DIFFICULTIES ENCOUNTERED WHEN ATTEMPTING TO VALIDATE THIN-WIRE FORMULATIONS FOR LINEAR DIPOLE ANTENNAS

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ABSTRACT: When analyzing wire antennas, the "thin-wire" kernel is often used as a convenient approximation to the exact singular kernel in the electric-field integral equation. In this paper, it is shown that the thin-wire kernel is a poor approximation to the true kernel, but its use does yield good values for input impedance over a significant range of parameters (e.g., wire radii, size of subsectional cells). The validity of the thin-wire kernel when used within the electric-field integral equation appears to be due to the fact that the approach is often a good approximation to the Extended Boundary Condition (EBC) formulation. Although it is often implied in the literature that use of the thin-wire kernel will produce "convergent" values for input impedance, in actuality there is no guarantee that results improve as more cells are taken along the wire. Despite widespread use of the "thin-wire" kernel, there are inherent difficulties in the validation of codes based on this approximation.

1. INTRODUCTION

One question of importance to the computational electromagnetics community is: *Do the results from moment-method codes using different basis and testing functions converge to the same solution?* In an attempt to answer this question, several integral equation formulations were used to generate numerical solutions for the input impedance of linear dipole antennas. As is common practice when modeling wire antennas, these formulations incorporated the "thin-wire" kernel within the electric-field integral equation. Unfortunately, it was impossible to determine if the results from these formulations converged to the same solution as the number of unknowns used within the numerical model increased, because the results for impedance failed to converge. In this paper, it is shown that the lack of convergence was due to the use of the thin-wire kernel within the electric-field integral equation. In addition, several conclusions are presented relevant to the validation of computer codes of this type.

The thin-wire kernel is presented in Section 2, followed by a description of the initial validation study in Section 3. In addition, Section 3 presents results for the input impedance of linear dipoles produced by two different codes that incorporate the thin-wire kernel. Section 4 presents data generated using the exact kernel. Finally, a discussion of the EBC method is presented in Section 5. Use of either the exact kernel or the EBC formulation appears to improve the convergence behavior of the numerical solutions.

2. THIN-WIRE APPROXIMATION TO SINGULAR KERNEL

During the past two decades, many publications concerning numerical solutions for impedance of wire antennas have employed the "thin-wire" kernel as a convenient approximation to the exact singular kernel in the electric-field integral equation. If we consider a hollow, linear dipole with constant radius a, excited by a ϕ-invariant feed, the exact form of the electric-field integral equation (EFIE) is

$$-E_z^{inc}(z) = \frac{1}{j\omega\varepsilon 4\pi}\left(\frac{\partial^2}{\partial z^2} + \beta^2\right)\int I(z')\, k(z-z')\, dz' \qquad (1)$$

where the kernel is given by

$$k(z) = \frac{1}{\pi}\int_0^{\pi} \frac{e^{-j\beta\sqrt{z^2 + 4a^2\sin^2(\phi/2)}}}{\sqrt{z^2 + 4a^2\sin^2(\phi/2)}}\, d\phi \qquad (2)$$

For simplicity, Equation (2) is often replaced by

$$k_{thin}(z) = \frac{e^{-j\beta\sqrt{z^2 + a^2}}}{\sqrt{z^2 + a^2}} \qquad (3)$$

In this paper, Equation (3) will be referred to as the "thin-wire" kernel, and the combination of Equations (1) and (3) as the "thin-wire" equation. Solutions to either the thin-wire equation or the exact equation are typically obtained via the method of moments [1-11]. The advantage to the use of (3) instead of (2) is the relative ease by which the moment-method matrix elements are computed. The use of (3) typically requires a one-dimensional numerical integration over a non-singular integrand for all the elements; the use of (2) usually requires two-dimensional numerical integration over an integrand that contains a singularity that must be dealt with as a special case. Thus, considerably more work is required to implement the exact kernel, both in terms of the numerical computation and in the complexity of the computer code.

The validity of the thin-wire equation has been discussed by many authors, who usually conclude that the equation is valid only when the wire radius a is much less than the wire length and the wavelength. King suggests that the equation is valid only when $\beta a < 0.1$ [3]. Tesche presents data showing a comparison between the thin-wire kernel of Equation (3) with the exact form of Equation (2), and clearly illustrates that results from the thin-wire equation diverge from those of the rigorous EFIE as the dipole size increases (length and radius) relative to the wavelength [6]. In order to develop guidelines for the use of the thin-wire equation in the context of the method of moments, it is necessary to incorporate information about the dipole radius with information describing the cell sizes in use. Imbriale and Ingerson applied the thin-wire equation to thicker wires, and introduced the concept of an equivalent radius that appears to correct for the rapid numerical divergence sometimes encountered [8]. Elliott concluded that the thin-wire equation is valid as long as the segment being integrated within the method of moments is at least several wire radii in length [10]. However, Poggio previously studied several different approximations to the exact kernel, and identified their regions of validity [5]. He concluded that the "blind" use of any approximate kernel was inappropriate. These and other publications identify limitations associated with the thin-wire kernel, but none appear to provide comprehensive guidelines for its use. This is in spite of the fact that the thin-wire kernel has a long history of use in computational electromagnetics.

3. INITIAL VALIDATION STUDY

To investigate the validity of numerical models for wire antennas, the input impedance results from several different moment-method codes were compared. The objective of the comparison was to determine whether the numerical solutions from each code converged to the same result for input impedance. To focus on the effect of different basis and testing functions, the geometry (linear dipole) and feed model (frill source [7,12]) were identical in each code.

Program CENFED from the text by Stutzman and Thiele was available for the case of pulse basis functions and Dirac delta testing functions [11], and a second code SINGAL was generated by the author which used subsectional sinusoidal-triangle functions for basis and testing. It is suggested that the type of basis-testing function pair used by SINGAL produces "rapidly converging" solutions [11]. Both of these codes were based on the exclusive use of the thin-wire kernel. One-dimensional numerical integration was required in each case to compute the entries of the moment-method matrix. The input impedance was defined as the reciprocal of the current at the feed location in the center of the dipole. The frill feed used in both codes was tested on the axis of the dipole rather than the actual surface, following the approach employed within CENFED. Both codes neglected currents on the dipole end caps.

Based on Figures 7-5 and 7-12 from Stutzman and Thiele [11], it was expected that both CENFED and SINGAL would produce input impedance solutions that began to converge to some value when the number N of basis functions along the dipole approached 100 per half wavelength (CENFED) or 25 per half wavelength (SINGAL). Typically, these figures are plotted on a linear scale which tends to flatten the curve as N is increased. Mittra and Klein recommend plotting impedance as a function of 1/N, in order to facilitate a better estimate of the limiting value as N increases [13]. Figures 1a and 1b show the input impedance produced by CENFED as a function of 1/N, for a dipole of length 0.47 λ and radius 0.005 λ (where λ denotes the wavelength). Of interest is the fact that there are two distinct curves in these figures, one obtained with N equal to an even number and another obtained with N odd. In Figure 1a, the even and odd curves appear to coalesce at approximately N=100. More important is the fact that the curves for input impedance do not appear to converge as N increases. This is especially evident in Figure 1b.

SINGAL was also used to model the identical dipole geometry, and results for input impedance as a function of N (for N odd) are displayed in Figures 2a and 2b. In this case, the results are plotted on a logarithmic scale, so that a successive doubling of N contributes to an equal increment along the horizontal axis of each figure. Although slightly different curves could be obtained for even values of N, there was little difference between SINGAL's results for N even and N odd. Noteworthy, however, is the fact that these curves appear to be straight lines; they lack any indication of convergence to a finite result.

It is clear that the above data do not converge as the number of expansion functions is increased. Yet, numerous examples of this type of data showing agreement with measured results are available in the literature. For example, Figure 7-6 from [11] shows excellent agreement between the output of CENFED with N=100 and measured impedance data. Thus, for certain parameters, valid results can be produced by these codes. However, based on Figure 1, it is obvious that data from CENFED for N less than 100 is questionable (because of the large difference between even and odd N). In addition, as N increases beyond 100 the numbers continue to change. As a result, the user of this type of moment-method code can not reliably get a "better" answer by increasing N.

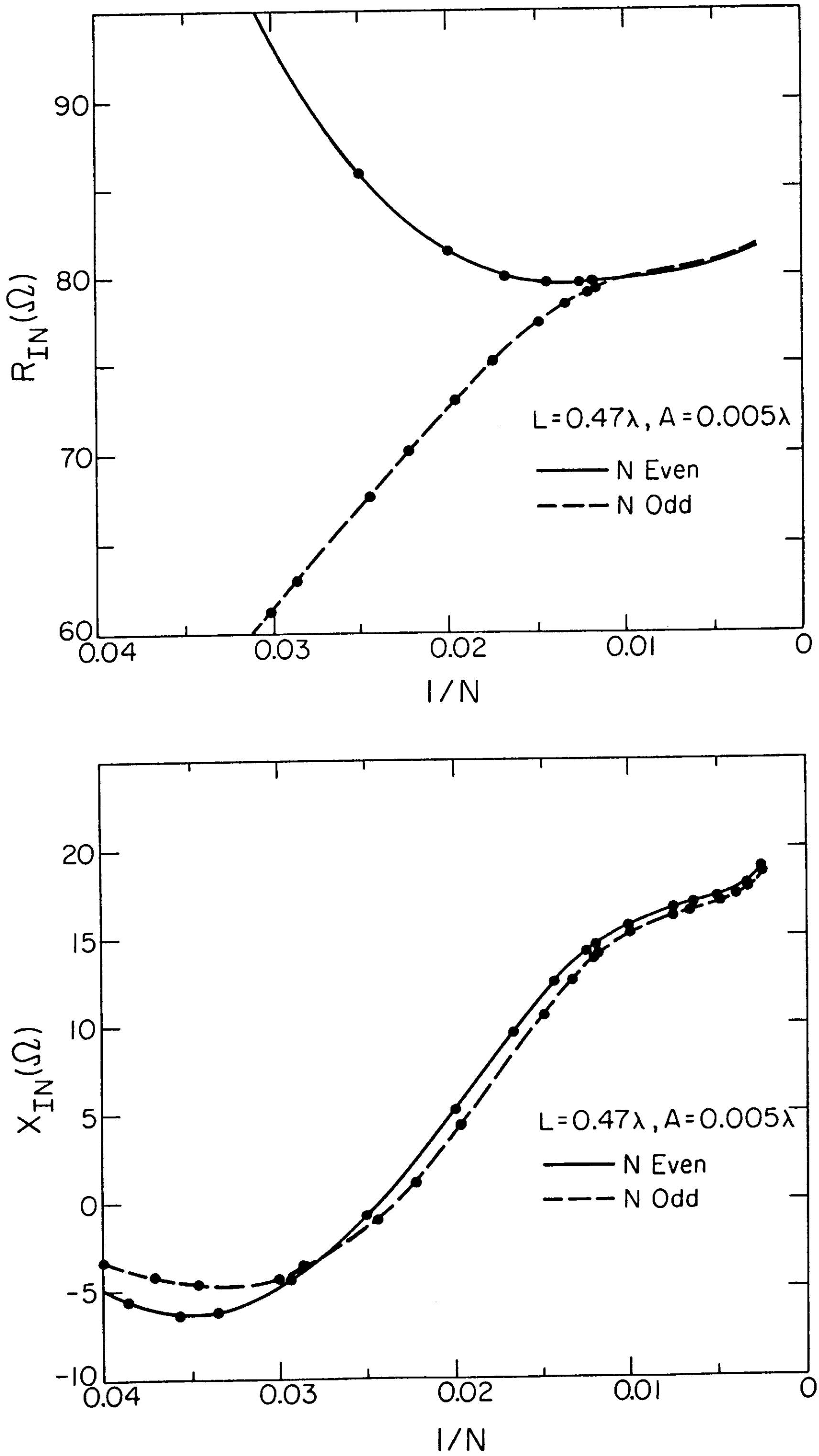

Figure 1. Input impedance produced by CENFED (thin-wire kernel; pulse basis functions and Dirac delta testing functions) for a dipole with length L=0.47 λ and a=0.005 λ. The data is plotted versus 1/N, where N is the number of basis functions. a) real part; b) imaginary part.

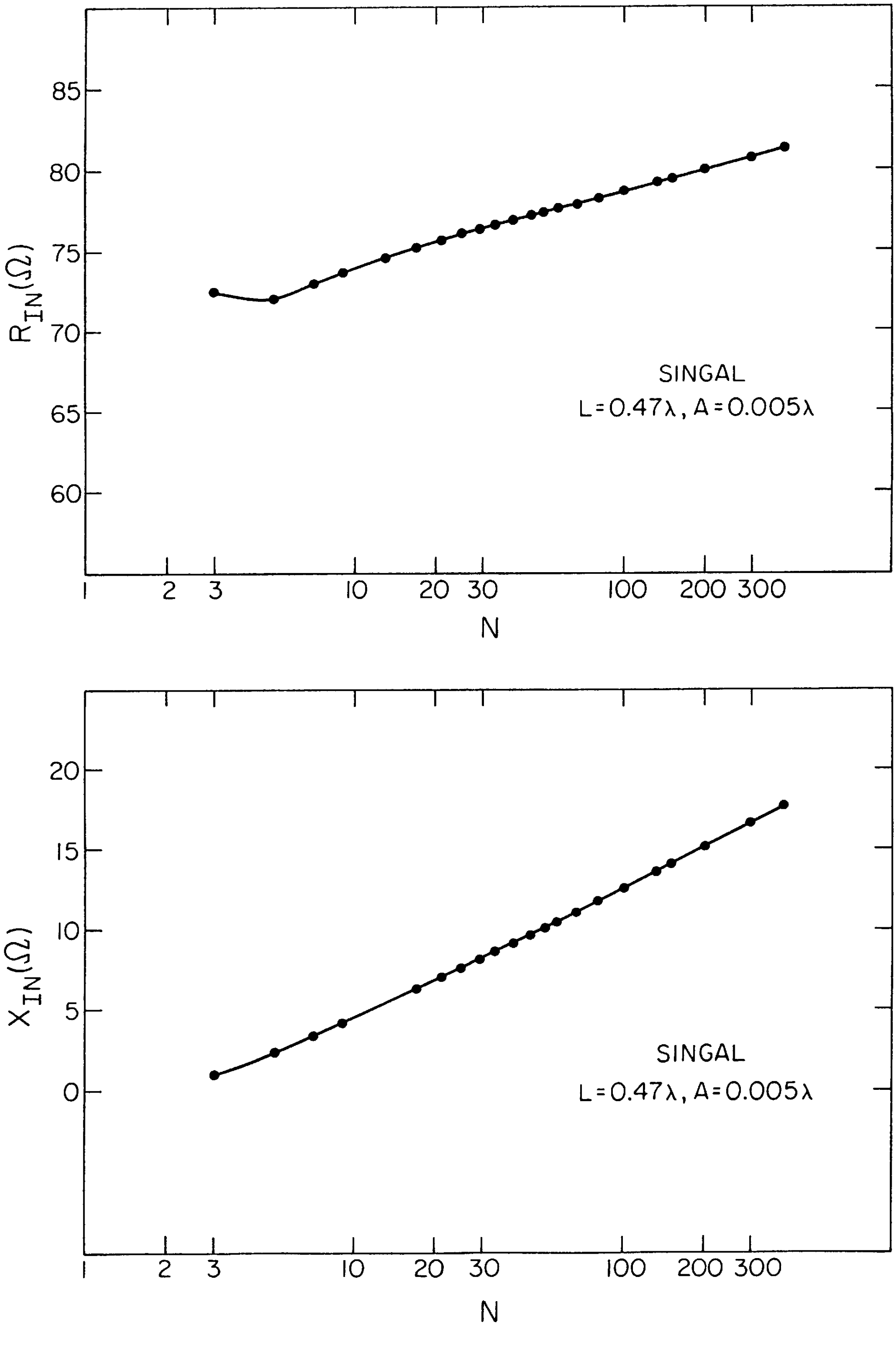

Figure 2. Input impedance produced by SINGAL (thin-wire kernel, sinusoidal-triangle basis and testing functions) for a dipole with length L=0.47 λ and a=0.005 λ. a) real part; b) imaginary part.

The source of the divergence in Figures 1 and 2 is believed to be due to the use of the thin-wire kernel. If viewed as an approximation to the exact kernel, the thin-wire kernel is fundamentally incompatible with a moment-method approach (which should improve in accuracy as the cell sizes are reduced) because it becomes less valid for smaller cells. The observed divergence of the impedance data is at odds with much of the literature, where the prevailing opinion seems to suggest (or assume) that the input impedance produced by the thin-wire equation converges as the number of basis functions used in a moment-method procedure is increased. In addition, Miller and Deadrick observed that the use of the thin-wire kernel resulted in a non-physical oscillation in the current at the ends and middle of a center-fed dipole [Figures 4.3 and 4.4 of reference 9]. They do not explain the oscillation other than to say that it is "numerically generated." Because the thin-wire kernel is usually considered a good approximation to the exact kernel, it is often suggested that one could improve the performance of a thin-wire computer code by replacing Equation (3) by Equation (2) for the closely-spaced terms in the matrix, and using Equation (3) for the remaining terms. This is sensible if, in fact, Equation (3) is a reasonable approximation to Equation (2). Below, we show that this is often not the case.

4. COMPARISON OF MATRIX ELEMENTS PRODUCED BY THIN-WIRE AND EXACT KERNELS

To study the difference between the exact and thin-wire kernel in detail, an additional computer code was created using the exact singular kernel from (2) and subsectional sinusoidal-triangle basis and testing functions. Care was taken to ensure that the matrix elements could be evaluated accurately (to 4 or 5 decimal places, if desired) so that meaningful comparisons could be obtained. The matrix elements were computed using adaptive numerical integration algorithms from the IMSL software library. When the source and observation regions within these integrals coincide, the singularity in the exact kernel is extracted and integrated analytically, following an approach similar to that implemented by Wilton and Butler [14]. The computer used for this study was the CDC CYBER 175, which uses 60 bit registers for arithmetic operations.

Although it is possible to directly compare the exact and thin-wire kernels, it is more meaningful to compare the entries of the moment-method matrix. Data for the diagonal impedance matrix elements produced by the use of Equations (2) and (3) are compared in Tables 1 to 8. These data are based on the use of sinusoidal-triangle basis and testing functions. It is clear that the numbers arising from the two kernels exhibit good agreement for the non-singular part of the equation (Tables 1 - 4) but are very different for the singular portion (Tables 5 - 8). In fact, there is a considerable degree of difference in these values even over the range of cell size often

TABLE 1

Re{Z_{mm}} as a function of cell length l for radius a=0.0025 λ.

l (λ)	thin-wire	exact
0.001	0.000197	0.000197
0.002	0.000789	0.000789
0.005	0.004932	0.004932
0.010	0.019732	0.019731
0.020	0.078957	0.078953
0.050	0.49485	0.49483
0.100	1.9992	1.9991
0.200	8.3296	8.3292
0.500	73.092	73.088

TABLE 2

Re{Z_{mm}} as a function of cell length l for radius a=0.01 λ.

l (λ)	thin-wire	exact
0.001	0.000197	0.000197
0.002	0.000789	0.000788
0.005	0.004929	0.004925
0.01	0.019717	0.019701
0.02	0.078899	0.078837
0.05	0.49449	0.49410
0.1	1.9977	1.9961
0.2	8.3234	8.3168
0.5	73.036	72.977

TABLE 3

Re{Z_{mm}} as a function of cell length l for radius a=0.05 λ.

l (λ)	thin-wire	exact
0.001	0.000193	0.000190
0.002	0.000774	0.000759
0.005	0.004836	0.004741
0.01	0.019345	0.018966
0.02	0.077411	0.075893
0.05	0.48516	0.47564
0.1	1.9600	1.9215
0.2	8.1658	8.0051
0.5	71.623	70.183

TABLE 4

Re{Z_{mm}} as a function of cell length l for radius a=0.2 λ.

l (λ)	thin-wire	exact
0.001	0.000140	0.000101
0.002	0.000560	0.000402
0.005	0.003502	0.002513
0.01	0.014008	0.010051
0.02	0.056052	0.040219
0.05	0.35127	0.25203
0.1	1.4187	1.0176
0.2	5.9045	4.2298
0.5	51.373	36.453

TABLE 5

Im{Z_{mm}} as a function of cell length l for radius a=0.0025 λ.

l (λ)	thin-wire	exact
0.001	-72.084	-1691.7
0.002	-248.48	-1704.1
0.005	-829.94	-1733.7
0.01	-1286.9	-1708.0
0.02	-1398.9	-1546.9
0.05	-1094.6	-1122.9
0.1	-759.73	-766.91
0.2	-440.82	-442.51
0.5	41.584	41.328

TABLE 6

Im{Z_{mm}} as a function of cell length l for radius a=0.01 λ.

l (λ)	thin-wire	exact
0.001	-1.186	-421.34
0.002	-4.693	-421.74
0.005	-27.261	-423.57
0.01	-88.072	-427.27
0.02	-206.67	-432.37
0.05	-337.17	-413.31
0.1	-325.19	-349.27
0.2	-230.22	-236.51
0.5	38.776	37.767

TABLE 7

Im{Z_{mm}} as a function of cell length *l* for radius *a*=0.05 λ.

l (λ)	thin-wire	exact
0.001	-0.0091	-90.448
0.002	-0.0364	-84.233
0.005	-0.2268	-84.246
0.01	-0.8969	-84.266
0.02	-3.4311	-84.287
0.05	-16.600	-83.961
0.1	-37.438	-81.361
0.2	-49.164	-68.174
0.5	24.257	19.822

TABLE 8

Im{Z_{mm}} as a function of cell length *l* for radius *a*=0.2 λ.

l (λ)	thin-wire	exact
0.001	-0.0002	-29.584
0.002	-0.0006	-22.263
0.005	-0.0038	-21.052
0.01	-0.0151	-21.046
0.02	-0.0604	-21.015
0.05	-0.3712	-20.852
0.1	-1.4031	-20.446
0.2	-4.6888	-19.454
0.5	-19.163	-20.190

TABLE 9

Z_{mm} and input impedance for dipole with *L*=0.5 λ and *a*=0.0025 λ.

N	101	201
matrix entry (thin-wire)	-0.00058+j39.350	-0.00008+j12.787
input impedance (thin-wire)	91.59+j48.45	92.44+j50.05
matrix entry (exact kernel)	-0.00058+j52.669	-0.00008+j26.959
input impedance (exact kernel)	90.57+j47.15	90.82+j47.94

described in the literature as the "accurate range" (i.e., the range where the thin-wire kernel is supposedly a good approximation to the true kernel). From a study of these tables, it appears that the cell length to radius ratio must exceed 10 for the thin-wire kernel and exact kernel to produce similar diagonal matrix elements.

To investigate the difference in input impedance, results from the code based on the exact kernel were compared to data from SINGAL. Both codes neglected end-cap currents, and the frill feed employed with the exact kernel code was tested on the dipole axis (an additional approximation). Figures 3a and 3b show the real and imaginary part of the input impedance of a dipole with length L=0.47λ and radius a=0.005λ as a function of the number of basis functions, for both the thin-wire equation and the exact EFIE. On a logarithmic scale, the exact kernel data do appear to be converging. Observe that the values for input impedance produced by these codes are very similar for N between 10 and 50. It is remarkable that the values for input impedance are so close, when for these parameters the values of the diagonal matrix elements differed by as much as a factor of 4. (Normally, the input impedance is very sensitive to slight changes in the diagonal matrix elements.) Table 9 shows a comparison of input impedance and diagonal matrix elements for a dipole with L=0.5λ and a=0.0025λ. There is clearly more difference in the matrix elements than in the input impedance.

5. DISCUSSION

The "thin-wire" equation has achieved widespread use because it produces useful results for many applications. Surprisingly however, Equation (3) is a often a poor approximation to Equation (2). In other words, numerical experimentation suggests that Equation (3) has a range of validity much greater than one would expect based on a comparison to Equation (2). The validity of the thin-wire kernel seems to be due to the fact that the thin-wire equation closely approximates a completely different formulation, the Extended Boundary Condition (EBC) method proposed by Waterman [15-16] and applied to dipoles by others [17]. (It is interesting that this point of view was adopted early in numerical electromagnetics by Richmond [2], although it does not appear to have achieved widespread appreciation. Recently, Burke [18] indicated that the EBC viewpoint was being employed in connection with upgrades to the Numerical Electromagnetics Code (NEC) for wires [19].) The EBC method requires the boundary conditions to be enforced interior to a solid body, and makes use of the analytical continuation of the fields to ensure their enforcement on the surface (thus, "extending" the boundary condition). An advantage of the EBC approach in comparison to the conventional surface integral equations is the nonsingular kernel that arises in the EBC method. A disadvantage is the inability of the rigorous EBC method to treat open structures

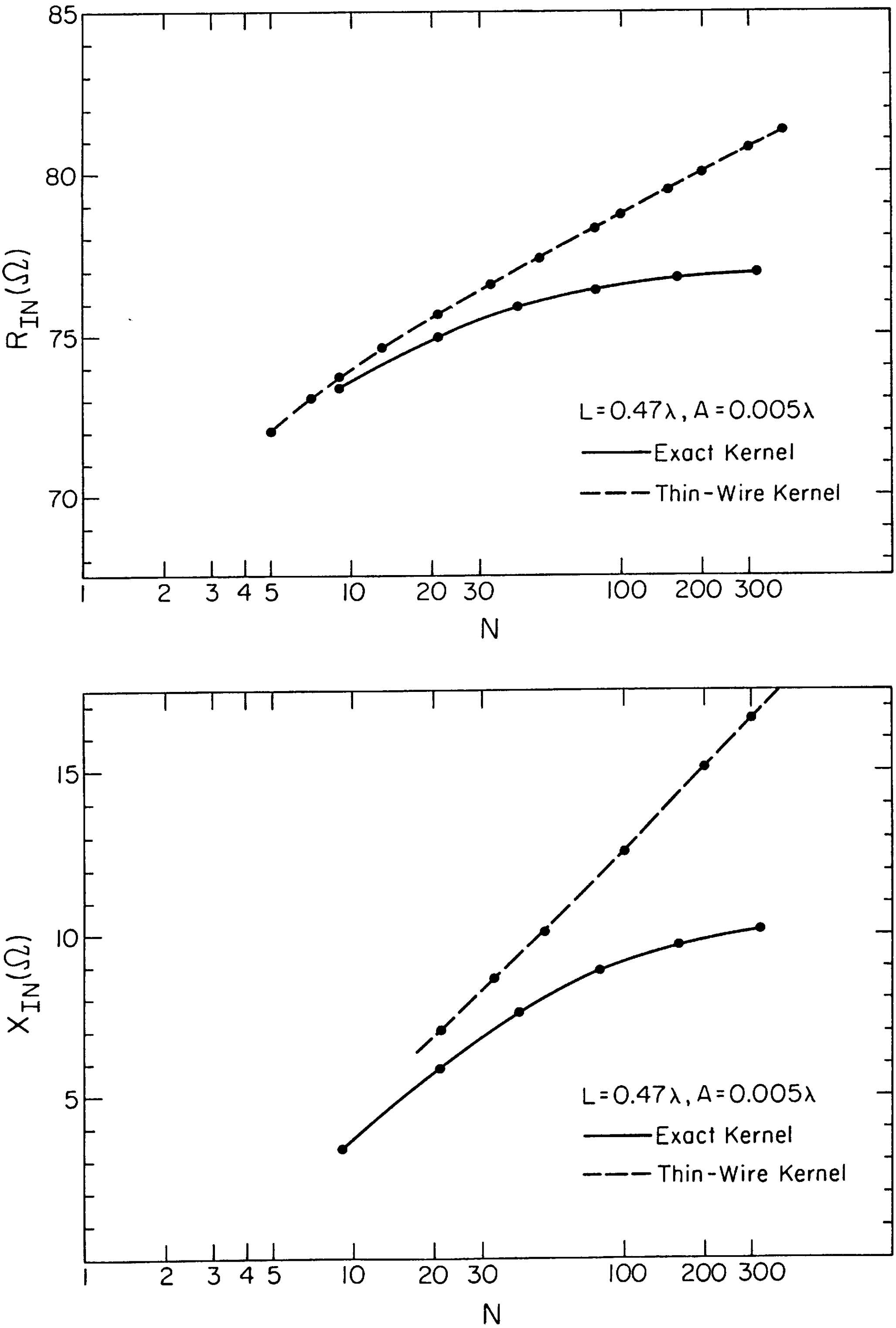

Figure 3. Input impedance produced by SINGAL compared to data from a formulation using the exact singular kernel for a dipole with length L=0.47 λ and a=0.005 λ. Both formulations use sinusoidal-triangle basis and testing functions. a) real part; b) imaginary part.

(including hollow dipoles). However, in many cases the computer modeling of thin wires does not distinguish between the solid and hollow case, as any end-cap currents are neglected. *The primary difference between the EBC and thin-wire equations for a solid wire is the absence of end-cap currents in the thin-wire equation.* In general, as long as the physical end-cap currents are in fact negligible, the thin-wire equation will be a good approximation to the EBC equation. Note that the exact implementation of the EFIE would require the feed field to be sampled on the dipole surface. In contrast, the EBC formulation requires the excitation to be sampled internal to the body. It is interesting that many codes based on the thin-wire kernel (including CENFED and SINGAL) employ the EBC treatment of the excitation.

There are many observed effects attributed to the thin-wire kernel that can be explained in terms of the EBC perspective. As compared to a rigorous EBC formulation, the apparent approximation in the thin-wire equation is to neglect the end-cap currents. Thus, it is not surprising that Miller and Deadrick observed a non-physical oscillation in that region of the dipole [9]. In unpublished numerical experimentation carried out by colleagues of the author, the modification of a thin-wire program to incorporate basis functions on the end caps reduced the type of oscillation observed by Miller and Deadrick by an order of magnitude. (The improved effect was observed after the addition of one basis function to model currents on flat end caps. This did not constitute a rigorous application of the EBC method, because many degrees of freedom are necessary in order to analytically continue the fields to a flat end cap. Thus, the oscillation was not eliminated entirely.) As hypothesized in Section 3, hybrid schemes involving a mixture of the thin-wire and exact kernels should only produce reasonable results if Equation (3) is a good approximation to Equation (2). Tests conducted by the author suggest that hybrid schemes sometimes completely fail to produce correct results. This is despite the fact that both the exact EFIE and the thin-wire equation, if not hybridized, produced very acceptable input impedance results on their own for the same parameters. These findings support the notion that Equation (3) does not have to be a good approximation to (2) in order for the thin-wire equation to produce reasonably accurate results.

Figures 4a and 4b show the effect on input impedance of including a single basis function to represent end cap currents. The curves clearly flatten compared to those of SINGAL. Although the results based on the improved end cap model do not appear to be converging, it is felt that this is primarily a consequence of the fact that only one basis function is employed at each end. Use of additional basis functions at the end caps should be investigated to verify this conclusion.

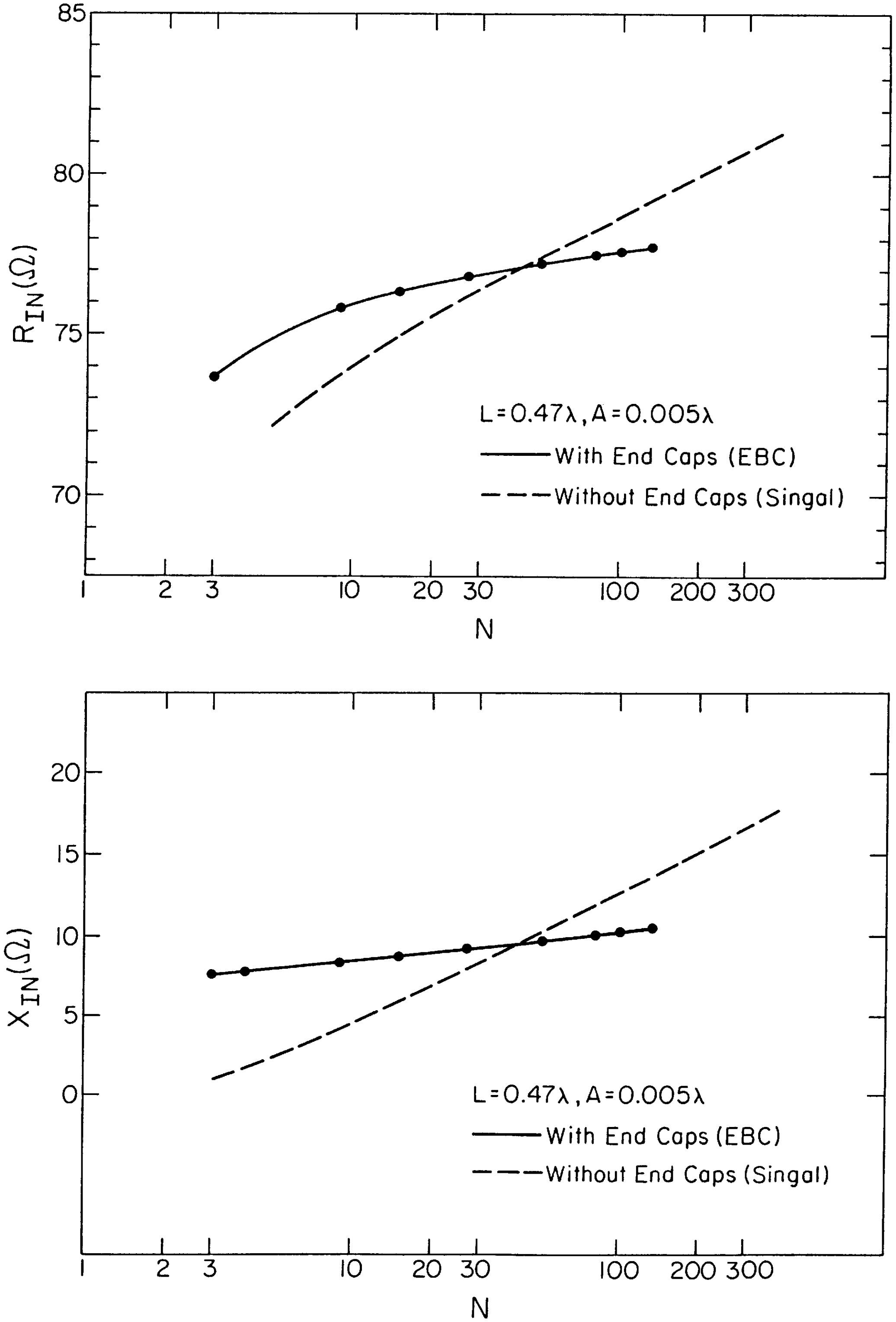

Figure 4. Input impedance produced by SINGAL compared to data from an approximate EBC formulation for a dipole with length L=0.47 λ and a=0.005 λ. The only difference between the two formulations is the addition of a single basis function to model each end-cap within the EBC code. a) real part; b) imaginary part.

6. CONCLUSIONS

The "thin-wire" kernel has been widely used in formulations for the input impedance of wire antennas. If used "blindly," it can produce reasonable results when an appropriate number of basis functions are employed. However, the numerical results will not converge as the number of functions is increased. This complicates the validation of codes employing the thin-wire kernel. The initial validation study described in Section 3 failed to determine if results from different moment-method codes converged to the same answers, because results did not converge when the thin-wire kernel was employed. To date we have not pursued this study further, but it is apparent that additional complexity will be required in the codes to obtain converging results. One possibility is to employ the exact singular kernel and the associated two-dimensional numerical integration. The alternative appears to be the incorporation of end cap currents within a rigorous EBC formulation, which would enable the use of the thin-wire kernel for most of the required matrix element computations. However, the EBC formulation may require many basis functions to properly represent the end cap currents.

7. ACKNOWLEDGEMENTS

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