-field error indicators are especially important when considering electromagnetic problems concerning radar-width and radiation intensity results, obtained with the FE/BE method solution.

The mathematical development of the error estimates and error indicators presented in this section, is based on the functional analysis formulation of the FE/BE method, presented in section 2 and, in more detail, in reference [16].

3.1 Error Estimates for the FEM

The Element Residual error estimate Method (ERM) for FE/BE method solutions of electromagnetic scattering problems will be formulated in this section. This is an *a posteriori* error estimate method, and can be applied after a FE/BE method solution has been obtained. Let us denote a "first" approximated electric field solution obtained in $\bar{\Omega}$ as E^1 and a "second" (higher order) field solution as E^2. (Note that $\bar{\Omega}$ is the closure of Ω, i.e. the domain Ω and its boundary Γ^{N_1}). The ERM will be applied after solution E^1 has been calculated and yields an estimate of the **relative** FE/BEM electric field error:

$$E^{12} = E^2 - E^1 \tag{9}$$

The above mentioned error will be quantified in terms of a norm (as required by the ERM). For static electric field problems an appropriate norm is the energy norm [19] which yields a quantitative error value of the stored energy in the region under consideration. For electromagnetic problems a similar norm *associated* [16, pp.248] with the stored electric and magnetic energy in the closed region $\bar{\Omega}$ will be employed. This norm will be called the electromagnetic energy norm (EM-norm) and is given by [16, pp.249]:

$$\left(||E^{12}||_{\bar{\Omega}}^{EM}\right)^2 = \sum_{k=1}^{M} \left(||E^{12}||_{\Omega_k}^{EM}\right)^2 \tag{10}$$

with M the number of finite elements and

$$\begin{aligned}\left(||E^{12}||_{\Omega_k}^{EM}\right)^2 \;=\; \int_{\Omega_k} \Big(& Re\{\frac{1}{\mu_r}\}|\nabla E^{12}|^2 + \\ & Re\{\epsilon_r k_0\}|E^{12}|^2\Big)\, d\Omega_k \end{aligned}\tag{11}$$

the local EM-norm associated with finite element Ω_k.

The ERM will thus yield a quantitative error estimate of the stored electric and magnetic energy in the region $\bar{\Omega}$. This error estimate will be denoted by $\left(||\Phi^{12}||_{\bar{\Omega}}^{EM}\right)^2$. The EM-norm satisfies the requirements for a norm generated by an inner product [16, pp.246] and is **not** unique for ERM error estimates applied to electromagnetic problems. Other norms could be developed and might lead to improved error estimate results. It will however be shown that the EM-norm introduced here results in reliable ERM error estimates for FE/BE method solutions of general (2D) electromagnetic problems (including problems involving lossy materials).

3.1.1 Local ERM error estimates

An estimate, $\left(||\Phi^{12}||_{\Omega_k}^{EM}\right)^2$, of the **actual** error $\left(||E^{12}||_{\Omega_k}^{EM}\right)^2$ in each finite element, Ω_k, can be obtained by solving the following local problem [10, pp.129][20][16], related to the EM-norm (a sub- or superscript k denotes that a quantity is associated with finite element Ω_k):

$$\begin{aligned}\int_{\Omega_k} \left(Re\{\frac{1}{\mu_r^k}\}\nabla\Phi_k^{12}\cdot\nabla v^{12} + Re\{\epsilon_r^k k_0\}\Phi_k^{12} v^{12}\right) d\Omega_k = \\ \int_{\Omega_k} r_k^1 v^{12}\, d\Omega_k + \frac{1}{2}\sum_{l=1}^{3}\int_{\Upsilon_{k(l)}} \left\{\frac{\partial E_k^1}{\partial n_{k(l)}}\right\} v^{12}\, d\Upsilon_{k(l)} + \\ \int_{\Upsilon_k \cap \Gamma^{N_1}} \left(g_1 - \frac{\partial E_k^1}{\partial n_k}\right) v^{12}\, d\Upsilon_k \quad for \;\; all \;\; v_k^{12}\end{aligned} \tag{12}$$

In this equation v_k^{12} is the weighting functions on Ω present in the FE/BEM solution E^2 but not in the solution E^1. $\Upsilon_{k(l)}$ is the side of the triangular finite element Ω_k connecting Ω_k with its l^{th} neighboring element $\Omega_{k(l)}$. Also, r_k^1 is the local element residual of the governing equation (equation 1) associated with E^1, with

$$r_k^1 = -\nabla\cdot\frac{1}{\mu_r^k}\nabla E_k^1 - \epsilon_r^k k_0^2 E_k^1 \tag{13}$$

$\left\{\frac{\partial E_k^1}{\partial n_{k(l)}}\right\}$ is the jump in normal derivative between element Ω_k and its neighboring element, element $\Omega_{k(l)}$, with

$$\left\{\frac{\partial E_k^1}{\partial n_{k(l)}}\right\} = \left(\frac{1}{\mu_r^{k(l)}}\frac{\partial E_{k(l)}^1}{\partial n_{k(l)}} - \frac{1}{\mu_r^k}\frac{\partial E_k^1}{\partial n_{k(l)}}\right) \tag{14}$$

Equation 12 appears formidable but can be written in the following matrix form for each finite element, Ω_k [16, pp.103]:

$$[S^{k(12)}][\Psi^{k(12)}] = [F^{k(12)}] \tag{15}$$

with $[S^{k(12)}]$ a square matrix with matrix elements which can be calculated using the integral on the left hand side of equation 12; $[\Psi^{k(12)}]$ is a column matrix with matrix elements the unknown local error estimate coefficients; $[F^{k(12)}]$ is a column matrix with matrix elements which can be calculated using the integrals on the right hand side of equation 12. The elements of $[S^{k(12)}]$ can be computed rapidly using tables for the result of integration over a prototypical element along the lines of [17, Chapter 4]. (Details are given in [16, Appendix F]). Solving the local matrix equation, equation 15, is equivalent to solving the local element residual error estimate equation, equation 12, yielding Φ_k^{12} and thus the local error estimate $\left(||\Phi^{12}||_{\Omega_k}^{EM}\right)^2$. Note that equation 15 is a very "small" matrix equation involving only one finite element (at a time). This equation can thus be solved within negligible computational time.

The specific form of equation 12 is important to ensure the following relation between the global error estimate, Φ^{12}, and actual global error in the EM-norm [20][10, pp.126,pp128]:

$$||\bar{E}^{12}||_{\Omega}^{EM} \leq C^{12}||\Phi^{12}||_{\Omega}^{EM} \tag{16}$$

with C^{12} the global error estimate inequality constant for the error estimate Φ^{12} in the EM-norm. It is obvious that the accuracy of the global error estimate depends on the value of the constant C^{12} in equation 16. C^{12} is dependent on the order of the basis functions used as well as the shape of the finite elements. In practice $C^{12} = 1$ is assumed with all error estimates. A better approximation of C^{12} would, however, improve the accuracy of the error estimates [10, pp.129][8].

A discussion of the contribution of each term of equation 12 to the error estimate will now follow. This should give a qualitative understanding of the ERM and specifically its application to EM problems.

The weighted residual integral term

The first term on the right hand side of equation 12 is a weighted residual term. The "first" solution E^1 results in a field solution E_k^1 in element Ω_k. The residual r_k^1 will be zero everywhere in Ω_k, if E^1 is the true (or exact) field solution. This is due to the fact that equation 13, with $r_k^1 = 0$, is a form of the governing equation which will be satisfied by the true field solution. If E^1 is an approximate finite element solution the residual r_k^1 will be non-zero and can, in general, vary in numerical value over the finite element Ω_k. The numerical value of the weighting functions, v_k^{12}, on Ω_k is zero at all nodal points on Ω_k associated with the basis functions of the solution E^1. The numerical value of v_k^{12} on Ω_k is **one** at all the nodal points on Ω_k associated with the solution E^2, but not with the solution in E^1. At non-nodal points v_k^{12} is a value between zero and one varying to the order of the basis functions used with E^2. Multiplication of the residual r_k^1 with v_k^{12} and integration over Ω_k is thus equivalent to weighting the residual with v_k^{12} over Ω_k. This weighted term will thus have a relatively large numerical value if the residual is, on average, large in the regions where the "new" nodal points, corresponding to the "new" basis functions of E^2 on Ω_k, will be situated. The weighted term will have a small numerical value if the residual is, on average, small in the regions where the "new" nodal points, corresponding to the "new" basis functions of E^2 on Ω_k, will be situated.

Weighted normal derivative continuity integral term

The second integral term on the right hand side of equation 12 is a weighted normal derivative continuity term on $\Upsilon_{k(l)}$. If E^1 is the true field solution, continuity between the terms $\frac{1}{\mu_r^{k(l)}}\frac{\partial E_{k(l)}^1}{\partial n_{k(l)}}$ and $\frac{1}{\mu_r^k}\frac{\partial E_k^1}{\partial n_{k(l)}}$ of equation 14 must hold on $\Upsilon_{k(l)}$. This continuity requirement is related to the zero divergence requirement [21, pp.127]. This means that $\left\{\frac{\partial E_k^1}{\partial n_{k(l)}}\right\}$ of equation 14 must be zero for E^1 a true field solution. The different local basis functions (or approximation functions) in each finite element Ω_k result in discontinuities on $\Upsilon_{k(l)}$ for E^1 an approximate finite element solution. In the integral of equation 12, $\left\{\frac{\partial E_k^1}{\partial n_{k(l)}}\right\}$ can, in general, vary on $\Upsilon_{k(l)}$. The term $\left\{\frac{\partial E_k^1}{\partial n_{k(l)}}\right\}$ is weighted by the restriction of the v_k^{12} to $\Upsilon_{k(l)}$.

Weighted Neumann boundary condition integral term

The third integral on the right hand side of equation 12 is a weighted Neumann boundary condition integral on $\Upsilon_k \cap \Gamma^{N_1}$. The Neumann boundary condition will be satisfied exactly with E^1 the true field solution, making the quantity $\left(g_1 - \frac{\partial E_k^1}{\partial n_k}\right)\Big|_{\Upsilon_k \cap \Gamma^{N_1}}$ zero. If E^1 is an approximate finite element solution, the Neumann boundary condition on $\Upsilon_k \cap \Gamma^{N_1}$ will be satisfied approximately and the term $\left(g_1 - \frac{\partial E_k^1}{\partial n_k}\right)\Big|_{\Upsilon_k \cap \Gamma^{N_1}}$ will be non-zero. In the integral of equation 12, $\left(g_1 - \frac{\partial E_k^1}{\partial n_k}\right)\Big|_{\Upsilon_k \cap \Gamma^{N_1}}$ can, in general, vary on $\Upsilon_k \cap \Gamma^{N_1}$. The term $\left(g_1 - \frac{\partial E_k^1}{\partial n_k}\right)\Big|_{\Upsilon_k \cap \Gamma^{N_1}}$ is weighted with the restriction of v_k^{12} to $\Upsilon_k \cap \Gamma^{N_1}$.

3.2 Error Estimates for the BEM

In this section, two L^2-norm [6, pp.50,72] error estimates for the FE/BE method solution, E^1, will be introduced. Both error estimates are associated with the accuracy of the field solution on Γ^{N_1}, and are related to **global** field and field derivative errors on Γ^{N_1}. These error estimates are estimates of the "true" errors; that is the errors in E^1 relative to the true field solution [14, 15]. Boundary errors appear to have been seldom used in computational electromagnetics as a measure of solution accuracy; the only example we are aware of is [22], which includes plots of the tangential electric (error) field along a wire antenna.

3.2.1 The boundary field L^2-norm residual error estimate

The first L^2-norm error estimate is associated with the accuracy with which the boundary element approximation functions can approximate the actual fields on the boundary Γ^{N_1}. The FE/BE method solution yields field and field derivative coefficients which, together with the boundary element basis functions, can be used to approximate the fields and field derivatives on Γ^{N_1}. These field approximations can be used to calculate the fields at any point in the exterior region Λ, using a numerical approximation of Huygens' principle [16, pp.57]. The field approximations can also be used to calculate the fields on Γ^{N_1} using the following equation [14, pp.40]:

$$\begin{aligned}\bar{E}^1(\vec{r}_o) \quad = \quad & 2\int_{\Gamma^{N_1}(\vec{r}_s)}\left[E^1(\vec{r}_s)\frac{\partial\Psi(\vec{r}_o,\vec{r}_s)}{\partial n'(\vec{r}_s)}\right. \\ & \left.-\frac{\partial\tilde{E}^1(\vec{r}_s)}{\partial n'(\vec{r}_s)}\Psi(\vec{r}_o,\vec{r}_s)\right] d\Gamma^{N_1}(\vec{r}_s) \\ & +2E^{inc}(\vec{r}_o) \end{aligned}\tag{17}$$

On the boundary, the FE/BE formulation ensures that $\tilde{E}^1(\vec{r}_s) = E^1(\vec{r}_s)$ and they can be used interchangeably here. The quantities $\vec{r}_o$ and $\vec{r}_s$ are the BEM observation and source points on Γ^{N_1}, $\Psi(\vec{r}_o,\vec{r}_s)$ is the 2D homogeneous Green's function and $E^{inc}(\vec{r}_o)$ is the incident electromagnetic field value at $\vec{r}_o$. $\bar{E}^1(\vec{r}_o)$ is the approximated field on Γ^{N_1} calculated using equation 17.

A boundary field residual can now be defined as[5]:

$$\bar{R}^1_{\Gamma^{N_1}}(\vec{r}_o) = E^1(\vec{r}_o) - \bar{E}^1(\vec{r}_o) \tag{18}$$

with $\bar{R}^1_{\Gamma^{N_1}}(\vec{r}_o)$ the BEM residual for the FE/BE method field solution, $E^1(\vec{r}_o)$, at $\vec{r}_o$, on Γ^{N_1}.

[5]Note that $E^1(\vec{r}_o)$ is the approximate field solution (at $\vec{r}_o$ on Γ^{N_1}) obtained with the "first" FE/BE solution. The function $E^1(\vec{r}_o)$ is thus of order equal to the order of the FEM and BEM basis functions. The function $\bar{E}^1(\vec{r}_o)$ is also an approximated field solution (at $\vec{r}_o$ on Γ^{N_1}) obtained with equation 17 (and thus indirectly from the "first" FE/BE method solution). The function $\bar{E}^1(\vec{r}_o)$ is of much higher order than $E^1(\vec{r}_o)$ due to the presence of the Green's function, $\Psi(\vec{r}_o,\vec{r}_s)$, and the incident field function, $E^{inc}(\vec{r}_o)$, in equation 17.

A global boundary field L^2-norm residual error estimate can be obtained using equation 18:

$$||\bar{\Theta}^1_{\Gamma^{N_1}}||^2_{L^2} = \int_{\Gamma^{N_1}} |\bar{R}^1_{\Gamma^{N_1}}(\vec{r}_o)|^2 \, d\Gamma^{N_1} \tag{19}$$

It is evident that the residual $\bar{R}^1_{\Gamma^{N_1}}(\vec{r}_o)$ will be zero if E^1 is the true field solution and $\frac{\partial\tilde{E}^1(\vec{r}_s)}{\partial n'(\vec{r}_s)}$ is the true field derivative solution on Γ^{N_1}, for then the approximate fields, $E^1(\vec{r}_o)$ and $\bar{E}^1(\vec{r}_o)$ will both be equal to the true field solution at $\vec{r}_o$. From equation 19 it is also evident that the boundary field error estimate, $||\bar{\Theta}^1_{\Gamma^{N_1}}||^2_{L^2}$, will be zero if E^1 is the true solution. Asymptotic exactness and upper and lower bounds for the error estimate $||\bar{\Theta}^1_{\Gamma^{N_1}}||^2_{L^2}$ have not yet been obtained for general FE/BE method solutions (although this has been done for some special cases [14, pp.56]) mainly due to the difficulties arising from the non-local nature of the BEM when applied to general electromagnetic problems.

Typically, this error estimate, which indicates the degree of accuracy of the BEM part of the solution, converges more rapidly than the FEM error estimates, as will be seen in the examples given in §3.4. However, it remains a useful estimate and is required for the far-field error indicators to be developed in §3.3.

3.2.2 The boundary field derivative L^2-norm residual error estimates

Procedures similar to those described in the previous subsection can be followed to obtain a BEM residual for the FE/BE method field **derivative** solution [16, pp.106]:

$$\tilde{R}^1_{\Gamma^{N_1}}(\vec{r}_o) \quad = \quad \frac{\partial\tilde{E}^1(\vec{r}_o)}{\partial n'(\vec{r}_o)} - \frac{\partial\overline{E}^1(\vec{r}_o)}{\partial n'(\vec{r}_o)} \tag{20}$$

$\frac{\partial\overline{E}^1(\vec{r}_o)}{\partial n'(\vec{r}_o)}$ is the normal derivative of $\bar{E}^1(\vec{r}_o)$ of equation 17. The first term on the right of equation 20 is the field derivative computed using the BEM part of the solution (and has the same order of accuracy as the FEM part, since the BEM expands both the field and field derivative to the same order); the second term is computed numerically using the Green's function of equation 17.

A residual for the FE/BE method **Neumann boundary condition** is [16, pp.108]:

$$\breve{R}^1_{\Gamma^{N_1}}(\vec{r}_o) = \frac{\partial E^1(\vec{r}_o)}{\partial n'(\vec{r}_o)} - \frac{\partial\tilde{E}^1(\vec{r}_o)}{\partial n'(\vec{r}_o)} \tag{21}$$

This estimate differs subtly from that in equation 20, in that the residual $\breve{R}^1_{\Gamma^{N_1}}(\vec{r}_o)$ is the difference (at $\vec{r}_o$, and on Γ^{N_1}) between the normal field derivative of $E^1(\vec{r}_o)$, obtained by numerically differentiating the FEM part of the solution, and the normal field derivative obtained explicitly from the BEM part of the solution (whereas the estimate of equation 20 is the difference between the solution obtained directly with the BEM and that obtained indirectly from the BEM via the Green's function integral of equation 17). The terms in equation 21 differ due to the indirect enforcement of the Neumann boundary conditions — see section 2.3 — and the different orders of approximation functions (the field derivative solution computed with the FEM is of one order less than that computed with the BEM).

Explicit numerical results for these estimates will not be given in this paper, but these estimates are used in §3.3 and implicitly contained in the results presented in §3.4 and have thus been defined here.

3.3 Far-field error indicators

The FE/BE method solution results in approximate values for the field and field derivatives on the boundary Γ^{N_1}. From this one can calculate the field value at any point (in the near, intermediate or far-field) in the exterior region, Λ, using a numerical approximation of Huygens' principle [16, pp.57] (the exterior BEM equation).

A far-field residual quantity can be defined as:

$$R^1(\vec{r}_o) = \int_{\Gamma^{N_1}(\vec{r}_s)} \left[\bar{R}^1_{\Gamma^{N_1}}(\vec{r}_s)\frac{\partial\Psi(\vec{r}_o,\vec{r}_s)}{\partial n'(\vec{r}_s)} - \breve{R}^1_{\Gamma^{N_1}}(\vec{r}_s)\Psi(\vec{r}_o,\vec{r}_s)\right] d\Gamma^{N_1}(\vec{r}_s) \quad (22)$$

where the residuals of equations 18 and 21 have been substituted into the BEM equation in the place of $E^1(\vec{r}_s)$ and $\frac{\partial \bar{E}^1(\vec{r}_s)}{\partial n'(\vec{r}_s)}$, respectively. $\tilde{R}^1_{\Gamma^{N_1}}(\vec{r}_s)$ of equation 20 can be used instead of $\breve{R}^1_{\Gamma^{N_1}}(\vec{r}_s)$, but the indicator would then be related to the accuracy of only the BEM part of the FE/BE method solution.

The far-field residual $R^1(\vec{r}_o)$ will be zero if E^1 is the true solution, for then both residuals, $\bar{R}^1_{\Gamma^{N_1}}(\vec{r}_s)$ and $\breve{R}^1_{\Gamma^{N_1}}(\vec{r}_s)$, used in the above equation, will be zero.

A radar-width error indicator can now be calculated using the far-field residual, $R^1(\vec{r}_o)$, in the radar-width equation [16, pp.57] instead of $E^{1(sca)}(\vec{r}_o)$:

$$\breve{\gamma}(\phi) = 2\pi|\vec{r}_o|\left|\frac{R^1(\vec{r}_o)}{E^{inc}}\right|^2 \quad (23)$$

A radiation intensity error indicator can also be calculated using the far-field residual, $R^1(\vec{r}_o)$, in the radiation intensity equation [16, pp.109] instead of $E^{1(rad)}(\vec{r}_o)$:

$$\breve{K}(\phi) = 2\pi|\vec{r}_o|\left|R^1(\vec{r}_o)\right|^2 \quad (24)$$

These error indicators estimate the possible error in the radar-width and radiation intensity due to the errors in the approximate FE/BE method field, E^1, and field derivative solution. Both these error indicators will be zero if E^1 is the true field solution.

3.4 Numerical Results

The error estimates and error indicators formulated in the previous sections have been applied to a variety of electromagnetic problems, with promising success [16]. Two examples will be considered in this section. It will be shown that a combination of the different error estimates and indicators can be used as a reliable *a posteriori* estimate of the accuracy and convergence of FE/BE method solutions.

3.4.1 A Scattering Example

The first numerical example which will be considered is for electromagnetic scattering from the lossy (using practical material parameters) dielectric right-circular cylinder of figure 2 excited by a TE_z polarized, plane incident field at a frequency of 2 GHz. (The TE_z error estimates are analogous to the TM_z estimates given in the preceding sections). Local error estimates and actual errors are compared with each other in figures 4 and 5[6]. In table 1 percentage global estimated errors (in the appropriate norms) are compared to actual percentage errors. Note that in tables 1 and 3, percentage estimated and actual errors of more than 100% are denoted by >100%. Percentage error estimates are, in general, not much larger than 100%, even if the actual percentage error is much larger than 100%. A percentage error near 100% does, however, indicate that the solution at hand is not at all close to the true, or converged, solution. It is evident that the global EM-norm error estimate is a good estimate of the actual error. The global percentage error estimates compare reasonably well with the actual global percentage errors. Important to note is that the

[6]The finite element numbers in all the "local EM-norm error comparison" figures in this section have been arranged in such a way that the corresponding actual EM-norm error values are presented in increasing order. The error estimate values plotted are local EM-norm error estimate values in the corresponding finite elements.

global EM-norm error estimate clearly identifies the solutions which are not at all close to convergence, and gives a reasonably accurate estimate of the percentage errors for the solutions which are close to convergence or have converged to the true solution.

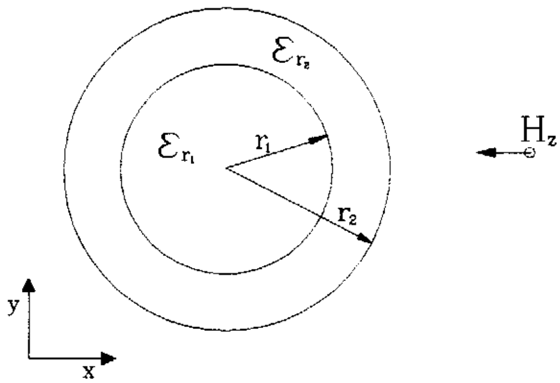

Figure 2: **Double dielectric right-circular cylinder ($\epsilon_{r1} = 1.2 + j1.5$, $\epsilon_{r2} = 1.5 + j2$, $r_1 = 0.2m$, $r_2 = 0.3m$). The cylinder is excited by a plane wave incident field with TE_z polarization. (Note that the time convention is $e^{-j\omega t}$.)**

A "dB" error margin can defined as:

$$\Delta\breve{\gamma}(\phi) = 10\left|log\{\gamma(\phi)\} - log\{\gamma(\phi) - \breve{\gamma}(\phi)\}\right| \qquad (25)$$

where $\gamma(\phi)$ is the radar-width calculated using the FE/BE method and $\breve{\gamma}(\phi)$ is the far-field error indicator of equation 23. Notice that $\breve{\gamma}(\phi)$ and thus $\Delta\breve{\gamma}(\phi)$ will be zero if E^1 is the true field solution.

The "dB" error margin for the first and second order FE/BE method solutions of the radar-width of the scattering problem under consideration are shown in figure 6. The first and second FE/BE method solution as well as the analytical solution of the radar-width are also shown.

All error estimates and error indicators indicate that the second order basis function FE/BE method solution for $M = 1502$ ($M/\lambda^2 = 67$) and $M_b = 90$ ($M_b/\lambda = 5.4$) has converged to an acceptably accurate solution (a maximum field error in the FEM region of around 5% and in the radar-width of 0.1dB). This is confirmed by the actual errors and the radar-width results presented. These error estimates and error indicators have been obtained within negligible computational times (see table 2) compared to the computationally expensive FE/BE method solutions. This is especially true for the FE/BE method

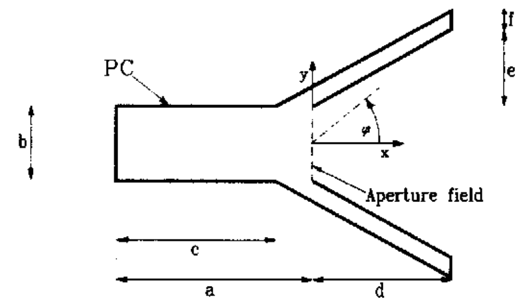

Figure 3: **Radiating 2D horn antenna, with $a = 0.55m$, $b = 0.2m$, $c = 0.45m$, $d = 0.35m$, $e = 0.2m$ and $f = 0.05m$.**

solutions concerning second and third order basis functions and relatively large numbers of finite and boundary elements.

3.4.2 A Radiation Example

The second numerical example which will be considered is for electromagnetic radiation from the 2D horn antenna of figure 3. The horn antenna is excited by the TE_1 mode aperture field (frequency: 750 MHz):

$$E_z = E_0 \cos(\frac{\pi y}{b}) \qquad (26)$$

Local error estimates and actual errors are compared with each other in figures 7 and 8 and it is again evident that these local error estimates clearly identify the regions where the largest EM-norm errors occur. In table 3, percentage estimated errors (in the appropriate norms) are compared to actual percentage errors. The L^2-norm boundary field error estimates indicate that enough boundary elements are used to approximate the fields on the boundary for all values of M_b considered. This is even true for $M_b = 35$ ($M_b/\lambda = 4.5$) with first order boundary element basis functions. (Note that in this case the error estimates show that enough boundary elements were used. The actual errors are quite large due to the fact that too few finite elements were used). The EM-norm ERM error estimates indicate that the second order basis function FE/BE method solution for $M = 859$ ($M/\lambda = 145$) and $M_b = 76$ ($M_b/\lambda = 9.8$) has converged to a satisfactorily accurate FE/BE method solution. (This corresponds to around 100 elements per square wavelength). This is confirmed by the actual EM-norm errors.

A "dB" error margin for the FE/BE method solution of the radiation intensity can be obtained using the radiation intensity error indicator of equation 24. This error margin, denoted by $\Delta\breve{K}(\phi)$, can be calculated as (this is similar to the radar-width error margin of equation 25):

$$\Delta\breve{K}(\phi) = 10\left|log\{K(\phi)\} - log\{K(\phi) - \breve{K}(\phi)\}\right| \quad (27)$$

Note that $\breve{K}(\phi)$ and thus $\Delta\breve{K}(\phi)$ will again be zero if E^1 is the true field solution. The "dB" error margins for the second and third order FE/BE method solutions of the radiation intensity of the problem under consideration are shown in figure 9. The second and third order FE/BE method solutions, as well as the converged solution of the radiation intensity are also shown.

It should be kept in mind that the local error estimates, obtained with equation 12, are dependent on: the frequency and polarization of the incident electromagnetic field; the field values in the finite element under consideration; the order of the approximation functions in the finite element under consideration; the shape of the finite element under consideration; the accuracy of the "first" FE/BE method solution E^1; and finally, the accuracy with which the fields on the boundary Γ^{N_1} are approximated by the BEM part of the FE/BE method solution E^1. The global error estimate is dependent on all local error estimates and the size of the finite element region under consideration.

Bearing all this in mind it is evident that the local and global ERM error estimates are, in general, acceptably accurate estimates of the actual local and global errors. Some of these above mentioned factors can be taken into account to improve the local and global error estimates [16, pp.143] [10, pp.129][8]

The results also indicate that the boundary field L^2-norm residual error estimate, $||\bar{\Theta}^1_{\Gamma^{N_1}}||^2_{L^2}$ is not a good quantitative estimate of the boundary field error. This is due to the fact that this error estimate is a measure of the error in the boundary field for the BEM part of the FE/BE method solution, without consideration of the actual coupling to the FEM part of the solution. The **actual** boundary field error is, however, dependent on the FEM and BEM part of the FE/BE method solution. ***As such, the boundary error estimates must be used in conjunction with the FEM error estimates.***

Only when the FEM part of the solution is of acceptable accuracy and the BEM part is highly inaccurate is this L^2-norm error estimate an accurate quantitative estimates of the boundary field error. This would occur if enough finite element basis functions but insufficient boundary element basis functions had been used — this has not occurred in the examples given in this paper.

It is evident from the far-field error indicator results (figures 6 and 9) that, *used together*, the error margins and error indicators can be very useful in determining the accuracy of FE/BE method radar-width or radiation intensity solutions.

4 Conclusions

The variational boundary-value problem formulation of the coupled FE/BE method for application to general 2D open boundary electromagnetic problems has been presented. This formulation has been developed over the past few decades and has been used successfully by engineers for numerically solving previously intractable electromagnetic problems. (This work is not new but was required as a basis for the rest of the paper.)

ERM local and global error estimates, L^2-norm boundary field error estimates and far-field error indicators have been developed and investigated. This was done for 2D FE/BE method solutions of electromagnetic scattering as well as radiation problems. The global EM-norm ERM error estimate results obtained for electromagnetic scattering and radiation problems were not as accurate as the ERM energy norm error estimates for static electric field problems [19] [16, pp.126]. This is due to a number of factors, including the dependency of the EM-norm error estimates on the frequency of the electromagnetic field, the dependency on the accuracy of the coupling of the BEM with the FEM, and the non-local nature of electromagnetic fields. It was, however, shown that the EM-norm error estimates provide valuable post-processed information regarding the accuracy and convergence of the FE/BE method solutions. This was shown to be true for electromagnetic radiating as well as scattering problems of arbitrary shaped (lossy and lossless) objects [16]. The local EM-norm ERM error estimate results obtained show that the estimated error distributions in the FEM region compared satisfactorily with the actual error distributions. Further investigation could lead to improvement of the local as well as global EM-norm ERM error estimates. (Improved element residual error estimate methods, for non-electromagnetic problems, have already been developed [9, 20] and seem to work well).

Results presented have show that the boundary **field**

L^2-norm error estimate is not a quantitatively accurate boundary field error estimate for the FE/BE method solution, in particular when the FEM part of the solution is considerably in error. This is because the BEM part of the solution may be accurately matching an inaccurate FEM solution at the boundary. The boundary error estimate must not be considered in isolation, but with due cognisance of the FEM error as well.

The radar-width and radiation intensity error indicators, developed for the FE/BE method solutions, were used to obtain "dB" error margins which proved to provide exceptionally useful post-processed radar-width and radiation intensity error information. (These do incorporate, via the boundary field derivative error estimates, information about the accuracy of the FEM part of the solution as well). We should comment that Lee in particular has recently emphasized the role of "global" errors, caused by dispersion error: we have not considered this here [23].

It was also shown that all these error estimates and indicators can be obtained within negligible computational times compared to the computational times of the FE/BE method solutions. This is due to the nature of the error estimate and error indicator methods as well as the highly efficient algorithms employed [16].

Adaptive finite element methods are closely linked to *a posteriori* error estimates and could be used to improve the efficiency of general FEM solutions. The *a posteriori* error estimate methods can be used to identify the regions where the fields need to be approximated more accurately (for example where the fields vary more rapidly), and the finite element mesh can thus be adapted to ensure superior basis function distributions in these regions [10]. Although not shown in this paper, the authors have also worked on this topic [16, pp.168].

In conclusion: the *a posteriori* error estimates and indicators show great promise, but can still be improved upon. It can also be extended to *a posteriori* error estimates for 3D FE/BE method solutions. The use of edge-based elements for 3D formulations would not pose special problems as far as error estimation is concerned, although hierarchical edge-based elements may prove more formidable than the nodal based equivalents if the error estimator is used to drive an adaptive meshing algorithm.

Ultimately one would like to develop a FE/BE method solver with error estimates and indicators which clearly identify the different inaccuracies in the solution at hand and use this information to automatically adapt itself accordingly. Such a solver should provide accuracy and convergence information to the user providing him with a very useful and reliable tool for solving practical electromagnetic engineering problems. That this problem remains a very pressing one, and in particular that ad-hoc estimates leave much to be desired, is clear from recent work [24]. It is hoped that the more rigorous estimate formulations presented in this paper will contribute to progress in this field.

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	Field error (estimated & actual)		**Boundary field error** (estimated & actual)	
$M : M_b$	$\frac{\|\Phi^{12}\|^{EM}}{\|E^1\|^{EM}}$	$\frac{\|E^{12}\|^{EM}}{\|E^1\|^{EM}}$	$\frac{\left\|\Theta^1_{\Gamma^{N_1}}\right\|^2_{L_2}}{\left\|E^1_{\Gamma^{N_1}}\right\|^2_{L_2}}$	$\frac{\left\|E^{12}_{\Gamma^{N_1}}\right\|^2_{L_2}}{\left\|E^1_{\Gamma^{N_1}}\right\|^2_{L_2}}$
	(%)	**(%)**	**(%)**	**(%)**
E^1 *first* **order approximation functions and** E^2 *second* **order approximation functions**				
170:24	**79.99**	**>100**	**>100**	**51.11**
501:43	**74.38**	**79.30**	**9.39**	**31.41**
977:73	**51.42**	**51.17**	**0.73**	**12.49**
1502:90	**44.09**	**35.61**	**0.16**	**2.99**
E^1 *second* **order approximation functions and** E^2 *third* **order approximation functions**				
170:24	**47.99**	**74.48**	**2.37**	**17.50**
501:43	**11.34**	**13.38**	**0.07**	**0.39**
977:73	**7.04**	**7.35**	**0.01**	**0.05**
1502:90	**5.52**	**4.24**	**0.00**	**0.03**

Table 1: **Percentage error estimates compared to actual errors. This is for scattering from the lossy dielectric right-circular cylinder of figure 2 excited by a TE_z polarized, plane incident field (frequency: 2 GHz).** M **is the number of finite elements and** M_b **the number of boundary element used.**

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E^1 *first* order approximation functions and E^2 *second* order approximation functions

M	Computational time (hours:min:sec)		
	FE/BE method solution		Error estimates
	E^1	E^2	
170	00:00:02	00:00:10	00:00:05
501	00:00:06	00:01:33	00:00:10
977	00:00:23	00:08:59	00:00:21
1502	00:00:42	00:18:28	00:00:29

E^1 *second* order approximation functions and E^2 *third* order approximation functions

M	Computational time (hours:min:sec)		
	FE/BE method solution		Error estimates
	E^1	E^2	
170	00:00:10	00:01:25	00:00:07
501	00:01:33	00:12:27	00:00:14
977	00:08:59	01:28:58	00:00:29
1502	00:18:28	02:27:31	00:00:40

Table 2: Computational times (on a HP-720 work station) for the FE/BE method solution and error estimates (all error estimates and error indicators combined) for the scattering problem concerning the right circular cylinder in figure 2. M is the number of finite element used.

	Field error		Boundary field error	
$M : M_b$	$\frac{\|\|\Phi^{12}\|\|^{EM}}{\|\|E^1\|\|^{EM}}$	$\frac{\|\|E^{12}\|\|^{EM}}{\|\|E^1\|\|^{EM}}$	$\frac{\left\|\Theta^1_{\Gamma^{N_1}}\right\|^2_{L_2}}{\left\|E^1_{\Gamma^{N_1}}\right\|^2_{L_2}}$	$\frac{\left\|E^{12}_{\Gamma^{N_1}}\right\|^2_{L_2}}{\left\|E^1_{\Gamma^{N_1}}\right\|^2_{L_2}}$
	(%)	(%)	(%)	(%)
E^1 *first* order approximation functions and E^2 *second* order approximation functions				
112:35	79.90	>100	0.04	>100
182:35	68.57	>100	0.05	>100
450:57	52.64	51.76	0.01	42.56
859:76	40.73	36.79	0.00	16.70
E^1 *second* order approximation functions and E^2 *third* order approximation functions				
112:35	16.03	57.55	0.00	3.40
182:35	11.74	15.78	0.00	1.78
450:57	10.16	9.63	0.00	0.35
859:76	6.25	5.92	0.00	0.16

Table 3: Percentage error estimates compared to actual errors. This is for the electromagnetic radiation problem concerning the horn antenna of figure 3. Frequency: 750 MHz, Polarization: TE_1 mode aperture source field. M is the number of finite elements and M_b the number of boundary element used.

E^1 *first* order approximation functions and E^2 *second* order approximation functions

E^1 *second* order approximation functions and E^2 *third* order approximation functions

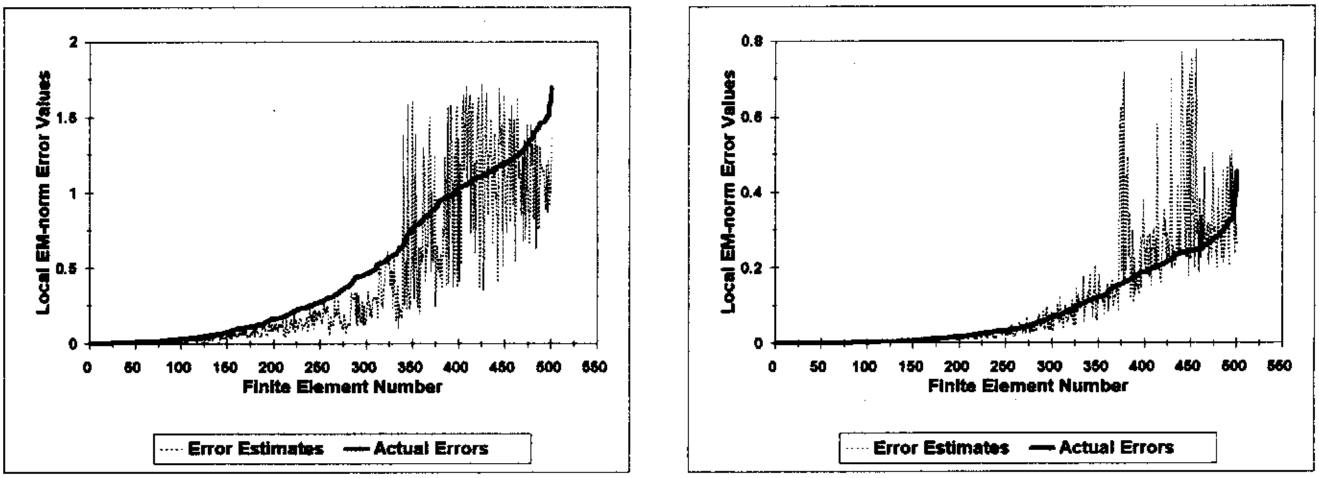

Figure 4: **Comparison of estimated and actual local EM-norm error values for the FE/BE method solution of the electromagnetic scattering problem concerning the right circular cylinder of figure 2. The incident electromagnetic field is TE_z polarized at frequency 2 GHz. $M = 501$ is the number of finite elements used.**

Estimated Error Distribution

Actual Error Distribution

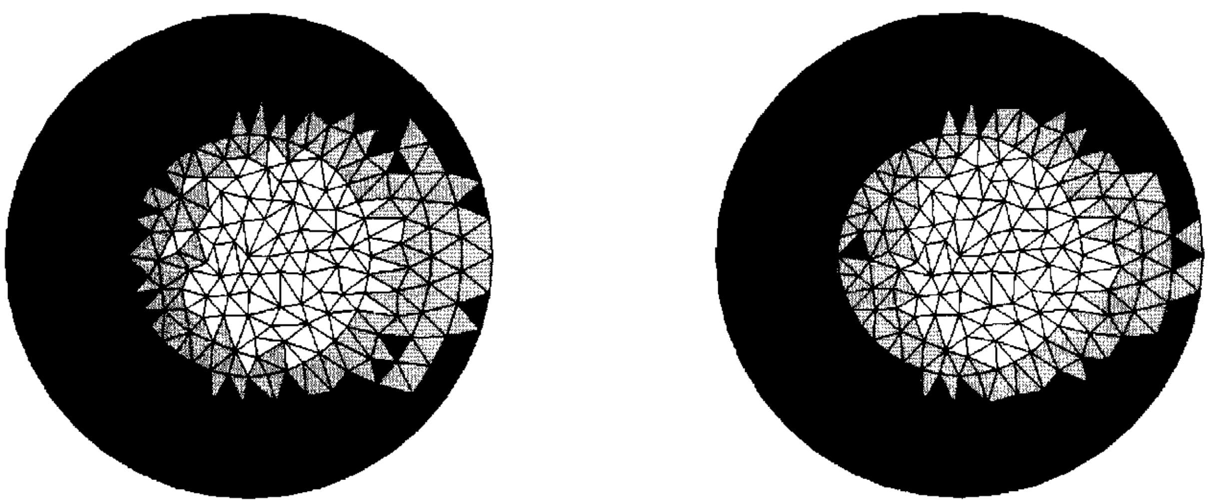

Figure 5: **Comparison of estimated and actual local EM-norm error distributions for the FE/BE method solution of the fields in and around the right circular cylinder of figure 2. The incident electromagnetic field is TE_z polarized at frequency 2 GHz. Dark elements indicate high error values and light elements indicate low error values. This is for E^1 *second* order approximation functions and E^2 *third* order approximation functions.**

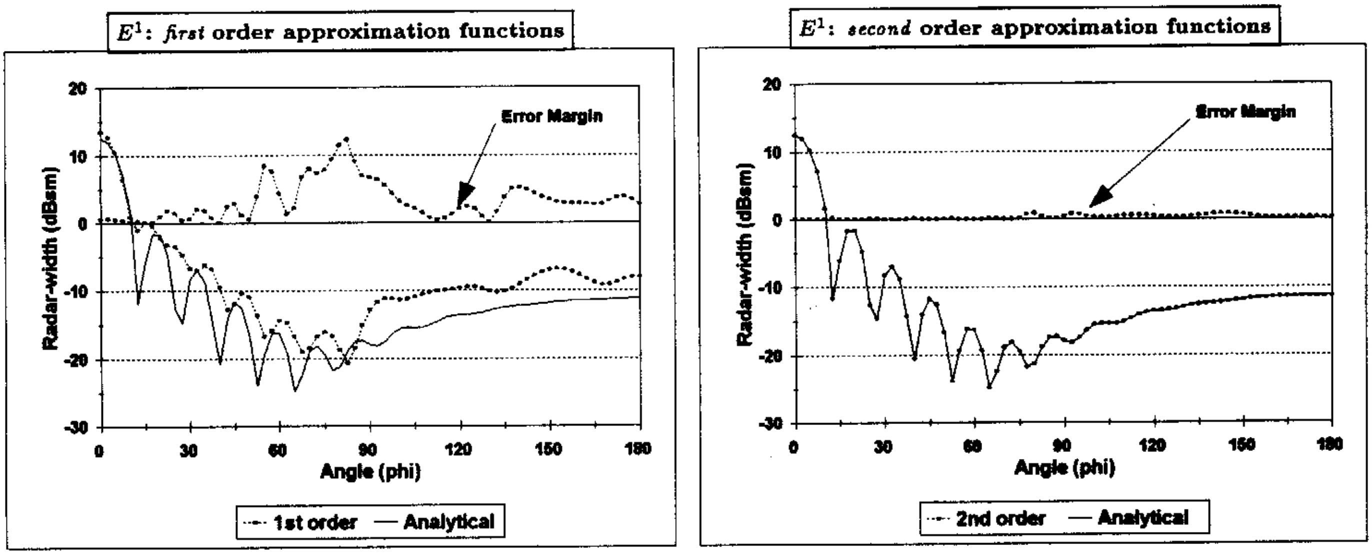

Figure 6: **Comparison of the radar-width values, obtained analytically and numerically (FE/BE method solution), for the scattering problem concerning the right circular cylinder in figure 2. The "dB" error margins, $\Delta\Upsilon(\phi)$, are also shown for first and second order FE/BE solutions. Frequency: 2 GHz. Polarization: TE_z. Number of finite elements used: $M = 1502$. Number of boundary elements used: $M_b = 90$.**

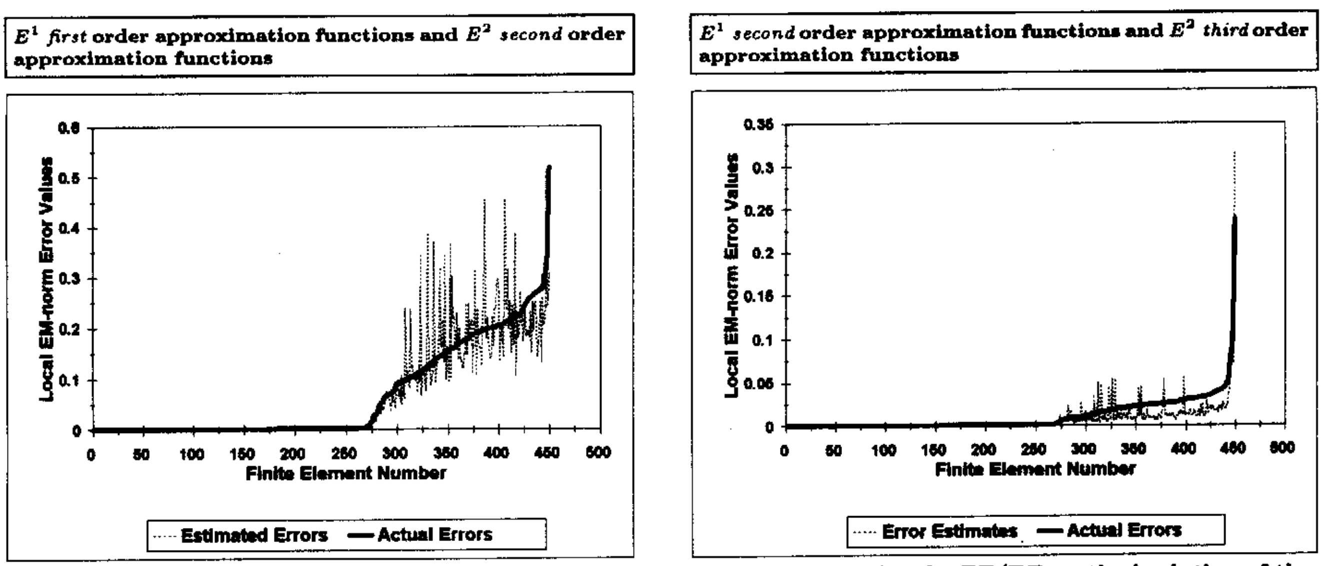

Figure 7: **Comparison of estimated and actual local EM-norm error values for the FE/BE method solution of the electromagnetic radiation problem concerning the horn antenna of figure 3. Frequency: 750 MHz, Polarization: TE_1 mode aperture source field. $M = 450$ is the number of finite elements used.**

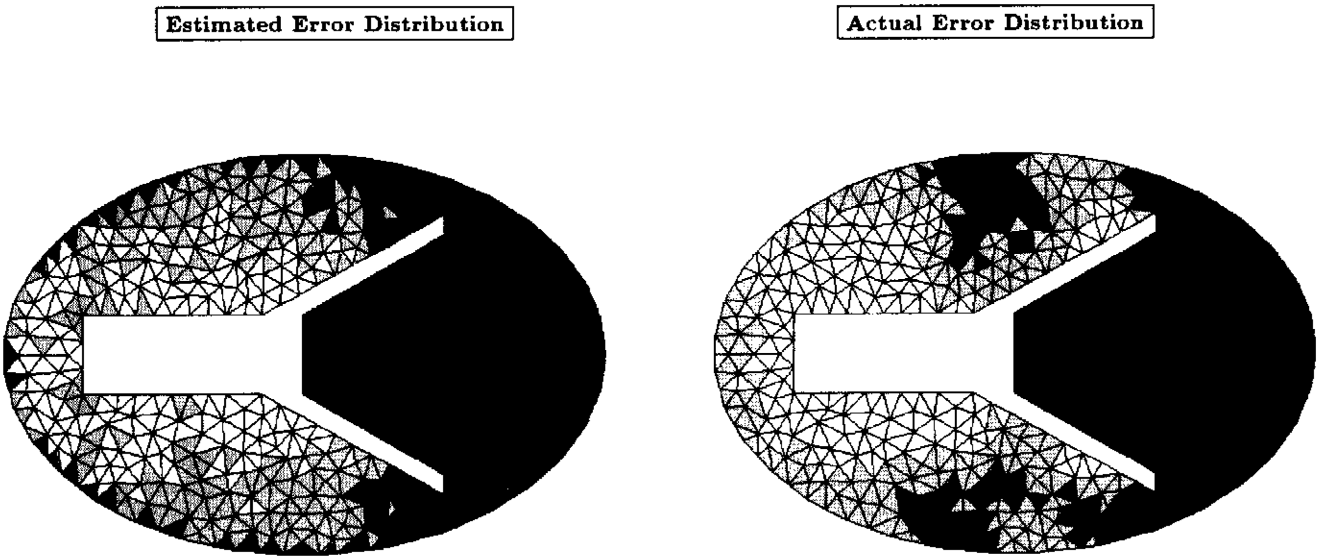

Figure 8: **Comparison of estimated and actual local EM-norm error distributions for the FE/BE method solution of the electromagnetic radiation problem concerning the horn antenna of figure 3. Frequency: 750 MHz, Polarization: TE_1 mode aperture source field. Dark elements indicate high error values and light elements indicate low error values. This is for E^1 *second* order approximation functions and E^2 *third* order approximation functions.**

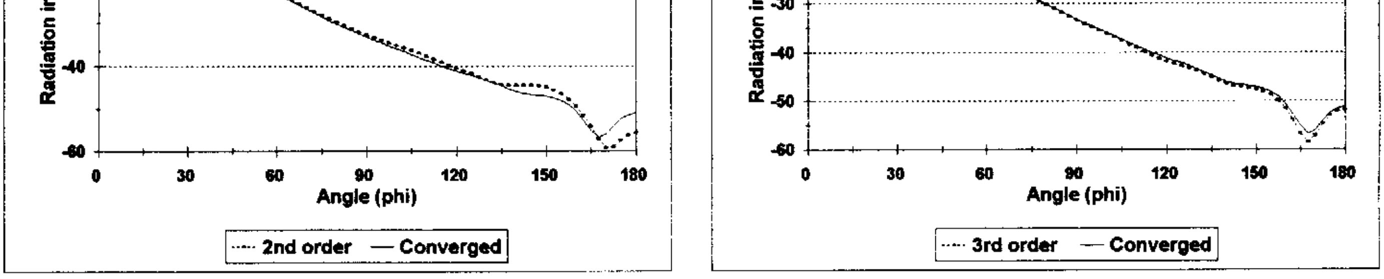

Figure 9: **Comparison of the radiation intensity values, obtained numerically (FE/BE method solution), for the radiation problem concerning the horn antenna of figure 3. Second and third order basis function solutions, the converged solution and the "dB" error margins, $\Delta\breve{K}(\phi)$, are shown. Frequency: 750 MHz, Polarization: TE_1 mode aperture source field. M and M_b are the number of finite elements and boundary elements used respectively.**